[9] Z. J. Jurek and J. Smalara, *On integrability with respect to infinitely divisible measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 29 (1981), 179–185.

[10] J. F. C. Kingman, *Completely random measures*, Pacific J. Math. 21, 1 (1967), 59–78.

[11] J. T. Lapreste, *Type et cotype, et mesures de Lévy sur les espaces de Banach*, Seminaire de l'Ecole Polytechnique, Expose n° XXIII, 1977/1978.

[12] W. Linde, *Operators generating stable measures on Banach spaces*, Friedrich-Schiller Universität Jena, G.D.R., Preprint 1980.

[13] W. Linde, V. Mandrekar and A. Weron, *p-stable measures and p-absolutely summing operators*, Lecture Notes in Math. 828 (1980), 167–178.

[14] M. B. Marcus and W. A. Woyczyński, *Stable measures and central limit theorems in spaces of stable type*, Trans. Amer. Math. Soc. 251 (1979), 71–102.

[15] Y. Okazaki, *Wiener integral by stable random measure*, Mem. Fac. Sci. Kyushu Univ. Ser. A 33, 1 (1979), 1–70.

[16] J. Rosiński, *On the convolution of cylindrical measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (1982), 379-383.

[17] — *Bilinear random integrals*, (to appear).

[18] J. Rosiński and E. Rowecka, *Random integrals and stable random measures on Banach spaces*, Preprint 1978.

[19] J. Rosiński and Z. Suchanecki, *On the space of vector-valued functions integrable with respect to the white noise*, Colloq. Math. 43, 1 (1980), 183–201.

[20] I. J. Schoenberg, *Metric spaces and positive define functions*, Trans. Amer. Math. Soc. 44 (1938), 522–536.

[21] R. Sztencel, *On boundedness and convergence of some Banach space valued random series*, Probab. Math. Statist. 2, 1 (1981), 83–88.

[22] D. H. Thang and N. Z. Tien, *On the extension of stable cylindrical measures*, Acta Math. Vietnamica 5, 1 (1980), 169–177.

[23] N. Z. Tien, *On the convergence of stable measures in a Banach space*, (to appear).

[24] K. Urbanik and W. A. Woyczyński, *Random integrals and Orlicz spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15 (1967), 161–169.

[25] W. A. Woyczyński, *On characteristic functional of linear processes with independent pieces*, ibid. 17 (1969), 551–557.

[26] — *Geometry and martingales in Banach spaces, Part* II: *Independent increments*, in: *Advances in probability*, Vol. IV, J. Kuelbs, Ed., Dekker, New York 1978.

[27] M. Yor, *Sur les intégrales stochastiques à valeurs dans un espace de Banach*, Ann. Inst. H. Poincaré, Séc. B 10 (1974), 31–36.

WROCŁAW UNIVERSITY
INSTITUTE OF MATHEMATICS
PL. Grunwaldzki 2/4
50-384 Wroclaw, Poland

Received May 10, 1982 (1758)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

On polynomial classification of locally convex spaces

by

DINAMÉRICO P. POMBO, Jr. (Rio de Janeiro)

Abstract. The purpose of this article is to develop a polynomial classification of locally convex spaces, analogous to the classical linear theory and to the holomorphic theory proposed recently by Nachbin.

1. Introduction. In this article we consider polynomially bornological, polynomially barreled, polynomially infrabarreled and polynomially Mackey locally convex spaces defined in [1] (see also [2] and [3]). Our purpose is to obtain a polynomial classification of locally convex spaces, analogous to the classical linear theory and to the holomorphic theory proposed by Nachbin in [15] and [16] (see also [4] and [17]). We must emphasize that, besides its intrinsic importance, the polynomial theory can clarify the holomorphic theory as was pointed out by Aragona in [1] (see also [2] and [3]). We now indicate briefly the organization of this article.

In Section 2 we study the $(\theta_1, \ldots, \theta_m)_\tau$-locally convex topologies in $\mathscr{L}(E_1, \ldots, E_m; F)$ and the θ_τ-locally convex topologies in $\mathscr{P}(^mE; F)$ (see [7], Chap. 3, for such a study in the linear case). We obtain an Alaoglu–Bourbaki theorem for homogeneous polynomials (Theorem 2.11) and Theorem 2.12, important tools in the subsequent sections.

In Section 3 we study the relationship among the above-mentioned polynomial concepts. As principal results of this section we obtain Theorem 3.34 and Theorem 3.37, both well known in the linear theory. As an application of such concepts, we prove Theorem 3.17, a generalization of a classical result of Bourbaki (see Remark 3.19).

In Section 4 we mention some examples of locally convex spaces which have such polynomial properties considered in the text, and observe that the linear notions are, really, more general than the corresponding polynomial ones.

This paper is based on part of my doctoral thesis ([18]), written under the guidance of Professor L. Nachbin, to whom I am sincerely indebted.

We shall adopt the notation and terminology of [4], [14], [15] and [16]. We will also use the following conventions. $\boldsymbol{N}$, $\boldsymbol{R}$ and $\boldsymbol{C}$, will denote the systems of natural integers, real numbers and complex numbers, respectively. All topological vector spaces will be assumed to be complex. If $E_1, \ldots, E_m$

and F are topological vector spaces, $\mathscr{L}_a(E_1, \ldots, E_m; F)$ (resp. $\mathscr{L}_{as}(E_1, \ldots, E_m; F)$) will denote the vector space of all m-linear mappings (resp. symmetric m-linear mappings) from $E_1 \times \ldots \times E_m$ into F, and $\mathscr{L}(E_1, \ldots, E_m; F)$ (resp. $\mathscr{L}_s(E_1, \ldots, E_m; F)$) will denote the vector subspace of all continuous m-linear mappings (resp. continuous symmetric m-linear mappings) from $E_1 \times \ldots \times E_m$ into F. If E and F are topological vector spaces and $m \in N$, then $\mathscr{P}_a(^mE; F)$ will denote the vector space of all m-homogeneous polynomials from E into F, and $P(^mE; F)$ will denote the vector subspace of all continuous m-homogeneous polynomials from E into F; $\mathscr{P}_a(E; F)$ will denote the vector space of all polynomials from E into F, and $\mathscr{P}(E; F)$ will denote the vector subspace of all continuous polynomials from E into F; $\mathscr{P}_M(^mE; F)$ will denote the vector subspace of $\mathscr{P}_a(^mE; F)$ of all m-homogeneous polynomials from E into F bounded on the bounded subsets of E, and $\mathscr{P}_M(E; F)$ will denote the vector subspace of $\mathscr{P}_a(E; F)$ of all polynomials from E into F bounded on the bounded subsets of E. When $F = C$, it is not included in the notation for function spaces; thus $\mathscr{P}(^mE)$ stands for $\mathscr{P}(^mE; C)$, etc.

2. Topologies on spaces of continuous multilinear mappings and spaces of continuous m-homogeneous polynomials. The following proposition is well known in the linear theory.

2.1. Proposition. *Let E and F be locally convex spaces, $m \in N^*$ and $P \in \mathscr{P}_a(^mE; F)$. The following conditions are equivalent:*

(i) *P is bounded on the compact subsets of E.*

(ii) *$P \in \mathscr{P}_M(^mE; F)$.*

(iii) *For every sequence $(x_n)_{n\in N}$ in E which converges to $x \in E$ in the Mackey sense, the sequence $(P(x_n))_{n\in N}$ converges to $P(x)$ in the Mackey sense.*

(iv) *For every sequence $(x_n)_{n\in N}$ in E which converges to zero in the Mackey sense, the sequence $(P(x_n))_{n\in N}$ converges to zero in the Mackey sense.*

(v) *For every sequence $(x_n)_{n\in N}$ in E which converges to zero in the Mackey sense, the sequence $(P(x_n))_{n\in N}$ converges to zero in F.*

(vi) *For every sequence $(x_n)_{n\in N}$ in E which converges to zero in the Mackey sense, the sequence $(P(x_n))_{n\in N}$ is bounded in F.*

Proof. We will prove (ii)$\Rightarrow$(iii) and (vi)$\Rightarrow$(i). It is obvious that (iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi), and the proof of (i)$\Rightarrow$(ii) is similar to the proof of (vi)$\Rightarrow$(i).

(ii)$\Rightarrow$(iii): Let $(x_n)_{n\in N}$ be a sequence in E converging to $x \in E$ in the Mackey sense. By definition, we can find a sequence $(\lambda_n)_{n\in N}$ of strictly positive real numbers, $(\lambda_n)_{n\in N} \to +\infty$, such that the sequence $(\lambda_n(x_n - x))_{n\in N}$ is bounded in E. Let $A \in \mathscr{L}_{as}(^mE; F)$ be such that $\hat{A} = P$. Then

$$P(x_n) - P(x) = \sum_{k=1}^{m} \binom{m}{k} A(x_n - x)^k (x)^{m-k},$$

where

$$A(x_n - x)^k (x)^{m-k} = A(\underbrace{x_n - x, \ldots, x_n - x}_{k \text{ times}}, \underbrace{x, \ldots, x}_{(m-k) \text{ times}}).$$

Since A is bounded on the compact subsets of E^m, the set

$$\{A(\lambda_n(x_n - x))^k (x)^{m-k};\ n \in N\} = \{\lambda_n^k A(x_n - x)^k (x)^{m-k};\ n \in N\}$$

is bounded in F, for each $1 \leqslant k \leqslant m$. Hence $(P(x_n))_{n\in N}$ converges to $P(x)$ in the Mackey sense.

(vi)$\Rightarrow$(i): Let $A \in \mathscr{L}_{as}(^mE; F)$ be such that $\hat{A} = P$. If there existed compact subsets $K_1, \ldots, K_m$ of E such that $A(K_1 \times \ldots \times K_m)$ were not bounded in F, we could find a balanced neighborhood V of zero in F such that for every $n \in N^*$, there is $y_n = (x_n^{(1)}, \ldots, x_n^{(m)}) \in K_1 \times \ldots \times K_m$, for which $A(y_n) \notin n^{m+1} V$. Hence the sequence

$$(z_n)_{n\in N^*} = \left(\frac{x_n^{(1)}}{n}, \ldots, \frac{x_n^{(m)}}{n}\right)_{n\in N^*}$$

converges to zero in E^m in the Mackey sense but the sequence $(A(z_n))_{n\in N^*}$ is not bounded in F since $A(z_n) \notin nV$ for each $n \in N^*$. ■

2.2. Definition. Let $E_1, \ldots, E_m$ be topological vector spaces and F a locally convex space. Let $\theta_1, \ldots, \theta_m$ be sets formed by bounded subsets of $E_1, \ldots, E_m$, respectively. The $(\theta_1, \ldots, \theta_m)_t$*-topology in* $\mathscr{L}(E_1, \ldots, E_m; F)$ is the locally convex topology defined by the family of seminorms:

$$A \in \mathscr{L}(E_1, \ldots, E_m; F) \mapsto \sup_{x_1 \in B_1, \ldots, x_m \in B_m} \beta(A(x_1, \ldots, x_m)) \in \boldsymbol{R}_+,$$

where β varies in the set of all continuous seminorms on F, B_1 varies in $\theta_1, \ldots, B_m$ varies in θ_m.

If $\theta_1, \ldots, \theta_m$ are the sets formed by all finite (resp. finite dimensional compact, or compact, or bounded) subsets of $E_1, \ldots, E_m$, respectively, we will denote $(\theta_1, \ldots, \theta_m)_t$ by τ_s (resp. τ_{of}, τ_o, τ_b).

In the same way, if E is a topological vector space, F is a locally convex, θ is a set formed by bounded subsets of E and $m \in N$, we define the θ_t-topology in $\mathscr{P}(^mE; F)$. We shall use the symbols τ_s, τ_{of}, τ_o and τ_b with the same meaning as in the multilinear case. Obviously, $\tau_s \leqslant \tau_{of} \leqslant \tau_o \leqslant \tau_b$ in both cases.

2.3. Remark. In Proposition 2.7 we shall prove that $\tau_s = \tau_{of}$ in $\mathscr{L}(E_1, \ldots, E_m; F)$ and $\mathscr{P}(^mE; F)$.

2.4. Proposition. *Let E be a topological vector space, F a locally convex space, $m \in N$ and $\mathscr{X} \subset \mathscr{P}(^mE; F)$ be equicontinuous. If θ is a set of bounded subsets of E, then $\mathscr{X}$ is bounded in $(\mathscr{P}(^mE; F), \theta_t)$.*

Proof. Fix β a continuous seminorm on F and $B \in \theta$. Since $\mathscr{X}$ is equicontinuous, there exists a neighborhood V of zero in E such that

$\beta(P(x)) \leqslant 1$, for every $x \in V$ and $P \in \mathscr{X}$. Since B is bounded, we can find $\lambda > 0$ such that $B \subset \lambda V$. Hence $\beta(P(x)) \leqslant \lambda^m$ for every $x \in B$ and $P \in \mathscr{X}$, and the proof is complete. ∎

2.5. Proposition. (a) *Let $E_1, \ldots, E_m$ be topological vector spaces, F a separated locally convex space and $\theta_1, \ldots, \theta_m$ coverings of $E_1, \ldots, E_m$ by bounded subsets of $E_1, \ldots, E_m$, respectively. Then $(\mathscr{L}(E_1, \ldots, E_m; F), (\theta_1, \ldots, \theta_m)_t)$ is a separated locally convex space.*

(b) *Let E be a topological vector space, F a separated locally convex space, θ a covering of E by bounded subsets of E and $m \in \boldsymbol{N}$. Then $(\mathscr{P}(^mE; F), \theta_t)$ is a separated locally convex space.*

Proof. We shall prove (a); the proof of (b) is similar. Let $A \in \mathscr{L}(E_1, \ldots, E_m; F)$, $A \neq 0$. There exists $x_1 \in E_1, \ldots, x_m \in E_m$, such that $A(x_1, \ldots, x_m) \neq 0$. Since $\theta_1, \ldots \theta_m$ are coverings of $E_1, \ldots, E_m$, respectively, we can find $B_1 \in \theta_1, \ldots, B_m \in \theta_m$ such that $x_1 \in B_1, \ldots, x_m \in B_m$. Since F is separated, there exists a continuous seminorm β on F such that $\beta(A(x_1, \ldots, x_m)) > 0$. Thus

$$\sup_{t_1 \in B_1, \ldots, t_m \in B_m} \beta(A(t_1, \ldots, t_m)) \geqslant \beta(A(x_1, \ldots, x_m)) > 0,$$

and the proof is complete. ∎

For $m = 0$ or $m = 1$, the mapping

$$A \in (\mathscr{L}_s(^mE; F), \theta_t) \mapsto \hat{A} \in (P(^mE; F), \theta_t)$$

is clearly a locally convex spaces isomorphism, where θ is an arbitrary set formed by bounded subsets of E. In general, the following proposition holds.

2.6. Proposition. *Let E be a topological vector space, F a locally convex space, $m \in \boldsymbol{N}^*$ and θ a set formed by bounded subsets of E such that $\lambda_1 \theta + \ldots + \lambda_m \theta \subset \theta$ for every $\lambda_1, \ldots, \lambda_m \in \boldsymbol{C}$. Under these assumptions, the vector spaces isomorphism*

$$\Phi\colon A \in \mathscr{L}_s(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$$

is a locally convex spaces isomorphism if $\mathscr{L}_s(^mE; F)$ is endowed with $(\theta, \ldots, \theta)_t$ and $\mathscr{P}(^mE; F)$ is endowed with θ_t. (We will also denote by $(\theta, \ldots, \theta)_t$ the locally convex topology induced by $(\mathscr{L}(^mE; F), (\theta, \ldots, \theta)_t)$ on $\mathscr{L}_s(^mE; F)$.)

Proof. Since Φ is obviously continuous, it remains to show that $\Phi^{-1}\colon (\mathscr{P}(^mE; F), \theta_t) \to (\mathscr{L}_s(^mE; F), (\theta, \ldots, \theta)_t)$ is continuous. Fix $B_1, \ldots, B_m \in \theta$ and β a continuous seminorm on F. For every $A \in \mathscr{L}_s(^mE; F)$,

$$(*) \quad \sup_{x_1 \in B_1, \ldots, x_m \in B_m} \beta(A(x_1, \ldots, x_m)) \leqslant \frac{1}{m!\,2^m} \sum_{\substack{\varepsilon_i = \pm 1 \\ 1 \leqslant i \leqslant m}} \sup_{x_1 \in B_1, \ldots, x_m \in B_m} \beta(\hat{A}(\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m)).$$

For each choice of $\varepsilon_i = \pm 1$ $(1 \leqslant i \leqslant m)$, the bounded set $\{\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m;\ x_1 \in B_1, \ldots, x_m \in B_m\} \in \theta$, by hypothesis. Hence the mapping

$$P \in \mathscr{P}(^mE; F) \mapsto \sup_{x_1 \in B_1, \ldots, x_m \in B_m} \beta(P(\varepsilon_1 x_1 + \ldots + \varepsilon_m x_m)) \in \boldsymbol{R}_+$$

is a continuous seminorm on $(\mathscr{P}(^mE; F), \theta_t)$, and inequality $(*)$ guarantees the continuity of Φ^{-1}. ∎

2.7. Proposition. (a) *Let $E_1, \ldots, E_m$ be topological vector spaces and F a locally convex space. Then $\tau_s = \tau_{of}$ on $\mathscr{L}(E_1, \ldots, E_m; F)$.*

(b) *Let E be a topological vector space, F a locally convex space and $m \in \boldsymbol{N}$. Then $\tau_s = \tau_{of}$ on $\mathscr{P}(^mE; F)$.*

Proof. (a): This follows from the multilinearity and from the fact that every finite-dimensional compact subset of a topological vector space is contained in a finite union of compact sets, each one of them being the convex hull of a finite set.

(b): If suffices to prove that $\tau_{of} \leqslant \tau_s$.

By Proposition 2.6, the mapping

$$\hat{A} \in \mathscr{P}(^mE; F) \mapsto A \in \mathscr{L}_s(^mE; F) \subset \mathscr{L}(^mE; F)$$

is continuous from τ_s to τ_s. By (a), $\tau_s = \tau_{of}$ on $\mathscr{L}(^mE; F)$. Hence the mapping $\hat{A} \in \mathscr{P}(^mE; F) \mapsto A \in \mathscr{L}_s(^mE; F) \subset \mathscr{L}(^mE; F)$ is continuous from τ_s to τ_{of}. Since the mapping $A \in \mathscr{L}(^mE; F) \mapsto \hat{A} \in \mathscr{P}(^mE; F)$ is continuous from τ_{of} to τ_{of}, we obtain by composition that the mapping $P \in \mathscr{P}(^mE; F) \mapsto P \in \mathscr{P}(^mE; F)$ is continuous from τ_s to τ_{of}. ∎

In the following proposition $\mathscr{F}(E_1 \times \ldots \times E_m; F)$ will denote the vector space of all mappings from $E_1 \times \ldots \times E_m$ into F, and τ_s will denote the topology of simple convergence on $\mathscr{F}(E_1 \times \ldots \times E_m; F)$ defined by the family of seminorms

$$f \in \mathscr{F}(E_1 \times \ldots \times E_m; F) \mapsto \beta(f(x_1, \ldots, x_m)) \in \boldsymbol{R}_+,$$

where β varies in the set of all continuous seminorms on F, x_1 varies in $E_1, \ldots, x_m$ varies in E_m.

2.8. Proposition. *Let $E_1, \ldots, E_m$ be topological vector spaces, F a locally convex space and $\mathscr{X} \subset \mathscr{L}(E_1, \ldots, E_m; F)$ be equicontinuous. Then $\bar{\mathscr{X}}$ (closure of $\mathscr{X}$ in $(\mathscr{F}(E_1 \times \ldots \times E_m; F), \tau_s)$) is equicontinuous and $\bar{\mathscr{X}} \subset \mathscr{L}(E_1, \ldots, E_m; F)$ if F is separated.*

Proof. The equicontinuity of $\bar{\mathscr{X}}$ follows from [8], chap. 10, p. 28, Proposition 6.

Now suppose F separated. Using the same argument as in [8], chap. 10, p. 37, Proposition 4, we get that $\mathscr{L}_a(E_1, \ldots, E_m; F)$ is closed in $(\mathscr{F}(E_1 \times \ldots \times E_m; F), \tau_s)$, and this implies that $\bar{\mathscr{X}} \subset \mathscr{L}(E_1, \ldots, E_m; F)$. ∎

2.9. Corollary. *Let $E_1, \ldots, E_m$ be topological vector spaces, F a separated locally convex space and $\mathscr{X} \subset \mathscr{L}(E_1, \ldots, E_m; F)$ be equicontinuous. Then $\mathscr{X}$ is relatively compact in $(\mathscr{L}(E_1, \ldots, E_m; F), \tau_s)$ if and only if $\mathscr{X}(x_1, \ldots, x_m) = \{A(x_1, \ldots, x_m);\ A \in \mathscr{X}\}$ is relatively compact in F for every $x_1 \in E_1, \ldots, x_m \in E_m$.*

Proof. Let us recall, at first, that $(\mathscr{F}(E_1 \times \ldots \times E_m; F), \tau_s)$ is isomorphic (algebraically and topologically) to the locally convex product space $F^{E_1 \times \ldots \times E_m}$.

By a corollary to Tychonoff's theorem ([8], Chap. 1, p. 100), we get that $T \subset \mathscr{F}(E_1 \times \ldots \times E_m; F)$ is relatively compact for τ_s if and only if $T(x_1, \ldots, x_m) = \{f(x_1, \ldots, x_m);\ f \in T\}$ is relatively compact in F, for every $x_1 \in E_1, \ldots, x_m \in E_m$. Thus, an application of Proposition 2.8 completes the proof. ■

2.10. Corollary. *Let E be a topological vector space, F a separated locally convex space, $m \in N$ and $X \subset \mathscr{P}(^mE; F)$ be equicontinuous. Then $\mathscr{X}$ is relatively compact in $(\mathscr{P}(^mE; F), \tau_s)$ if and only if $\mathscr{X}(x) = \{P(x);\ P \in \mathscr{X}\}$ is relatively compact in F for every $x \in E$.*

Proof. The corollary is clear for $m = 0$. Suppose $m \geqslant 1$. Necessity follows from Proposition 2.6 and Corollary 2.9. Let us prove sufficiency. By using the Polarization Formula and the fact that every finite sum of relatively compact subsets of a topological vector space is again relatively compact, we get that $\check{\mathscr{X}}(x_1, \ldots, x_m) = \{A(x_1, \ldots, x_m);\ A \in \check{\mathscr{X}}\}$ is relatively compact in F, for every $x_1 \in E_1, \ldots, x_m \in E_m$. Applying, successively, Corollary 2.9 and Proposition 2.6 we finish the proof. ■

We are now able to prove an Alaoglu–Bourbaki theorem for homogeneous polynomials.

2.11. Theorem. *Let E be a topological vector space, F a separated semi-Montel locally convex space, $m \in N$ and $X \subset \mathscr{P}(^mE; F)$ equicontinuous. Then X is relatively compact in $(\mathscr{P}(^mE; F), \tau_s)$.*

Proof. By Proposition 2.4, $\mathscr{X}(x) = \{P(x);\ P \in \mathscr{X}\}$ is bounded in F for every $x \in E$. Since F is semi-Montel, $\mathscr{X}(x)$ is relatively compact in F, and Corollary 2.10 guarantees that $\mathscr{X}$ is relatively compact in $(\mathscr{P}(^mE; F), \tau_s)$. ■

2.12. Theorem. *Let $E_1, \ldots, E_m$ be topological vector spaces and F a separated quasi-complete locally convex space. Let $\theta_1, \ldots, \theta_m$ be coverings of $E_1, \ldots, E_m$ by bounded subsets of $E_1, \ldots, E_m$, respectively, and $\mathscr{X} \subset \mathscr{L}(E_1, \ldots, E_m; F)$ be equicontinuous and closed in $(\mathscr{L}(E_1, \ldots, E_m; F), (\theta_1, \ldots, \theta_m)_t)$. Then $\mathscr{X}$ is a complete subset of $(\mathscr{L}(E_1, \ldots, E_m; F), (\theta_1, \ldots, \theta_m)_t)$.*

Proof. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a Cauchy net in $\mathscr{X}$. We must find $A \in \mathscr{X}$ such that $(A_\lambda)_{\lambda \in \Lambda}$ converges to A in $\mathscr{X}$. Since $\theta_1, \ldots, \theta_m$ are coverings of $E_1, \ldots, E_m$, respectively, the mapping

$$A \in (\mathscr{L}(E_1, \ldots, E_m; F), (\theta_1, \ldots, \theta_m)_t) \mapsto A(x_1, \ldots, x_m) \in F$$

is uniformly continuous for each $x_1 \in E_1, \ldots, x_m \in E_m$. Thus, using the equicontinuity of $\mathscr{X}$ and the fact that F is quasi-complete, we get that for each $x_1 \in E_1, \ldots, x_m \in E_m$ the net $(A_\lambda(x_1, \ldots, x_m))_{\lambda \in \Lambda}$ converges to a point $A(x_1, \ldots, x_m) \in F$. Now we need to verify that the mapping

$$A\colon (x_1, \ldots, x_m) \in E_1 \times \ldots \times E_m \mapsto A(x_1, \ldots, x_m) \in F$$

belongs to $\mathscr{L}(E_1, \ldots, E_m; F)$.

In fact, by the definition of A,

$$A \in \overline{(A_\lambda)_{\lambda \in \Lambda}} \subset \bar{\mathscr{X}}$$

(closures taken in $(\mathscr{F}(E_1 \times \ldots \times E_m; F), \tau_s)$)

By Proposition 2.8, $A \in \mathscr{L}(E_1, \ldots, E_m; F)$.

Finally, it is easy to verify that $(A_\lambda)_{\lambda \in \Lambda}$ converges to A in $(\mathscr{L}(E_1, \ldots, E_m; F), (\theta_1, \ldots, \theta_m)_t)$. Thus $A \in \mathscr{X}$ since

$$A \in \overline{(A_\lambda)_{\lambda \in \Lambda}}^{(\theta_1, \ldots, \theta_m)_t} \subset \bar{\mathscr{X}}^{(\theta_1, \ldots, \theta_m)_t} = \mathscr{X}$$

(by hypothesis). ■

2.13. Remark. Theorem 2.12 is well known in the linear case ([7], Chap. 3, p. 30, Theorem 4).

2.14. Corollary. *Let E be a topological vector space, F a separated quasi-complete locally convex space, $m \in N, m \geqslant 1$, and θ a covering of E formed by bounded subsets of E such that $\lambda_1 \theta + \ldots + \lambda_m \theta \subset \theta$ for every $\lambda_1, \ldots, \lambda_m \in C$. If $\mathscr{X} \subset \mathscr{P}(^mE; F)$ is equicontinuous and closed in $(\mathscr{P}(^mE; F), \theta_t)$, then X is a complete subset of $(\mathscr{P}(^mE; F), \theta_t)$.*

Proof. If $m = 1$, the corollary holds for every covering θ of E. Suppose $m \geqslant 2$. The corresponding $\check{\mathscr{X}} \subset \mathscr{L}_s(^mE; F)$ is equicontinuous and closed in $(\mathscr{L}_s(^mE; F), (\theta, \ldots, \theta)_t)$ (Proposition 2.6). By Theorem 2.12, $\check{\mathscr{X}}$ is complete for $(\theta, \ldots, \theta)_t$, and a new application of Proposition 2.6 completes the proof. ■

3. Polynomially bornological, polynomially barreled, polynomially infrabarreled and polynomially Mackey spaces.

3.1. Definition ([1]). A locally convex space E is said to be *polynomially bornological* if for every locally convex space F and every $m \in N$, $\mathscr{P}_M(^mE; F) = \mathscr{P}(^mE; F)$.

3.2. Remark. (a) It suffices to let F be a normed space in Definition 3.1.

(b) Every holomorphically bornological space ([16], Definition 3) is polynomially bornological but there are polynomially bornological spaces which are not holomorphically bornological ([4], Example 22). Every polynomially bornological space is bornological, but there are bornological spaces which are not polynomially bornological (Example 4.8).

3.3. PROPOSITION. *Every metrizable locally convex space E is polynomially bornological.*

Proof. Let F be a locally convex space, $m \in N$ and $P \in \mathscr{P}_M({}^mE; F)$. Let $(x_n)_{n\in N}$ be a sequence in E converging to zero in E. By [12], p. 149, $(x_n)_{n\in N}$ converges to zero in the Mackey sense. By Proposition 2.1, $(P(x_n))_{n\in N}$ converges to zero in F and hence P is continuous. ∎

3.4. Remark. More precisely, one can prove that every metrizable locally convex space is holomorphically bornological ([16], Proposition 1).

3.5. PROPOSITION. *For a locally convex space E, the following conditions are equivalent:*

(i) *E is polynomially bornological.*

(ii) *For every locally convex space F, $\mathscr{P}_M(E; F) = \mathscr{P}(E; F)$.*

(iii) *For every locally convex space F and for every $m \in N$, each $P \in \mathscr{P}_a({}^mE; F)$, which is continuous from E_B into F for every absolutely convex closed and bounded subset B of E, is necessarily continuous.*

(iv) *For every locally convex space F and for every $m \in N$, each $P \in \mathscr{P}_a({}^mE; F)$, which is bounded on the sequences in E which converge to zero, is necessarily continuous.*

Proof. (i) ⇒ (ii): This follows from the known fact that $\mathscr{P}_M(E; F) = \bigoplus_{m\in N} \mathscr{P}_M({}^mE; F)$.

(iv) ⇒ (i): Obvious.

(ii) ⇒ (iii): Since the assertion is obvious for $m = 0$, let us suppose $m \geqslant 1$. Let $P \in \mathscr{P}_a({}^mE; F)$ be such that $P \in \mathscr{P}({}^m(E_B); F)$, for each absolutely convex closed and bounded subset B of E. If $(x_n)_{n\in N}$ is a sequence in E which converges to zero in the Mackey sense, there is an absolutely convex closed and bounded subset B of E such that $(x_n)_{n\in N}$ converges to zero in the normed space E_B ([13], p. 382). Hence the sequence $(P(x_n))_{n\in N}$ is bounded in F. By Proposition 2.1, $P \in \mathscr{P}_M({}^mE; F)$, and (ii) assures the continuity of P.

(iii) ⇒ (iv): Since the assertion is obvious for $m = 0$, let us suppose $m \geqslant 1$. Let $P \in \mathscr{P}_a({}^mE; F)$ be bounded on the sequences in E which converge to zero, and let B be an absolutely convex, closed and bounded subset of E. If $(x_n)_{n\in N}$ converges to zero in E_B, then $(x_n)_{n\in N}$ converges to zero in the Mackey sense. Hence the sequence $(P(x_n))_{n\in N}$ is bounded in F. By Proposition 2.1, $P \in \mathscr{P}_M({}^m(E_B); F)$, and an application of Proposition 3.3 and (iii) guarantees the continuity of P. ∎

3.6. PROPOSITION. *If E is a polynomially bornological space, F is a complete, locally convex space and $m \in N$, then $(\mathscr{P}({}^mE; F), \tau_o)$ and $(\mathscr{P}({}^mE; F), \tau_b)$ are complete.*

Proof. We will prove it for τ_b. The proof is analogous for the other case. Let $(P_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in $(\mathscr{P}({}^mE; F), \tau_b)$. Thus for each $x \in E$, the net $(P_\lambda(x))_{\lambda\in\Lambda}$ is a Cauchy net in F and, by completeness of F, converges to $P(x) \in F$. It is easy to verify that the mapping P: $x \in E \mapsto P(x) \in F$ is an m-homogeneous polynomial. By using the facts that $(P_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $(\mathscr{P}({}^mE; F), \tau_b)$ and that each $P_\lambda \in \mathscr{P}({}^mE; F)$, we obtain that P is bounded on the bounded subsets of E and hence $P \in \mathscr{P}({}^mE; F)$ since E is polynomially bornological. Finally, using again the fact that $(P_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in $(\mathscr{P}({}^mE; F), \tau_b)$, we get that $(P_\lambda)_{\lambda\in\Lambda}$ converges to P in $(\mathscr{P}({}^mE; F), \tau_b)$ as it was to be proved. ∎

3.7. Remark. (a) The argument used in the proof of Proposition 3.6 guarantees that, for each $m \in N$, $(\mathscr{P}({}^mE, F), \theta_t)$ is complete if E is a polynomially bornological space, F is a complete locally convex space and θ is a set formed by bounded subsets of E which contains all compact subsets of E.

(b) In the linear and holomorphic cases Proposition 3.6 is well known ([6], corollary of Theorem 3, and [16], Proposition 3).

3.8. DEFINITION ([1]). A locally convex space E is said to be *polynomially infrabarreled* if for every locally convex space F and for every $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}({}^mE; F)$ is equicontinuous if $\mathscr{X}$ is bounded on the compact subsets of E.

3.9. Remark. Every holomorphically infrabarreled space ([16], Definition 9) is polynomially infrabarreled but there are polynomially infrabarreled spaces which are not holomorphically infrabarreled ([1], p. 29). Every polynomially infrabarreled space is infrabarreled but there are infrabarreled spaces which are not polynomially infrabarreled (Example 4.8).

3.10. PROPOSITION. *A locally convex space E is polynomially infrabarreled if and only if for each $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}({}^mE)$ is equicontinuous if $\mathscr{X}$ is bounded on the compact subsets of E.*

Proof. Necessity being clear, let us prove sufficiency. By [12], p. 158, exercise 1, it suffices to prove that for every equicontinuous subset T of F', the set $T \circ \mathscr{X} = \{\varphi \circ P;\ \varphi \in F',\ P \in \mathscr{X}\}$ is an equicontinuous subset of $\mathscr{P}({}^mE)$. In fact, if T is an equicontinuous subset of F', $T \circ \mathscr{X}$ is bounded on the compact subsets of E, and hence equicontinuous by hypothesis. Thus E is polynomially infrabarreled. ∎

As in the linear and holomorphic cases, we have the following.

3.11. PROPOSITION. *A locally convex space E is polynomially bornological if and only if E is polynomially infrabarreled and for each $m \in N$, $\mathscr{P}_M({}^mE) = \mathscr{P}({}^mE)$.*

Proof. *Necessity.* Let $m \in N$ and $\mathscr{X} \subset \mathscr{P}({}^mE)$ be bounded on the compact subsets of E. We will prove that the corresponding $\check{\mathscr{X}} \subset \mathscr{L}_s({}^mE)$ is equicontinuous. By Proposition 2.6, $\check{\mathscr{X}}$ is bounded on the compact subsets of E^m.

Define g: $E^m \to l^\infty(\check{\mathscr{X}})$ by $g(x_1, \ldots, x_m)(A) = A(x_1, \ldots, x_m)$, if $x_1, \ldots, x_m \in E$ and $A \in \check{\mathscr{X}}$. Obviously, $g \in \mathscr{L}_{as}({}^mE; l^\infty(\check{\mathscr{X}}))$. Moreover, g is bounded on the compact subsets $K_1 \times \ldots \times K_m$ of E^m since

$$\sup_{x_1 \in K_1, \ldots, x_m \in K_m} \|g(x_1, \ldots, x_m)\|_{l^\infty(\tilde{\mathscr{X}})} = \sup_{x_1 \in K_1, \ldots, x_m \in K_m, A \in \tilde{\mathscr{X}}} |A(x_1, \ldots, x_m)|,$$

and the last supremum is finite. Since E is polynomially bornological, g is continuous and the continuity of g implies immediately the equicontinuity of $\tilde{\mathscr{X}}$. By Proposition 3.10, E is polynomially infrabarreled. The rest of necessity is clear.

Sufficiency. Let F be a locally convex space, $m \in N$ and $P \in \mathscr{P}_M(^mE; F)$. Fix a continuous seminorm β on F. Consider $T = \{\varphi \in F'; |\varphi(y)| \leqslant \beta(y)$ for every $y \in F\}$ and $\mathscr{X} = T \circ P$. By hypothesis, $\mathscr{X} \subset \mathscr{P}(^mE)$, and $\mathscr{X}$ is bounded on the compact subsets of E. Thus $\mathscr{X}$ is equicontinuous since E is polynomially infrabarreled. By the Hahn–Banach Theorem, $\beta \circ P$ is locally bounded, and since β is arbitrary, P is continuous. ■

3.12. Remark. As we have seen in Proposition 3.11, every polynomially bornological space is polynomially infrabarreled but there are polynomially infrabarreled spaces which are not polynomially bornological (Example 4.5). Under certains conditions, the converse is true.

3.13. Proposition. *Let E be a polynomially infrabarreled space and $(A_n)_{n \in N}$ a sequence in $\mathscr{L}(E; E)$ satisfying the following conditions*:

(a) *For each $x \in E$, $(A_n(x))_{n \in N}$ converges to x in the Mackey sense.*

(b) *For each bounded subset B of E, $\bigcup_{n \in N} A_n(B)$ is bounded in E.*

(c) *For each $n \in N$, there exists a polynomially bornological space E_n, and continuous linear mappings $L_n \in \mathscr{L}(E_n; E)$, $T_n \in \mathscr{L}(E; E_n)$ such that $A_n = L_n \circ T_n$. Under these assumptions, E is polynomially bornological.*

Proof. By Proposition 3.11, it suffices to show that for each $m \in N$, $\mathscr{P}_M(^mE) = \mathscr{P}(^mE)$. Fix $P \in \mathscr{P}_M(^mE)$. For each $n \in N$, let $Q_n = P \circ A_n \in \mathscr{P}_a(^mE)$. Since E_n is polynomially bornological, $Q_n \in \mathscr{P}(^mE)$. Applying (a) we obtain that $(Q_n)_{n \in N}$ converges to P in $(\mathscr{F}(E), \tau_s)$. To finish the proof it suffices to observe that $(Q_n)_{n \in N}$ is equicontinuous but this follows from (b) and from the fact that E is polynomially infrabarreled. ■

3.14. Proposition. *If E is a polynomially infrabarreled space, F is a separated quasi-complete locally convex space and $m \in N$, then $(\mathscr{P}(^mE; F), \tau_b)$ is quasi-complete.*

Proof. For $m = 0$ the proposition is clear. If $m = 1$, $(\mathscr{P}(^mE; F), \tau_b) = (\mathscr{L}(E; F), \tau_b)$ is quasi-complete since E is infrabarreled.

Suppose $m \geqslant 2$ and let $\mathscr{X} \subset \mathscr{P}(^mE; F)$ be closed and bounded for τ_b. Since E is polynomially infrabarreled, $\mathscr{X}$ is equicontinuous, and an application of Corollary 2.14 completes the proof. ■

3.15. Proposition. *Every polynomially infrabarreled space E satisfies the following property*:

For each locally convex space F and for each $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}(^mE; F)$ is relatively compact for τ_0 if $\mathscr{X}$ is bounded for τ_0 and for each $x \in E$, $\mathscr{X}(x) = \{P(x); P \in \mathscr{X}\}$ is relatively compact in F.

Proof. First let us suppose that F is a separated locally convex space, $m \in N$ and $\mathscr{X}$ is a bounded subset of $\mathscr{P}(^mE; F)$ for τ_0 such that for each $x \in E$, $\mathscr{X}(x)$ is relatively compact in F. Since E is polynomially infrabarreled, we get that $\mathscr{X}$ is equicontinuous, and Ascoli's theorem garantees that $\mathscr{X}$ is relatively compact in $(C(E; F), \tau_0)$. Moreover, $\mathscr{P}(^mE; F)$ is closed in $C(E; F)$ for τ_0, and this implies that $\mathscr{X}$ is relatively compact in $\mathscr{P}(^mE; F)$ for τ_0.

Finally, if F is an arbitrary locally convex space, we are reduced to the first case by considering the separated locally convex space associated to F. ■

To prove Theorem 3.17, we will need the following lemma.

3.16. Lemma. *Let E be a polynomially infrabarreled space, F a locally convex space, $m \in N$ and $\mathscr{X} \subset \mathscr{P}(^mE; F)$. Then $\mathscr{X}$ is equicontinuous if and only if for every sequence $(x_n)_{n \in N}$ in E which converges to zero, the set $\{P(x_n); P \in \mathscr{X}, n \in N\}$ is bounded in F.*

Proof. Necessity follows immediately from Proposition 2.4. Let us turn to sufficiency. By the polarization formula, the corresponding $\tilde{\mathscr{X}} \subset \mathscr{L}_s(^mE; F)$ is bounded on the sequences in E^m which converge to zero. Using the argument of the proof of Proposition 2.1, we obtain that $\tilde{\mathscr{X}}$ is bounded on the compact subsets of E^m and hence equicontinuous since E is polynomially infrabarreled. Thus $\mathscr{X}$ is equicontinuous. ■

3.17. Theorem. *Let E be a barreled locally convex space, F a locally convex space such that $E \times F$ is polynomially infrabarreled and G a locally convex space. If $\mathscr{X} \subset \mathscr{L}(E, F; G)$ is separately equicontinuous, then $\mathscr{X}$ is equicontinuous.*

Proof. Let $((x_n, y_n))_{n \in N}$ be a sequence in $E \times F$ which converges to $(0, 0) \in E \times F$. We shall prove that $B = \{A(x_n, y_n); A \in \mathscr{X}, n \in N\}$ is bounded in G. Fix V to be a balanced neighborhood of zero in G. We claim that for each $x \in E$, the set $B_x = \{A(x, y_n); A \in \mathscr{X}, n \in N\}$ is bounded in G. By hypothesis, the family of continuous linear mappings $\{t \in F \mapsto A(x, t) \in G; A \in \mathscr{X}\} \subset \mathscr{L}(F; G)$ is equicontinuous and the sequence $(y_n)_{n \in N}$ converges to zero in F. Thus there exists $n_0 \in N$ such that $A(x, y_n) \in V$, for every $A \in \mathscr{X}$ and $n \geqslant n_0$. Moreover, for each $j \in \{0, \ldots, n_0 - 1\}$, the set $\{A(x, y_j); A \in \mathscr{X}\}$ is bounded in F (Proposition 2.4). Hence there exists $\lambda \geqslant 1$ such that $A(x, y_j) \in \lambda V$ for each $A \in X$ and $j \in \{0, \ldots, n_0 - 1\}$, and therefore $B_x \subset \lambda V$ since V is balanced. We have verified that the set of continuous linear mappings $\{z \in E \mapsto A(z, y_n) \in G; A \in \mathscr{X}, n \in N\}$ is bounded in $(\mathscr{L}(E; G), \tau_s)$ and, consequently, equicontinuous since E is barreled. Thus there exists $n_1 \in N$ such that $A(x_n, y_n) \in V$ for every $A \in \mathscr{X}$ and $n \geqslant n_1$. Using the same argument as before, we get $\mu \geqslant 1$ such that $B \subset \mu V$, and so $\mathscr{X}$ is equicontinuous (Lemma 3.16). ■

3.18. Example. The condition that $E \times F$ is polynomially infrabarreled cannot be omitted from the hypothesis of Theorem 3.17.

In fact, let $E = C^N \times C^{(N)}$ be the cartesian product of the infinite denumerable cartesian power of C by the infinite denumerable direct sum of C; C^N is a Fréchet space, hence barreled. For each $n \in N$, define $P_n \in \mathscr{P}(^2E)$ by

$$P_n((x_m)_{m\in N}, (y_m)_{m\in N}) = x_n y_n \quad \text{if} \quad (x_m)_{m\in N} \in C^N \quad \text{and} \quad (y_m)_{m\in N} \in C^{(N)}.$$

The sequence $(P_n)_{n\in N}$ is bounded on the compact subsets of E but $(P_n)_{n\in N}$ is not equicontinuous (Example 4.8). Hence E is not polynomially infrabarreled. Finally, it is easy to prove that $\mathscr{X}$ is separately equicontinuous, where $\mathscr{X} = \{P_n;\ n \in N\}$.

3.19. Remark. (a) Theorem 3.18 generalizes a classical result of Bourbaki ([7], chap. 3, p. 28, Theorem 3).

(b) The locally convex spaces E and F which appear in the hypothesis of Theorem 3.18 are necessarily polynomially infrabarreled (Proposition 3.27).

3.20. Definition ([1]). A locally convex space E is said to be *polynomially barreled* if for every locally convex space F and for every $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}(^mE; F)$ is equicontinuous if $\mathscr{X}$ is bounded on the finite-dimensional compact subsets of E.

3.21. Remark. (a) By Proposition 2.7, a locally convex space E is polynomially barreled if for every locally convex space F and for every $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}(^mE; F)$ is equicontinuous if $\mathscr{X}$ is pointwise bounded.

(b) Every holomorphically barreled space ([16], Definition 6) is polynomially barreled but there are polynomially barreled spaces which are not holomorphically barreled ([1], p. 29). Every polynomially barreled space is barreled but there are barreled spaces which are not polynomially barreled (Example 4.8).

(c) Every polynomially barreled space is polynomially infrabarreled but there are polynomially infrabarreled spaces which are not polynomially barreled (Example 3.22).

3.22. Example. Let $E = C^{(N)}$ be endowed with the supremum norm, E is polynomially bornological (Proposition 3.3) and hence polynomially infrabarreled (Proposition 3.11). Since E is not barreled, E is not polynomially barreled.

As in the infrabarreled case, we have the following

3.23. Proposition. *A locally convex space E is polynomially barreled if and only if for every $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}(^mE)$ is equicontinuous if $\mathscr{X}$ is pointwise bounded.*

Proof. Similar to the proof of Proposition 3.10, noting Remark 3.21 (a).

3.24. Proposition. *If E is a separated polynomially barreled (resp. polynomially infrabarreled) space, then $\hat{E}$ (a completion of E) is polynomially barreled (resp. polynomially infrabarreled).*

Proof. By a classical result, every $P \in \mathscr{P}(^mE)$ can be extended to $\bar{P} \in \mathscr{P}(^m(\hat{E}))$ (in a unique way). The proof follows from the fact that $\mathscr{X} \subset \mathscr{P}(^mE)$ is equicontinuous if and only if the corresponding $\bar{\mathscr{X}} = \{\bar{P};\ P \in \mathscr{X}\} \subset \mathscr{P}(^m(\hat{E}))$ is equicontinuous.

3.25. Proposition. *A locally convex space E is polynomially barreled if and only if E is polynomially infrabarreled and E verifies the following property:*

For every locally convex space F and for every $m \in N$, a subset $\mathscr{X} \subset \mathscr{P}(^mE; F)$ is relatively compact for τ_0 if X is pointwise bounded and for each $x \in E$, $\mathscr{X}(x) = \{P(x);\ P \in \mathscr{X}\}$ is relatively compact in F.

Proof. *Necessity.* The property is proved using the argument of the proof of Proposition 3.15. The rest of necessity is clear (Remark 3.21 (c)).

Sufficiency. Let us suppose that E is a polynomially infrabarreled space which satisfies the property. Let $m \in N$ and let $\mathscr{X} \subset \mathscr{P}(^mE)$ be pointwise bounded. By the property, $\mathscr{X}$ is relatively compact in $(\mathscr{P}(^mE), \tau_0)$ and hence equicontinuous since E is polynomially infrabarreled. An application of Proposition 3.23 completes the proof. ■

If E is a barreled space, F is a separated quasi-complete locally convex space, θ is a covering of E by bounded subsets of E, and $m = 0$ or $m = 1$, then $(\mathscr{P}(^mE; F), \theta_t)$ is quasi-complete. In general, we have the following

3.26. Proposition. *Let E be a polynomially barreled space, F a separated quasi-complete locally convex space, $m \in N^*$ and θ a covering of E by bounded subsets of E such that $\lambda_1 \theta + \ldots + \lambda_m \theta \subset \theta$ for every $\lambda_1, \ldots, \lambda_m \in C$. Then $(\mathscr{P}(^mE, F), \theta_t)$ is quasi-complete.*

Proof. If $m = 1$, the proposition holds without any restriction on the covering θ. Suppose $m \geqslant 2$, and take $\mathscr{X} \subset \mathscr{P}(^mE; F)$ closed and bounded in $(\mathscr{P}(^mE; F), \theta_t)$. Since $\tau_s \leqslant \theta_t$, $\mathscr{X}$ is a fortiori bounded in $(\mathscr{P}(^mE; F), \tau_s)$ and hence equicontinuous because E is polynomially barreled. By Corollary 2.14, $\mathscr{X}$ is a complete subset of $(\mathscr{P}(^mE; F), \theta_t)$. ■

3.27. Proposition. *Let E and F be locally convex spaces such that $E \times F$ is polynomially bornological (polynomially barreled, polynomially infrabarreled). Then E and F are of the same type.*

Proof. We shall prove the bornological case. Let G be a locally convex space, $m \in N$ and $P \in \mathscr{P}_M(^mE; G)$.

Let pr_E be the continuous projection of $E \times F$ onto E. Since $E \times F$ is polynomially bornological, $Q = P \circ \mathrm{pr}_E \in \mathscr{P}(^m(E \times F); G)$, and this implies immediately that P is continuous. Hence E is polynomially bornological. In a similar way we prove that F is polynomially bornological, and the other cases are analogous. ■

3.28. Proposition. *Let $E_1, \ldots, E_m$ be locally convex spaces such that $E_1 \times \ldots \times E_m$ is polynomially barreled and F a locally convex space. Then every separately equicontinuous subset $\mathscr{X} \subset \mathscr{L}(E_1, \ldots, E_m; F)$ is equicontinuous.*

Proof. It suffices to use the argument from the proof of Theorem 3.17 and Proposition 3.27. ■

3.29. Proposition. *Every Baire locally convex space E is polynomially barreled.*

Proof. Let $m \in N$ and $\mathscr{X} \subset \mathscr{P}(^m E)$ be pointwise bounded. By a classical remark, there exists $x_0 \in E$ and a neighborhood V of x_0 in E such that $\mathscr{X}$ is uniformly bounded in V. Thus $\mathscr{X}$ is equicontinuous and Proposition 3.23 guarantees that E is polynomially barreled. ■

3.30. Remark. More precisely, one can prove that every Baire locally convex space is holomorphically barreled ([16], Proposition 5).

3.31. Proposition. *Let E be a polynomially barreled space, $m \in N$ and $(P_n)_{n \in N}$ be a sequence of continuous m-homogeneous polynomials from E into a locally, convex space F, pointwise convergent to a mapping $P: E \to F$. Then $P \in \mathscr{P}(^m E; F)$.*

Proof. It is easy to verify that $P \in \mathscr{P}_a(^m E; F)$. Moreover, since $(P_n)_{n \in N}$ is pointwise convergent to P, $(P_n)_{n \in N}$ is pointwise bounded and hence equicontinuous because E is polynomially barreled. Then $P \in \mathscr{P}(^m E; F)$. ■

3.32. Remark. Proposition 3.31, a Banach–Steinhaus theorem for continuous m-homogeneous polynomials, generalizes Theorem 2 of [5] which holds in the category of Baire spaces. We must observe that there are barreled DF spaces ([7], Chap. 5, p. 157, Exercise 10 and [12], p. 165, Exercise 1) hence polynomially barreled spaces ([1], Proposition 1.22), which are not Baire spaces.

It is known in the linear theory that every quasi-complete bornological space is a barreled space. To prove this result in the polynomial context, we need the following lemma.

3.33. Lemma. *Let E be a polynomially bornological space, F a seminormed space and $m \in N$. Let $\mathscr{X} \subset \mathscr{P}(^m E; F)$ be such that for every absolutely convex bounded and closed subset B of E, $\mathscr{X}|_{E_B} \subset \mathscr{P}(^m(E_B); F)$ is locally bounded (or, equivalently, equicontinuous). Then $\mathscr{X}$ is locally bounded (or, equivalently, equicontinuous).*

Proof. Let $\tilde{\mathscr{X}} \subset \mathscr{L}_s(^m E; F)$ be the corresponding set of continuous symmetric m-linear mappings. Then $\tilde{\mathscr{X}}|(E_B)^m$ is locally bounded for every absolutely convex bounded and closed subset B of E. Define $g: E^m \to l^\infty(\mathscr{X}; F)$ by

$$g(x_1, \ldots, x_m)(A) = A(x_1, \ldots, x_m) \quad \text{if} \quad x_1, \ldots, x_m \in E \quad \text{and} \quad A \in \tilde{\mathscr{X}}.$$

Obviously, $g \in \mathscr{L}_{as}(^m E; l^\infty(\tilde{\mathscr{X}}; F))$. Moreover, $g|(E_B)^m$ is locally bounded because $\tilde{\mathscr{X}}$ is locally bounded on $(E_B)^m$. Hence $g|(E_B)^m$ is continuous for every such B and by Proposition 3.5, g is continuous. Thus $\mathscr{X}$ is locally bounded. ■

3.34. Theorem. *Every quasi-complete polynomially bornological space E is polynomially barreled.*

Proof. Let $m \in N$ and $\mathscr{X} \subset \mathscr{P}(^m E)$ be pointwise bounded. For every absolutely convex bounded and closed subset B of E, E_B is a Banach space since E is quasi-complete. By Proposition 3.29, E_B is polynomially barreled, and since $\mathscr{X}|E_B$ is clearly pointwise bounded, we get that $\mathscr{X}|E_B$ is equicontinuous. An application of Lemma 3.33 and Proposition 3.23 gives the proof.

3.35. Remark. Theorem 3.34 can be false if E is not quasi-complete, even in the normed case (Example 3.22). In the next proposition we will see that under certain conditions a metrizable space is polynomially barreled.

3.36. Proposition. *Let E be a metrizable locally convex space such that for each $m \in N$, $(\mathscr{P}(^m E), \tau_s)$ is sequentially complete. Then E is polynomially barreled.*

Proof. The argument is a minor modification of the proof of the corresponding result in the linear case ([19], Theorem 2.7).

3.37. Theorem. *Let E be a polynomially barreled space, F a separated semi-Montel locally convex space, $m \in N$, and $\mathscr{X} \subset \mathscr{P}(^m E; F)$. The following conditions are equivalent:*

(i) *$\mathscr{X}$ is equicontinuous.*

(ii) *$\mathscr{X}$ is relatively compact in $(\mathscr{P}(^m E; F), \tau_s)$.*

(iii) *$\mathscr{X}$ is bounded in $(\mathscr{P}(^m E; F), \tau_b)$.*

(iv) *$\mathscr{X}$ is bounded in $(\mathscr{P}(^m E; F), \tau_s)$.*

Proof. (i) ⇒ (ii): This follows from Theorem 2.11.

(ii) ⇒ (iii): Since $\mathscr{X}$ is relatively compact in $(\mathscr{P}(^m E; F), \tau_s)$, X is bounded in $(\mathscr{P}(^m E; F), \tau_s)$ and hence equicontinuous because E is polynomially barreled. By Proposition 2.4, $\mathscr{X}$ is bounded in $(\mathscr{P}(^m E; F), \tau_b)$.

(iii) ⇒ (iv) is clear since $\tau_s \leqslant \tau_b$, and (iv) ⇒ (i) is also clear since E is polynomially barreled.

3.38. Definition ([1]). A locally convex space E is said to be *polynomially Mackey* if for every locally convex space F and for every $m \in N$, an m-homogeneous polynomial $P: E \to F$ is continuous if $\varphi \circ P \in \mathscr{P}(^m E)$ for every $\varphi \in F'$.

3.39. Remark. (a) In Definition 3.38 it suffices to take F a normed space.

b) Every holomorphically Mackey space ([15], Definition 11) is polynomially Mackey, and every polynomially Mackey space is a Mackey space.

c) Since $\mathscr{P}(E; F) = \bigoplus_{m \in N} \mathscr{P}(^m E; F)$, a locally convex space E is polynomially Mackey if and only if for every locally convex space F, a polynomial $P: E \to F$ is continuous if $\varphi \circ P \in \mathscr{P}(E)$ for every $\varphi \in F'$.

3.40. Proposition. *Every polynomially infrabarreled space E is polynomially Mackey.*

Proof. Let $(F, \|\ \|)$ be a normed space, $m \in N$ and $P \in \mathscr{P}_a(^mE; F)$ be such that $\varphi \circ P \in \mathscr{P}(^mE)$ for every $\varphi \in F'$. Then $P(K)$ is weakly bounded in F if K is a compact subset of E, and by Mackey's theorem ([7], chap. 4, p. 70, Theorem 3) $P(K)$ is bounded in F. Let $Y = \{\Phi \in F';\ |\Phi(y)| \leqslant \|y\| \text{ for every } y \in F\}$.

Since Y is bounded in $(F', \beta(F', F))$, $\mathscr{X} = Y \circ P \subset \mathscr{P}(^mE)$ is bounded on the compact subsets of E, and consequently equicontinuous because E is polynomially infrabarreled.

By the Hahn–Banach theorem, $\|y\| = \sup_{\varphi \in y} |\varphi(y)|$ for every $y \in F$. Hence $P \in \mathscr{P}(^mE; F)$, and by Remark 3.39 (a), E is polynomially Mackey. ■

3.41. Example. The space $E = C^N \times C^{(N)}$ considered in Example 3.18 is a polynomially Mackey space ([16], Example 7), which is not polynomially infrabarreled (Example 4.8).

3.42. Proposition. *A locally convex space E is polynomially bornological if and only if E is polynomially Mackey and for every $m \in N$, $\mathscr{P}(^mE) = \mathscr{P}_M(^mE)$.*

Proof. Necessity follows from Proposition 3.11 and 3.40. To prove sufficiency, let F be a locally convex space, $m \in N$ and $P \in \mathscr{P}_M(^mE; F)$. By hypothesis, $\varphi \circ P \in \mathscr{P}(^mE)$ for every $\varphi \in F'$. Since E is polynomially Mackey, $P \in \mathscr{P}(^mE; F)$. ■

4. Examples.

4.1. Proposition. *Every bornological* (*resp. infrabarreled*) DF *locally convex space E is polynomially bornological* (*resp. polynomially infrabarreled*).

Proof. We prove the bornological case. The other case is analogous. Let F be a locally convex space, $m \in N$, $m \geqslant 2$, and $P \in \mathscr{P}_M(^mE; F)$. Let $A \in \mathscr{L}_{as}(^mE; F)$ be such that $\hat{A} = P$. By the polarization formula, A is bounded on the bounded subsets of E^m. Since E is bornological, the linear mapping $x \in E \mapsto A(x_1, x_2, \ldots, x_m) \in F$ is continuous if $x_2, \ldots, x_m \in E$. To prove the continuity of A it suffices to prove that A is hypocontinuous ([10], p. 64, Theorem 2). To do it, fix $m-1$ bounded subsets $B_2, \ldots, B_m$ of E. We must prove that the family of continuous linear mappings

$$\mathscr{X} = \{x \in E \mapsto A(x, x_2, \ldots, x_m) \in F;\ x_2 \in B_2, \ldots, x_m \in B_m\}$$

is equicontinuous. But this follows from the facts that A is bounded on the bounded subsets of E^m, and that E is infrabarreled. ■

4.2. Corollary. *If E is a metrizable locally convex space, the following conditions are equivalent:*

(i) *E is distinguished.*

(ii) *$(E', \beta(E', E))$ is polynomially bornological.*

(iii) *$(E', \beta(E', E))$ is polynomially barreled.*

(iv) *$(E', \beta(E', E))$ is polynomially infrabarreled.*

Proof. (i) ⇒ (ii): This follows from Proposition 4.1, from [10], p. 61, Theorem 1, and from [10], p. 73, Theorem 7.

(ii) ⇒ (iii): Since E is metrizable, $(E', \beta(E', E))$ is complete. Hence (iii) follows from Theorem 3.34.

(iii) ⇒ (iv): Clear.

(iv) ⇒ (i): This follows from [10], p. 73, Theorem 7, and from the fact that every separated infrabarreled locally convex space is barreled. ■

4.3. Corollary. *If E and F are bornological* (*resp. barreled, infrabarreled*) DF *locally convex spaces, then $E \otimes_\pi F$* (*the projective topological tensor product of E and F*) *is polynomially bornological* (*resp. polynomially barreled, polynomially infrabarreled*).

Proof. It suffices to apply [11], p. 44, Corollary 1, Proposition 4.1 (in the bornological and infrabarreled cases), and [1], Proposition 1.22 (in the barreled case).

4.4. Remark. In [1], Proposition 1.22, Aragona proves that every barreled and bornological DF locally convex space is polynomially bornological and every barreled DF locally convex space is polynomially barreled.

4.5. Example. In [20], Valdivia gives an example of a barreled DF locally convex space (hence polynomially barreled) which is not bornological. This gives an example of a polynomially infrabarreled space which is not polynomially bornological.

4.6. Example. Let X be a non-compact completely regular Lindelöf topological space. Then $(C(X), \tau_0)$ is a polynomially bornological space (use the argument of [9], Theorem 2) which is not a DF space ([21], p. 276, Corollary 1).

In the next proposition we shall prove that the converse of Corollary 4.3 is true without the condition E and F being DF spaces.

4.7. Proposition. *Let E and F be separated locally convex spaces, $E \neq 0$, $F \neq 0$, such that $E \otimes_\pi F$ is polynomially bornological* (*polynomially barreled, polynomially infrabarreled*). *Then E and F are of the same type.*

Proof. We shall prove that E is polynomially bornological. The other cases are similar to this one.

Let $y \in F$, $y \neq 0$, and consider the one dimensional vector subspace of F

generated by y, say M. By [7], chap. 2, p. 68, Corollary 2, there is a subspace N of F such that $F = M \oplus N$, algebraically and topologically. Hence $E \otimes_\pi F = E \otimes_\pi (M \oplus N) = (E \otimes_\pi M) \oplus (E \otimes_\pi N)$ ([11], p. 46, Proposition 6), and an application of Proposition 3.27 gives the proof. ∎

4.8. EXAMPLE. Let $E = \boldsymbol{C}^{\boldsymbol{N}} \times \boldsymbol{C}^{(\boldsymbol{N})}$ as in Example 3.18. E is barreled and bornological as a product of two barreled and bornological spaces. We shall show that E is not polynomially infrabarreled, by proving that the sequence $(P_n)_{n \in \boldsymbol{N}}$ defined in Example 3.18 is bounded on the compact subsets of E but is not equicontinuous. If K is a compact subset of E, there exists a compact subset K_1 of $\boldsymbol{C}^{\boldsymbol{N}}$ and a compact subset K_2 of $\boldsymbol{C}^{(\boldsymbol{N})}$ such that $K \subset K_1 \times K_2$. Since K_2 is a compact subset of $\boldsymbol{C}^{(\boldsymbol{N})}$, there exists $m_0 \in \boldsymbol{N}$ such that if $y = (y_m)_{m \in \boldsymbol{N}} \in K_2$ and $m \geqslant m_0$, then $y_m = 0$. From this fact we get that

$$\sup_{(x,y) \in K, n \in \boldsymbol{N}} |P_n(x, y)| = \sup_{(x,y) \in K, 0 \leqslant r \leqslant m_0 - 1} |P_r(x, y)|,$$

and hence $(P_n)_{n \in \boldsymbol{N}}$ is bounded on K. Now let us prove that $(P_n)_{n \in \boldsymbol{N}}$ is not equicontinuous. In fact, let V be a neighborhood of zero in $\boldsymbol{C}^{\boldsymbol{N}}$ and let $(\lambda_m)_{m \in \boldsymbol{N}}$ be a sequence of strictly positive real numbers. We shall prove that $(P_n)_{n \in \boldsymbol{N}}$ is not locally bounded on $W = \{(x, y) \in E;\ x \in V, |y_m| \leqslant \lambda_m, m \in \boldsymbol{N}\}$ (a basic neighborhood of zero in E). Since V is a neighborhood of zero in $\boldsymbol{C}^{\boldsymbol{N}}$, there exists $m_1 \in \boldsymbol{N}$ such that the projection $\mathrm{pr}_{m_1}\colon \boldsymbol{C}^{\boldsymbol{N}} \to \boldsymbol{C}$ is not bounded in V. Hence P_{m_1} is not bounded on the subset $T = \{(x, y) \in E;\ x \in V;\ y_m = 0, m \neq m_1, y_{m_1} = \lambda_{m_1}\}$ of W since $P_{m_1}(T) = \lambda_{m_1} \mathrm{pr}_{m_1}(V)$. Hence $(P_n)_{n \in \boldsymbol{N}}$ is not equicontinuous. Moreover, $\boldsymbol{C}^{\boldsymbol{N}}$ and $\boldsymbol{C}^{(\boldsymbol{N})}$ are polynomially bornological and polynomially barreled spaces (Propositions 3.3 and 3.29 and [1], Proposition 1.22), whose product is not polynomially infrabarreled. This example also shows that an inductive limit of polynomially bornological and polynomially barreled spaces may fail to be a polynomially infrabarreled space.

References

[1] J. Aragona, *Sobre classificação holomorfa de espaços de germes holomorfos*, Tese de Doutorado, Universidade de São Paulo, Brasil, 1977.

[2] — *Holomorphically significant properties of spaces of holomorphic germs*, in: *Advances in Holomorphy* (Editor: J. A. Barroso), North-Holland Publishing Company, 1979, 31–46.

[3] — *On the holomorphical classification of spaces of holomorphic germs*, Nagoya Math. J., to appear.

[4] J. A. Barroso, M. C. Matos, L. Nachbin, *On holomorphy versus linearity in classifying locally convex spaces*, in: *Infinite Dimensional Holomorphy and Applications* (Editor: M. C. Matos), North-Holland Publishing Company, 1977, 31–74.

[5] J. Bochnak, J. Siciak, *Polynomials and multilinear mappings in topological vector spaces*, Studia Math. 39 (1971), 59–76.

[6] N. Bourbaki, *Sur certains espaces vectoriels topologiques*, Ann. Inst. Fourier 2 (1950), 5–16.

[7] — *Espaces vectoriels topologiques*, Hermann.

[8] — *Topologie générale*, Chapitres 1 et 10, Hermann.

[9] S. Dineen, *Equivalent definitions of holomorphic functions in infinite dimensions*, Séminaire P. Lelong (1973/1974), Lecture Notes in Math. 474 (1975), 114–122.

[10] A. Grothendieck, *Sur les espaces F et DF*, Summa Brasiliensis Math. 3 (1954), 57–123.

[11] — *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).

[12] — *Espaces vectoriels topologiques*, Publicação da Sociedade de Matemática de São Paulo, 3ª edição, 1966.

[13] G. Köthe, *Topological vector spaces I*, Springer-Verlag.

[14] L. Nachbin, *Topology on spaces of holomorphic mappings*, Ergebnisse der Math. 47 (1969), Springer-Verlag.

[15] — *A glimpse at infinite dimensional holomorphy*, in: *Proceedings on Infinite Dimensional Holomorphy* (Editors: T. L. Hayden and T. S. Suffridge), Lecture Notes in Math. 364 (1974), 69–79.

[16] — *Some holomorphically significant properties of locally convex spaces*, in: *Functional Analysis* (Editor: D. de Figueiredo), Marcel Dekker, 1976, 251–277.

[17] — *Analogies entre l'holomorphie et la linéarité*, Séminaire P. Krée 1977/1978, Institut H. Poincaré, 1/1–10.

[18] D. P. Pombo, Jr., *Classificação polinomial de espaços localmente convexos*, Tese de Doutorado, Universidade Federal do Rio de Janeiro, Brasil 1980.

[19] S. A. Saxon, *Nuclear and product spaces. Baire-like spaces and the strongest locally convex topology*, Math. Ann. 197 (1972), 87–106.

[20] M. Valdivia, *A characterization of Echelon Köthe–Schwartz spaces*, in: *Approximation Theory and Functional Analysis* (Editor: J. B. Prolla), North-Holland Publishing Company, 1979, 409–419.

[21] S. Warner, *The topology of compact convergence on continuous functions spaces*, Duke Math. J. 25 (1958), 265–282.

Received May 24, 1982 (1762)