On the convergence in $L^1$ of singular integrals

by

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Abstract. It is shown that if a singular integral operator such as in [1], see references, acting on a function in $L^1$ is in $L^1$, then the truncated operator converges to its limit in $L^1$.

We begin by stating a known theorem (Theorem A), which plays an essential role in our article. Let us assume that $k(x), x \in \mathbb{R}^n$, is a Lebesgue measurable function such that

(i) for $a > 0$,

$$\int_{|x| < a} |x|^2 |k(x)| \, dx \leq b_1$$

(ii) for $|x| > 2a$,

$$\int_{|x| > 2a} |k(x-y) - k(x)| \, dx \leq b_2$$

(iii) for $0 < \varepsilon < \lambda$,

$$|\int_{|x| < \varepsilon} k(x)\, dx| \leq b_3$$

(iv) for $\varepsilon \rightarrow 0$,

$$\int_{|x| = \varepsilon} k(x)\, dx$$

converges as $\varepsilon \rightarrow 0$.

Set $k_{\varepsilon,l}(x) = k(x)$ if $\varepsilon < |x| < \lambda$ and $k_{\varepsilon,l}(x) = 0$ elsewhere. For $f \in L^1(\mathbb{R}^n)$, $1 \leq p < \infty$, let

$$K_{\varepsilon,l}(f)(x) = (k_{\varepsilon,l} * f)(x) = \int_{\mathbb{R}^n} k_{\varepsilon,l}(x-y)f(y)\, dy.$$  

The convolution is well-defined almost everywhere and belongs to $L^p(\mathbb{R}^n)$.

**Theorem A.** Let $k(x)$ be a singular kernel which satisfies the above conditions and suppose that $f \in L^1(\mathbb{R}^n)$, $1 \leq p < \infty$. Then the limit

$$\lim_{\varepsilon \rightarrow 0} K_{\varepsilon,l}(f)(x) = K(f)(x) = \overline{f}(x)$$

exists almost everywhere. Moreover,
If \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), then \( \int f = K(f) \in L^p(\mathbb{R}^n) \), \( ||f||_p \leq c_p ||f||_p \), where \( c_p \) is a constant, and \( ||f * h_n - \int f||_p \to 0 \) as \( n \to \infty \) and \( \lambda \to \infty \).

If \( f \in L^1(\mathbb{R}^n) \), then there exists a constant \( c > 0 \) such that
\[
\left| \int_{x \in E} f(x) \right| \leq c ||f||_{L^1}
\]
for any \( E \subset \mathbb{R}^n \) of finite measure. In other words, the operator \( K \) is of weak type \((1, 1)\).

Proof. See Benedek–Calderón–Panzone [1] and Rivière [3] (Theorem 4.1 and Theorem 5.1, respectively). For the homogeneous case see [2].

The main purpose of this note is to prove the following statement in which the kernel \( k_\lambda(x) \) is defined, for any \( \varepsilon > 0 \), by the formula \( k_\lambda(x) = k(x) \) if \( |x| > \varepsilon \) and \( k_\lambda(x) = 0 \) if \( |x| \leq \varepsilon \).

**Theorem 1.** If \( f \in L^1(\mathbb{R}^n) \) and \( \tilde{f} \in L^1(\mathbb{R}^n) \), then \( f * k_\lambda \in L^1(\mathbb{R}^n) \) for each \( \varepsilon > 0 \), and \( ||f * k_\lambda - f||_1 \to 0 \) as \( \varepsilon \to 0 \).

For the proof we need the following definition and lemmas.

**Definition.** Suppose that \( \varphi(x) \) is a fixed function of \( C_0(\mathbb{R}^n) \) (here \( C_0(\mathbb{R}^n) \) denotes the set of all continuous functions with compact support) such that \( \varphi(x) \geq 0 \), \( \supp \varphi \subset \{ |x| \leq 1 \} \) and \( \int \varphi(x)dx = 1 \). Let \( \varepsilon > 0 \) and put \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon) \). We define, for each \( \varepsilon > 0 \),

\[
\delta_\varepsilon(x) = \varphi_\varepsilon(x) - k_\lambda(x) \quad \text{a.e.}
\]

**Lemma 1.** There exists a constant \( c > 0 \), such that
\[
||\delta_\varepsilon||_1 \leq \int_{|x| \leq \varepsilon} |\delta_\varepsilon(x)|dx \leq c
\]
for every \( \varepsilon > 0 \).

**Proof.** We first suppose that \( |x| > 2\varepsilon \). Then, by the Lebesgue dominated convergence theorem we have
\[
\delta_\varepsilon(x) = K(\varphi_\varepsilon)(x) = \lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon} k_{2\varepsilon}(x-y) \varphi_\varepsilon(y)dy = \int_{|y| \leq \varepsilon} k(x-y) \varphi_\varepsilon(y)dy.
\]

Therefore, for \( |x| > 2\varepsilon \),
\[
\delta_\varepsilon(x) = \int_{|y| \leq \varepsilon} [k(x-2\varepsilon - y) - k(x)] \varphi_\varepsilon(y)dy.
\]

Hence, by Fubini's theorem and condition (ii) of the kernel \( k(x) \), we obtain
\[
\int_{|x| > 2\varepsilon} |\delta_\varepsilon(x)|dx \leq \int_{|y| \leq \varepsilon} \int_{|x| > 2\varepsilon} |k(x-2\varepsilon - y)| \varphi_\varepsilon(y)dydx \leq c_1
\]

On the other hand
\[
\int_{|x| \leq 2\varepsilon} |\delta_\varepsilon(x)|dx \leq \int_{|x| \leq 2\varepsilon} |\varphi_\varepsilon(x)|dx + \int_{|x| > 2\varepsilon} |k(x)|dx = I_1 + I_2,
\]

By Schwarz's inequality
\[
I_1 \leq 2^{n/2} \rho_k^{1/2} \rho_{2k}^{1/2} \int |x| \varphi_\varepsilon(x)^2 dx \leq \frac{\rho_k}{\rho_{2k}} \int \varphi_\varepsilon(x)^2 dx \leq c_k,
\]

where \( \rho_k \) denotes the volume of the unit ball of \( \mathbb{R}^n \). Whence, taking into account that the operator \( K \) is of type \((2, 2)\) we obtain
\[
I_1 \leq 2^{n/2} \rho_k^{1/2} \rho_{2k}^{1/2} C_2 \int |x| \varphi_\varepsilon(x)^2 dx \leq b_k,
\]

where \( b_k \) is a constant.

Moreover, by (4) and by condition (i) satisfied by the kernel \( k(x) \), we have
\[
I_2 \leq \rho_{2k} \int |x| \varphi_\varepsilon(x)dx \leq 2b_k.
\]

Finally, from (3), (4), (5) and (6), we obtain (2) with \( c = 2b_1 + b_2 + b_k \).

**Lemma 2.** (i) If \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < \infty \), then, for each \( \varepsilon > 0 \),
\[
\int_{|x| < 2\varepsilon} |f(x-\varepsilon)| |k_\varepsilon(x)|dx < \infty.
\]

(ii) If \( f \in L^1(\mathbb{R}^n) \), \( 1 \leq p < \infty \), then, for almost every \( x \),
\[
\int_{|x| < 2\varepsilon} |f(x-\varepsilon)| |k_\varepsilon(x)|dx \leq c_p \int |f(x)|dx.
\]

(iii) If \( f \in L^1(\mathbb{R}^n) \), \( 1 < p < \infty \), then
\[
||f * k_\lambda||_p \leq c_p ||f||_p \quad \tilde{f} \in L^p(\mathbb{R}^n), \quad \text{and} \quad \lim_{\varepsilon \to 0} ||f * k_\varepsilon - f||_p = 0.
\]

**Proof.** (i) By formula (1), \( k_\varepsilon(x) = \varphi_\varepsilon(x) - k_\lambda(x) \). Therefore
\[
||f * k_\varepsilon||_p \leq c_p ||f||_p \quad \tilde{f} \in L^p(\mathbb{R}^n), \quad \text{and} \quad \lim_{\varepsilon \to 0} ||f * k_\varepsilon - f||_p = 0.
\]

We suppose first that \( p = 1 \). Then, by Young's convolution theorem and by Lemma 1, we have \( ||\delta_\varepsilon * f||_1 \leq c ||f||_1 \). Therefore \( ||\delta_\varepsilon * f||_1 \leq c ||f||_1 \), a.e.

The second convolution which appears on the right-hand member of (10) is also finite almost everywhere. Indeed
\[
||f ||_{L^p} * ||\varphi_\varepsilon||_2 \leq c_2 ||f ||_{L^p} ||\varphi_\varepsilon||_2.
\]

Now, we suppose that \( 1 < p < \infty \). Then
\[
||\delta_\varepsilon||_p * ||f||_p \leq c ||f||_p \quad \text{and} \quad ||f * \varphi_\varepsilon||_p \leq c ||f||_p.
\]
where $q$ is the conjugate exponent of $p$. Therefore, both convolutions which appear on the right-hand member of (10) are finite almost everywhere.

(ii) By the Lebesgue dominated convergence theorem, taking into account (7), we have
\[
\lim_{\lambda \to \infty} K_{\lambda,t}(f)(x) = \lim_{\lambda \to 0} \int_{\lambda^{-1} \cdot \mathbb{R}^n} k_{\lambda,t}(x-t)f(t)dt = (f * k_t)(x) \quad \text{a.e.}
\]
Hence, letting $\varepsilon \to 0$, we obtain
\[
\overline{f}(x) = \lim_{\lambda \to 0} K_{\lambda,t}(f)(x) = \lim_{\lambda \to 0} (f * k_t)(x) \quad \text{a.e.}
\]
(iii) By Theorem A, we have
\[
\lim_{\lambda \to 0} \int_{\lambda^{-1} \cdot \mathbb{R}^n} |(k_{\lambda,t} * f)(x) - \overline{f}(x)|^\delta dx = 0.
\]
Choose now, given $\eta > 0$, a $\delta$ ($0 < \delta < 1$), such that
\[
\int_{\mathbb{R}^n} |(k_{\lambda,t} * f)(x) - \overline{f}(x)|^\delta dx \leq \eta
\]
if $0 < \varepsilon < \delta$ and $\lambda > \delta^{-1}$. Letting $\lambda \to \infty$ and using part (ii) of the lemma and Fatou’s lemma we conclude that
\[
\int_{\mathbb{R}^n} |(f * g)(x) - \overline{f}(x)|^\delta dx \leq \eta
\]
for $0 < \varepsilon < \delta$. This proves (9).

**Lemma 3.** If $f \in L^p(\mathbb{R}^n)$ and $g \in C_0(\mathbb{R}^n)$, then
\[
(f * g)(x) = (f * g)(x) \quad \text{a.e.}
\]

**Proof.** By the associative property of the convolution product we have
\[
(f * g)(x) = (f * g)(x) = (f * g)(x).
\]
By Theorem A, since $f * g \in L^2(\mathbb{R}^n)$, we have
\[
\lim_{\lambda \to 0} \| (f * g)(x) - (f * g)(x) \|_2 = 0.
\]

**Lemma 4.** If $f \in L^p(\mathbb{R}^n)$ and $\overline{f} \in L^p(\mathbb{R}^n)$, then for each $g \in C_0(\mathbb{R}^n)$
\[
(f * g)(x) = (f * g)(x) \quad \text{a.e.}
\]

**Proof.** For every positive integer $m$ and for $x \in \mathbb{R}^n$, we define
\[
h_m(x) = \sum_{k \in \mathbb{Z}^n} \int_{Q_k^m} g(t)dt,
\]
where $Q_k^m = k/2^m$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ ($\mathbb{Z}$ is the set of the integers) and
\[
\mathcal{Q}_m^n = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n : \frac{k_1}{2^m} < t_1 < \frac{k_1 + 1}{2^m}, \ldots, \frac{k_n}{2^m} < t_n < \frac{k_n + 1}{2^m} \right\}
\]
We note first that, since the support of $g$ is compact, for any $x \in \mathbb{R}^n$, only finitely many terms of the series on the right of (15) are non-zero.

We claim that
\[
\lim_{m \to \infty} \| h_m - f * g \|_1 = 0.
\]
In fact, as it is easy to see, for any given $\varepsilon > 0$, there exists an $m_0$ such that
\[
\| h_m - f * g \|_1 \leq \sum_{k \in \mathbb{Z}^n} \int_{Q_k^m} |f(x - t)| \int_{Q_k^m} g(t)dt dt \leq \varepsilon \| g \|_1
\]
if $m \geq m_0$. From formula (15), since the operator $K$ commutes with translations we obtain
\[
h_m(x) = \sum_{k \in \mathbb{Z}^n} \int_{Q_k^m} g(t)dt.
\]
Then, arguing just as in the proof of (16) we conclude that
\[
\lim_{m \to \infty} \| h_m - f * g \|_1 = 0.
\]
On the other hand, taking into account the weak type $(1, 1)$ of the operator $K$, it follows from (16) that the sequence $h_m$ converges in measure to $(f * g)$. Therefore, taking into account formula (17), we see that there exists a subsequence $h_{m_j}$ of $h_m$ such that
\[
(f * g)(x) = \lim_{j \to \infty} h_{m_j}(x) = (f * g)(x)
\]
for almost every $x$. This proves the lemma.

**Lemma 5.** If $g \in C_0(\mathbb{R}^n)$, then
\[
\lim_{\varepsilon \to 0} \| g \ast \delta_{\varepsilon} \|_1 = 0,
\]
where $\delta_{\varepsilon}(x) = \delta_{\varepsilon}(x) - k_{\varepsilon}(x)$, a.e.

**Proof.** We first prove that
\[
\lim_{\varepsilon \to 0} \| g \ast \delta_{\varepsilon} \|_1 = 0.
\]
In fact, by Lemma 3
\[ (g \ast \delta_\varepsilon)(x) = (g \ast \phi_\varepsilon)(x) - (k_\varepsilon \ast g)(x), \quad \text{a.e.} \]
Hence, \[ ||g \ast \delta_\varepsilon||_2 \leq ||g \ast \phi_\varepsilon||_2 \leq ||g \ast \phi_\varepsilon - g||_2 + ||g - k_\varepsilon \ast g||_2 \to 0 \quad \text{as} \quad \varepsilon \to 0. \]
Indeed \[ ||g \ast \phi_\varepsilon - g||_2 \to 0, \] by the fact that \( \phi_\varepsilon \) is an approximation of the identity, and \[ ||g - k_\varepsilon \ast g||_2 \to 0 \] by Lemma 2.

Now, we suppose that the support of \( g \) is included in the ball \( |x| \leq N \) and that \( 0 < \varepsilon < N \). Then, the support of \( g \ast \phi_\varepsilon \) is included in the ball \(|x| \leq 2N\). Therefore, by Lemma 3, for \(|x| \geq 4N\) and \(0 < \varepsilon < N\), we have
\begin{equation}
( \delta_\varepsilon \ast g)(x) = (\phi_\varepsilon \ast g)(x) = \int_{|y| \leq \varepsilon} k(x-y)(\phi_\varepsilon \ast g)(y) \, dy.
\end{equation}
Moreover, for \(|x| \geq 4N\) and \(0 < \varepsilon < N\),
\begin{equation}
(k_\varepsilon \ast g)(x) = \int_{|y| \leq \varepsilon} k(x-y)y \, dy.
\end{equation}
From (20) and (21), it follows that
\begin{equation}
(g \ast \delta_\varepsilon)(x) = \int_{|y| \leq \varepsilon} [k(x-y) - k(x)](\phi_\varepsilon \ast g)(y) - g(y) \, dy
\end{equation}
for \(|x| \geq 4N\) and \(0 < \varepsilon < N\). Taking into account that
\[ \int_{|y| \leq \varepsilon} (\phi_\varepsilon \ast g)(y) - g(y) \, dy = 0, \]
we obtain from (22)
\[ (g \ast \delta_\varepsilon)(x) = \int_{|y| \leq \varepsilon} [k(x-y) - k(x)] \phi_\varepsilon \ast g(y) \, dy, \]
for \(|x| \geq 4N\) and \(0 < \varepsilon < N\). Hence, by Fubini's Theorem, we have
\[ \int_{|x| \geq 4N} ||g \ast \delta_\varepsilon||_2 \, dx \leq \int_{|y| \leq \varepsilon} \int_{|x| \geq 4N} [k(x-y) - k(x)] ||\phi_\varepsilon \ast g||_2 \, dx \, dy. \]

Then, by condition (ii) satisfied by the kernel,
\begin{equation}
\int_{|x| \geq 4N} ||g \ast \delta_\varepsilon||_2 \, dx \leq b_2 ||\phi_\varepsilon \ast g - g||_2 \to 0,
\end{equation}
as \( \varepsilon \to 0 \). On the other hand, by Schwarz's inequality and (19), we obtain
\begin{equation}
\int_{|x| \geq 4N} ||g \ast \delta_\varepsilon||_2 \, dx \leq (4N)^d \int_{|x| \geq 4N} ||\phi_\varepsilon \ast \delta_\varepsilon||_2 \, dx \to 0,
\end{equation}
as \( \varepsilon \to 0 \). Finally, formula (18) of the thesis follows from (23) and (24).

Proof of Theorem 1. We prove first that \( f \ast k_\varepsilon \in L^1(\mathbb{R}^d) \), for every \( \varepsilon > 0 \). In fact, by formula (1) and Lemma 4 we have
\begin{equation}
(f \ast k_\varepsilon)(x) = (f \ast \delta_\varepsilon)(x) - (f \ast \phi_\varepsilon)(x), \quad \text{a.e.}
\end{equation}
Hence, by Lemma 1, we conclude that
\[ ||f \ast k_\varepsilon||_1 \leq ||f \ast \phi_\varepsilon||_1 + c ||f \ast \delta_\varepsilon||_1. \]
We now prove that \( ||f \ast k_\varepsilon - f \ast \delta_\varepsilon||_1 \to 0 \) as \( \varepsilon \to 0 \). To this end choose an \( \eta > 0 \), then there exists a function \( g \in C(\mathbb{R}^d) \) such that \( ||g - f||_1 \leq \eta \). Then, by formula (25)
\[ ||f \ast k_\varepsilon - f \ast \delta_\varepsilon||_1 \leq ||f \ast \phi_\varepsilon - f \ast \delta_\varepsilon||_1 + ||f - g||_1 ||\phi_\varepsilon \ast \delta_\varepsilon||_1 + ||g \ast \delta_\varepsilon||_1. \]
Hence, by Lemma 1 and Lemma 5, we obtain
\[ \limsup_{\varepsilon \to 0} ||f \ast k_\varepsilon - f \ast \delta_\varepsilon||_1 \leq \lim_{\varepsilon \to 0} ||f \ast \phi_\varepsilon - f \ast \delta_\varepsilon||_1 + c\eta + \lim_{\varepsilon \to 0} ||g \ast \delta_\varepsilon||_1 = c\eta. \]
Then the theorem follows by the arbitrariness of \( \eta > 0 \).

References

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