STUDIA MATHEMATICA, T. LXXVIII. (1984)

On the convergence in L^1 of singular integrals

by

ALBERTO P. CALDERÓN and OSVALDO N. CAPRI (Buenos Aires, Argentina)

Abstract. It is shown that if a singular integral operator such as in [1], see references, acting on a function in L^1 is in L^1, then the truncated operator converges to its limit in L^1.

We begin by stating a known theorem (Theorem A), which plays an essential role in our article. Let us assume that $k(x)$, $x \in \mathbf{R}^n$, is a Lebesgue measurable function such that

(i) for $\varrho > 0$,

$$\int_{|x|<\varrho} |x|\,|k(x)|\,dx \leqslant b_1 \varrho,$$

(ii)
$$\int_{|x|>2|y|} |k(x-y)-k(x)|\,dx \leqslant b_2,$$

(iii) for $0<\varepsilon<\lambda$,

$$\Big|\int_{\varepsilon<|x|<\lambda} k(x)\,dx\Big| \leqslant b_3,$$

(iv)
$$\int_{\varepsilon<|x|<1} k(x)\,dx$$

converges as $\varepsilon \to 0$.

Set $k_{\varepsilon,\lambda}(x) = k(x)$ if $\varepsilon<|x|<\lambda$ and $k_{\varepsilon,\lambda}(x)=0$ elsewhere. For $f\in L^p(\mathbf{R}^n)$, $1\leqslant p<\infty$, let

$$K_{\varepsilon,\lambda}(f)(x) = (k_{\varepsilon,\lambda} * f)(x) = \int_{\mathbf{R}^n} k_{\varepsilon,\lambda}(x-y) f(y)\,dy.$$

The convolution is well-defined almost everywhere and belongs to $L^p(\mathbf{R}^n)$.

THEOREM A. *Let $k(x)$ be a singular kernel which satisfies the above conditions and suppose that $f\in L^p(\mathbf{R}^n)$, $1\leqslant p<\infty$. Then the limit*

$$\lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}} K_{\varepsilon,\lambda}(f)(x) = K(f)(x) = \tilde{f}(x)$$

exists almost everywhere. Moreover,

(1) *If* $f \in L^p(\boldsymbol{R}^n)$, $1 < p < \infty$, *then* $\tilde{f} = K(f) \in L^p(\boldsymbol{R}^n)$, $\|\tilde{f}\|_p \leqslant c_p \|f\|_p$, *where* c_p *is a constant, and* $\|f * k_{\varepsilon,\lambda} - \tilde{f}\|_p \to 0$ *as* $\varepsilon \to 0$ *and* $\lambda \to \infty$.

(2) *If* $f \in L^1(\boldsymbol{R}^n)$, *then there exists a constant* $c_1 > 0$ *such that*

$$\left|\{x \in \boldsymbol{R}^n \colon |\tilde{f}(x)| > t\}\right| \leqslant (c_1/t)\|f\|_1,$$

for any $t > 0$ ($|E|$ *denotes the Lebesgue measure of the set* E). *In other words, the operator* K *is of weak type* (1, 1).

Proof. See Benedek–Calderón–Panzone [1] and Rivière [3] (Theorem 4.1 and Theorem 5.1, respectively). For the homogeneous case see [2].

The main purpose of this note is to prove the following theorem in which the kernel $k_\varepsilon(x)$ is defined, for any $\varepsilon > 0$, by the formula $k_\varepsilon(x) = k(x)$ if $|x| > \varepsilon$ and $k_\varepsilon(x) = 0$ if $|x| \leqslant \varepsilon$.

THEOREM 1. *If* $f \in L^1(\boldsymbol{R}^n)$ *and* $\tilde{f} \in L^1(\boldsymbol{R}^n)$, *then* $f * k_\varepsilon \in L^1(\boldsymbol{R}^n)$ *for each* $\varepsilon > 0$, *and* $\|f * k_\varepsilon - \tilde{f}\|_1 \to 0$ *as* $\varepsilon \to 0$.

For the proof we need the following definition and lemmas.

DEFINITION. Suppose that $\varphi(x)$ is a fixed function of $C_0(\boldsymbol{R}^n)$ (here $C_0(\boldsymbol{R}^n)$ denotes the set of all continuous functions with compact support) such that $\varphi(x) \geqslant 0$, $\operatorname{supp} \varphi \subset \{|x| \leqslant 1\}$ and $\int_{\boldsymbol{R}^n} \varphi(x)\,dx = 1$. Let $\varepsilon > 0$ and put $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. We define, for each $\varepsilon > 0$,

$$\delta_\varepsilon(x) = \tilde{\varphi}_\varepsilon(x) - k_\varepsilon(x) \quad \text{a.e.} \tag{1}$$

LEMMA 1. *There exists a constant* $c > 0$, *such that*

$$\|\delta_\varepsilon\|_1 = \int_{\boldsymbol{R}^n} |\delta_\varepsilon(x)|\,dx \leqslant c \tag{2}$$

for every $\varepsilon > 0$.

Proof. We first suppose that $|x| \geqslant 2\varepsilon$. Then, by the Lebesgue dominated convergence theorem we have

$$\tilde{\varphi}_\varepsilon(x) = K(\varphi_\varepsilon)(x) = \lim_{\substack{\delta \to 0 \\ \lambda \to \infty}} \int_{\boldsymbol{R}^n} k_{\delta,\lambda}(x-y)\,\varphi_\varepsilon(y)\,dy = \int_{|y| \leqslant \varepsilon} k(x-y)\,\varphi_\varepsilon(y)\,dy.$$

Therefore, for $|x| \geqslant 2\varepsilon$,

$$\delta_\varepsilon(x) = \int_{|y| \leqslant \varepsilon} [k(x-y) - k(x)]\,\varphi_\varepsilon(y)\,dy.$$

Hence, by Fubini's theorem and condition (ii) of the kernel $k(x)$, we obtain

$$\int_{|x| > 2\varepsilon} |\delta_\varepsilon(x)|\,dx \leqslant \int_{|y| \leqslant \varepsilon} \Big\{ \int_{|x| > 2\varepsilon} |k(x-y) - k(x)|\,dx \Big\}\,\varphi_\varepsilon(y)\,dy \leqslant b_2. \tag{3}$$

On the other hand

$$\int_{|x| \leqslant 2\varepsilon} |\delta_\varepsilon(x)|\,dx \leqslant \int_{|x| \leqslant 2\varepsilon} |\tilde{\varphi}_\varepsilon(x)|\,dx + \int_{\varepsilon < |x| \leqslant 2\varepsilon} |k(x)|\,dx = \mathrm{I}_1 + \mathrm{I}_2. \tag{4}$$

By Schwarz's inequality

$$\mathrm{I}_1 \leqslant 2^{n/2}\,\Omega_n^{1/2}\,\varepsilon^{n/2} \Big\{ \int_{|x| \leqslant 2\varepsilon} |\tilde{\varphi}_\varepsilon(x)|^2\,dx \Big\}^{1/2},$$

where Ω_n denotes the volume of the unit ball of $\boldsymbol{R}^n$. Whence, taking into account that the operator K is of type (2, 2) we obtain

$$\mathrm{I}_1 \leqslant 2^{n/2}\,\Omega_n^{1/2}\,\varepsilon^{n/2} \cdot C_2 \Big\{ \int_{\boldsymbol{R}^n} [\varphi_\varepsilon(x)]^2\,dx \Big\}^{1/2} = b_4, \tag{5}$$

where b_4 is a constant.

Moreover, by (4) and by condition (i) satisfied by the kernel $k(x)$, we have

$$\mathrm{I}_2 \leqslant (1/\varepsilon) \int_{|x| \leqslant 2\varepsilon} |x|\,|k(x)|\,dx \leqslant 2b_1. \tag{6}$$

Finally, from (3), (4), (5) and (6), we obtain (2) with $c = 2b_1 + b_2 + b_4$.

LEMMA 2. (i) *If* $f \in L^p(\boldsymbol{R}^n)$, $1 \leqslant p < \infty$, *then, for each* $\varepsilon > 0$,

$$\int_{\boldsymbol{R}^n} |f(x-t)|\,|k_\varepsilon(t)|\,dt < \infty, \tag{7}$$

for almost every x.

(ii) *If* $f \in L^p(\boldsymbol{R}^n)$, $1 \leqslant p < \infty$, *then, for almost every* x,

$$\tilde{f}(x) = K(f)(x) = \lim_{\varepsilon \to 0} (f * k_\varepsilon)(x). \tag{8}$$

(iii) *If* $f \in L^p(\boldsymbol{R}^n)$, $1 < p < \infty$, *then*

$$\|f * k_\varepsilon\|_p \leqslant c_p \|f\|_p, \quad \tilde{f} \in L^p(\boldsymbol{R}^n), \quad \text{and} \quad \lim_{\varepsilon \to 0} \|f * k_\varepsilon - \tilde{f}\|_p = 0. \tag{9}$$

Proof. (i) By formula (1), $k_\varepsilon(x) = \tilde{\varphi}_\varepsilon(x) - \delta_\varepsilon(x)$. Therefore

$$(|f| * |k_\varepsilon|)(x) \leqslant (|\delta_\varepsilon| * |f|)(x) + (|\tilde{\varphi}_\varepsilon| * |f|)(x). \tag{10}$$

We suppose first that $p = 1$. Then, by Young's convolution theorem and by Lemma 1, we have $\||\delta_\varepsilon| * |f|\|_1 \leqslant c\|f\|_1 < \infty$. Therefore $(|\delta_\varepsilon| * |f|)(x) < \infty$, a.e.

The second convolution which appears on the right-hand member of (10) is also finite almost everywhere. Indeed

$$\||f| * |\tilde{\varphi}_\varepsilon|\|_2 \leqslant c_2 \|f\|_1 \|\varphi_\varepsilon\|_2.$$

Now, we suppose that $1 < p < \infty$. Then

$$\||\delta_\varepsilon| * |f|\|_p \leqslant \|\delta_\varepsilon\|_1 \|f\|_p \leqslant c\|f\|_p$$

and

$$\||f| * |\tilde{\varphi}_\varepsilon|\|_\infty \leqslant \|f\|_p \|\tilde{\varphi}_\varepsilon\|_q \leqslant c_q \|\varphi_\varepsilon\|_q \|f\|_p,$$

where q is the conjugate exponent of p. Therefore, both convolutions which appear on the right-hand member of (10) are finite almost everywhere.

(ii) By the Lebesgue dominated convergence theorem, taking into account (7), we have

$$\lim_{\lambda\to\infty} K_{\varepsilon,\lambda}(f)(x) = \lim_{\lambda\to\infty} \int_{R^n} k_{\varepsilon,\lambda}(x-t) f(t)\,dt = (f*k_\varepsilon)(x), \quad \text{a.e.}$$

Hence, letting $\varepsilon\to 0$, we obtain

$$\tilde{f}(x) = \lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}} K_{\varepsilon,\lambda}(f)(x) = \lim_{\varepsilon\to 0}(f*k_\varepsilon)(x) \quad \text{a.e.}$$

(iii) By Theorem A, we have

$$\lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}} \int_{R^n} |(k_{\varepsilon,\lambda}*f)(x)-\tilde{f}(x)|^p\,dx = 0.$$

Choose now, given $\eta>0$, a δ $(0<\delta<1)$, such that

$$\int_{R^n} |(k_{\varepsilon,\lambda}*f)(x)-\tilde{f}(x)|^p\,dx \leqslant \eta$$

if $0<\varepsilon<\delta$ and $\lambda>\delta^{-1}$. Letting $\lambda\to\infty$ and using part (ii) of the lemma and Fatou's lemma we conclude that

$$\int_{R^n} |(k_\varepsilon*f)(x)-\tilde{f}(x)|^p\,dx \leqslant \eta$$

for $0<\varepsilon<\delta$. This proves (9).

Lemma 3. *If $f\in L^1(R^n)$ and $g\in C_0(R^n)$, then*

$$(f*g)\tilde{}(x) = (f*\tilde{g})(x) \quad \text{a.e.} \tag{11}$$

Proof. By the associative property of the convolution product we have

$$(f*g)*k_{\varepsilon,\lambda} = f*(g*k_{\varepsilon,\lambda}). \tag{12}$$

By Theorem A, since $f*g\in L^2(R^n)$, we have

$$\lim_{\substack{\varepsilon\to 0\\ \lambda\to\infty}} \|(f*g)*k_{\varepsilon,\lambda}-(f*g)\tilde{}(x)\|_2 = 0. \tag{13}$$

On the other hand, by Young's convolution theorem and Theorem A, we have

$$\|f*(g*k_{\varepsilon,\lambda})-(f*\tilde{g})\|_2 \leqslant \|f\|_1\|\tilde{g}-g*k_{\varepsilon,\lambda}\|_2 \to 0, \tag{13'}$$

as $\varepsilon\to 0$ and $\lambda\to 0$.

Finally, formula (11) follows from (12), (13) and (13′).

Lemma 4. *If $f\in L^1(R^n)$ and $\tilde{f}\in L^1(R^n)$, then for each $g\in C_0(R^n)$*

$$(f*g)\tilde{}(x) = (\tilde{f}*g)(x) \quad \text{a.e.} \tag{14}$$

Proof. For every positive integer m and for $x\in R^n$, we define

$$h_m(x) = \sum_{k\in Z^n} f(x-t_k^m)\int_{Q_k^m} g(t)\,dt, \tag{15}$$

where $t_k^m = k/2^m$, $k=(k_1,\ldots,k_n)\in Z^n$ (Z is the set of the integers) and

$$Q_k^m = \left\{(t_1,\ldots,t_n)\in R^n:\ \frac{k_1}{2^m}\leqslant t_1<\frac{k_1+1}{2^m},\ \ldots,\ \frac{k_n}{2^m}\leqslant t_n<\frac{k_n+1}{2^m}\right\}.$$

We note first that, since the support of g is compact, for any $x\in R^n$, only finitely many terms of the series on the right of (15) are non-zero.

We claim that

$$\lim_{m\to\infty}\|h_m-f*g\|_1 = 0. \tag{16}$$

In fact, as it is easy to see, for any given $\varepsilon>0$, there exists an m_0 such that

$$\|h_m-f*g\|_1 \leqslant \sum_{k\in Z^n}\int_{Q_k^m}\Big\{\int_{R^n}|f(x-t_k^m)-f(x-t)|\,dx\Big\}|g(t)|\,dt \leqslant \varepsilon\|g\|_1,$$

if $m\geqslant m_0$. From formula (15), since the operator K commutes with translations we obtain

$$\tilde{h}_m(x) = \sum_{k\in Z^n}\tilde{f}(x-t_k^m)\int_{Q_k^m} g(t)\,dt.$$

Then, arguing just as in the proof of (16) we conclude that

$$\lim_{m\to\infty}\|\tilde{h}_m-\tilde{f}*g\|_1 = 0. \tag{17}$$

On the other hand, taking into account the weak type (1, 1) of the operator K, it follows from (16) that the sequence $\tilde{h}_m$ converges in measure to $(f*g)\tilde{}$. Therefore, taking into account formula (17), we see that there exists an subsequence h_{m_j} of h_m such that

$$(f*g)\tilde{}(x) = \lim_{j\to\infty}\tilde{h}_{m_j}(x) = (\tilde{f}*g)(x)$$

for almost every x. This proves the lemma.

Lemma 5. *If $g\in C_0(R^n)$, then*

$$\lim_{\varepsilon\to 0}\|g*\delta_\varepsilon\|_1 = 0, \tag{18}$$

where $\delta_\varepsilon(x) = \tilde{\varphi}_\varepsilon(x)-k_\varepsilon(x)$, a.e.

Proof. We first prove that

$$\lim_{\varepsilon\to 0}\|g*\delta_\varepsilon\|_2 = 0. \tag{19}$$

In fact, by Lemma 3

$$(g*\delta_\varepsilon)(x) = (\tilde{g}*\varphi_\varepsilon)(x) - (k_\varepsilon * g)(x), \quad \text{a.e.}$$

Hence, $\|g*\delta_\varepsilon\|_2 \leqslant \|g*\delta_\varepsilon\|_2 \leqslant \|\tilde{g}*\varphi_\varepsilon - \tilde{g}\|_2 + \|\tilde{g} - k_\varepsilon * g\|_2 \to 0$ as $\varepsilon \to 0$. Indeed $\|\tilde{g}*\varphi_\varepsilon - \tilde{g}\|_2 \to 0$, by the fact that φ_ε is an approximation of the identity, and $\|\tilde{g} - k_\varepsilon * g\|_2 \to 0$ by Lemma 2.

Now, we suppose that the support of g is included in the ball $|x| \leqslant N$ and that $0 < \varepsilon < N$. Then, the support of $g*\varphi_\varepsilon$ is included in the ball $|x| \leqslant 2N$. Therefore, by Lemma 3, for $|x| \geqslant 4N$ and $0 < \varepsilon < N$, we have

$$(20) \qquad (\tilde{\varphi}_\varepsilon * g)(x) = (\varphi_\varepsilon * g)\tilde{}(x) = \int_{|y| \leqslant 2N} k(x-y)(\varphi_\varepsilon * g)(y)\,dy.$$

Moreover, for $|x| \geqslant 4N$ and $0 < \varepsilon < N$,

$$(21) \qquad (k_\varepsilon * g)(x) = \int_{|y| \leqslant 2N} k(x-y)\,g(y)\,dy.$$

From (20) and (21), it follows that

$$(22) \qquad (g*\delta_\varepsilon)(x) = \int_{|y| \leqslant 2N} k(x-y)\,[(\varphi_\varepsilon * g)(y) - g(y)]\,dy$$

for $|x| \geqslant 4N$ and $0 < \varepsilon < N$. Taking into account that

$$\int_{|y| \leqslant 2N} [(\varphi_\varepsilon * g)(y) - g(y)]\,dy = 0,$$

we obtain from (22)

$$(g*\delta_\varepsilon)(x) = \int_{|y| \leqslant 2N} [k(x-y) - k(x)]\,[\varphi_\varepsilon * g)(y) - g(y)]\,dy,$$

for $|x| \geqslant 4N$ and $0 < \varepsilon < N$. Hence, by Fubini's Theorem, we have

$$\int_{|x| \geqslant 4N} |(g*\delta_\varepsilon)(x)|\,dx \leqslant \int_{|y| \leqslant 2N} \Big\{ \int_{|x| \geqslant 4N} |k(x-y) - k(x)|\,dx \Big\} |(\varphi_\varepsilon * g)(y) - g(y)|\,dy.$$

Then, by condition (ii) satisfied by the kernel,

$$(23) \qquad \int_{|x| \geqslant 4N} |(g*\delta_\varepsilon)|(x)\,dx \leqslant b_2 \|\varphi_\varepsilon * g - g\|_1 \to 0,$$

as $\varepsilon \to 0$. On the other hand, by Schwarz's inequality and (19), we obtain

$$(24) \qquad \int_{|x| < 4N} |(g*\delta_\varepsilon)(x)|\,dx \leqslant (4N)^{n/2}\,\Omega_n^{1/2}\,\|g*\delta_\varepsilon\|_2 \to 0,$$

as $\varepsilon \to 0$. Finally, formula (18) of the thesis follows from (23) and (24).

Proof of Theorem 1. We prove first that $f*k_\varepsilon \in L^1(\boldsymbol{R}^n)$, for every $\varepsilon > 0$. In fact, by formula (1) and Lemma 4 we have

$$(25) \qquad (f*k_\varepsilon)(x) = (\tilde{f}*\varphi_\varepsilon)(x) - (f*\delta_\varepsilon)(x), \quad \text{a.e.}$$

Hence, by Lemma 1, we conclude that

$$\|f*k_\varepsilon\|_1 \leqslant \|\tilde{f}\|_1 + c\|f\|_1.$$

We now prove that $\|f*k_\varepsilon - \tilde{f}\|_1 \to 0$ as $\varepsilon \to 0$. To this end choose an $\eta > 0$, then there exists a function $g \in C_0(\boldsymbol{R}^n)$ such that $\|g - f\|_1 \leqslant \eta$. Then, by formula (25)

$$\|f*k_\varepsilon - \tilde{f}\|_1 \leqslant \|\tilde{f}*\varphi_\varepsilon - \tilde{f}\|_1 + \|f-g\|_1\,\|\delta_\varepsilon\|_1 + \|g*\delta_\varepsilon\|_1.$$

Hence, by Lemma 1 and Lemma 5, we obtain

$$\limsup_{\varepsilon\to 0} \|f*k_\varepsilon - \tilde{f}\|_1 \leqslant \lim_{\varepsilon\to 0} \|\tilde{f}*\varphi_\varepsilon - \tilde{f}\|_1 + c\eta + \lim_{\varepsilon\to 0} \|g*\delta_\varepsilon\|_1 = c\eta.$$

Then the theorem follows by the arbitrariness of $\eta > 0$.

References

[1] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. 48 (1962), 356–365.

[2] A. P. Calderón, M. Weiss and A. Zygmund, *On the existence of singular integrals*, Proc. Symp. Pure Math. 10 (1967), 56–73.

[3] N. M. Rivière, *Singular integrals and multipliers operators*, Arkiv for Mat. 9 (1971), 243–278.

INSTITUTO ARGENTINO DE MATEMÁTICA
CONSEJO NACIONAL DE INVESTIGACIONES CIENTÍFICAS Y TÉCNICAS
and
FACULTAD DE CIENCIAS EXACTAS Y NATURALES
CIUDAD UNIVERSITARIA, NÚÑEZ, BUENOS AIRES, ARGENTINA

Received October 13, 1982 (1817)