References

[1] E. Atencia and A. de la Torre, *A dominated ergodic estimate for L_p spaces with weights*, Studia Math. 74 (1982), 35–47.

[2] E. Atencia and F. J. Martin-Reyes, *The maximal ergodic Hilbert transform with weights*, Pacific J. Math. 108 (2) (1983), 257–263.

[3] C. Fefferman and E. M. Stein, *Some maximal inequalities*, Amer. J. Math. 1 (1971), 107–115.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.

[5] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, 1970.

DEPARTAMENTO DE ANALISIS MATEMATICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE MALAGA, SPAIN

Received May 4, 1982 (1756)

STUDIA MATHEMATICA, T. LXXVIII. (1984)

On a generalized Carleson inequality

by

D. G. DENG (Peking)

Abstract. In this note we prove a generalized Carleson inequality

$$\Big|\iint_{R^2_+} F(x, t)v(x, t)\,dx\,dt\Big| \leqslant C \int_R A_p(F)(x)\, v_{*p'}(x)\,dx,$$

where $1/p+1/p' = 1$, $1 \leqslant p \leqslant \infty$,

$$A_p(F)(x) = \Big(\iint_{\Gamma(x)} |F(y, t)|^p \frac{dy\,dt}{t}\Big)^{1/p}, \quad v_{*p'}(x) = \sup_{x\in I}\Big(\frac{1}{|I|}\iint_{\tilde I} |v(y, t)|^{p'}\,dy\,dt\Big)^{1/p'}.$$

Moreover, $v_{*p'}$ belongs to the Muckenhoupt class A_1 for $p' > 1$.

1. Introduction. The inequality

$$\Big|\iint_{R^2_+} F(x, t)v(x, t)\,dx\,dt\Big| \leqslant C \int_R F^*(x)\,dx \ (*) \tag{1}$$

is known as the Carleson inequality ([4], [5], p. 236), where $F^*(x)$ is the non-tangential maximal function of $F(x, t)$, i.e.,

$$F^*(x) = \sup_{|y-x|<t} |F(y, t)|,$$

and $v(x, t)\,dx\,dt$ is a Carleson measure on R^2_+, i.e., $v(x, t) \geqslant 0$ and

$$\frac{1}{|I|}\int_{I\times[0,|I|]} v(x, t)\,dx\,dt \leqslant C$$

for any interval I on R. The purpose of this note is to give a more general form of inequality (1). To prove this we need to prove that a new kind of a maximal function gives rise to weights in A_1. This is of independent interest. Our inequality incorporates various inequalities proved by C. Fefferman and E. M. Stein and easily extends to R^n or, more generally, to the spaces of homogeneous type.

(*) As usual, throughout this note C will denote a constant not necessarily the same at each occurrence.

Let $O = \bigcup I_j$ be an open set on $\boldsymbol{R}$, where I_j are disjoint intervals. Let $\hat{O} = \bigcup \hat{I}_j$ be the open set on $\boldsymbol{R}^2_+$, defined by

$$\hat{I}_j = \{(x, t) \in \boldsymbol{R}^2_+ :\ t > 0,\ x \in I_j \text{ s.t. } (x-t, x+t) \subset I_j\}.$$

For any measurable function $v(x, t)$ defined on $\boldsymbol{R}^2_+$ and satisfying $v(x, t) \geqslant 0$ with $v \in L^p_{\text{loc}}(\boldsymbol{R}^2_+)$, we introduce

$$v_{*p}(x) = \sup_{x \in I} \left(\frac{1}{|I|} \iint_{\hat{I}} |v(y, t)|^p \, dy\, dt \right)^{1/p}. \tag{2}$$

Let $F(x, t)$ be given on $\boldsymbol{R}^2_+$; we define a p-area function as follows:

$$A_p(F)(x) = \Big(\iint_{\Gamma(x)} |F(y, t)|^p \, dy\, dt/t \Big)^{1/p}, \tag{3}$$

where $\Gamma(x)$ is a *cone* with vertex at x:

$$\Gamma(x) = \{(y, t) \in \boldsymbol{R}^2_+ :\ |y - x| < t\},$$

and

$$A_\infty(F)(x) = F^*(x) = \sup_{\Gamma(x)} |F(y, t)|.$$

We have

THEOREM 1. *If* $1/p + 1/p' = 1$, $1 \leqslant p \leqslant \infty$, *then*

$$\Big| \iint_{\boldsymbol{R}^2_+} F(x, t)\, v(x, t)\, dx\, dt \Big| \leqslant C \int_{\boldsymbol{R}} A_p(F)(x)\, v_{*p'}(x)\, dx, \tag{4}$$

where $A_p(F)(x)$ *and* $v_{*p'}(x)$ *are defined by* (2), (3), *respectively.*

In particular, if $v_{*p'}(x) \leqslant C$, *then*

$$\Big| \iint_{\boldsymbol{R}^2_+} F(x, t)\, v(x, t)\, dx\, dt \Big| \leqslant C \int_{\boldsymbol{R}} A_p(F)(x)\, dx.$$

In the case $p' = 1$, the condition $v_{*p'}(x) \leqslant C$ means that $v(x, t)\, dx\, dt$ is a Carleson measure, and the area function becomes the non-tangential maximal function $A_\infty(F)(x) = F^*(x)$, this reduces to the Carleson inequality.

To prove Theorem 1, we need

THEOREM 2. *For* $p > 1$, $v_{*p}(x)$ *is always in the class* A_1 [1], *i.e.,*

$$\frac{1}{|I|} \int_I v_{*p}(x)\, dx \leqslant C \inf_{x \in I} v_{*p}(x).$$

For examples of applications of these results, let $\psi(x)$ be a C^1 function defined on $\boldsymbol{R}$ satisfying

$$|\psi(x)| \leqslant \frac{C}{1 + x^2}, \quad \int_{\boldsymbol{R}} \psi(x)\, dx = 0.$$

We introduce an area function of f

$$\bar{A}(f)(x) = \Big(\iint_{\Gamma(x)} |(\psi_t * f)(y)|^2 \, dy\, dt/t^2 \Big)^{1/2},$$

where $\psi_t(\cdot) = t^{-1} \psi(\cdot/t)$. From [2], [5] we know that $|\psi_t * a|^q\, dx\, dt/t$ is a Carleson measure if $a \in BMO$ and $q \geqslant 2$ and $\|\bar{A}(f)\|_p \leqslant C \|f\|_p$ if $1 < p < \infty$. Thus by using Theorem 1, for $1 < p < 2$ we have

$$\iint_{\boldsymbol{R}^2_+} |\psi_t * f|^p |\psi_t * a|^\alpha \, dx\, dt/t \leqslant C \int_{\boldsymbol{R}} \bar{A}(f)^p \, dx \leqslant C \int_{\boldsymbol{R}} |f|^p \, dx, \tag{5}$$

provided $2\alpha/(2-p) \geqslant 2$, i.e., $\alpha \geqslant 2 - p$ (clearly, (5) is valid for $p \geqslant 2$, $\alpha \geqslant 0$). In particular, pick $\alpha = 1$, we have

$$\iint_{\boldsymbol{R}^2_+} |\psi_t * f|^p |\psi_t * a| \, dx\, dt/t \leqslant C \int_{\boldsymbol{R}} |f|^p \, dx.$$

This is not a consequence of Carleson's inequality since $|\psi_t * a|\, dx\, dt/t$ may not be a Carleson measure.

Another easy consequence of Theorem 1 is the Fefferman–Stein inequality [3]

$$\iint_{\boldsymbol{R}^2_+} |\psi_t * f|\, |\psi_t * a| \, dx\, dt/t \leqslant C \int_{\boldsymbol{R}} \bar{A}(f)\, dx, \qquad a \in BMO.$$

Finally, I would like to thank Professor R. R. Coifman for his effective suggestions in this work.

2. Proof of Theorem 1. First of all, assume $1 \leqslant p < \infty$. Consider

$$\Omega_k = \{x :\ A_p(F)(x) > 2^k\} = \bigcup_j J_j^{(k)},$$

where $J_j^{(k)}$ are disjoint open intervals, and

$$\Omega_k^* = \{x :\ \chi^*_{\Omega_k}(x) > \tfrac{1}{2}\} = \bigcup_j I_j^{(k)},$$

where χ_{Ω_k} is the characteristic function of Ω_k, $\chi^*_{\Omega_k}$ is the Hardy–Littlewood maximal function of χ_{Ω_k}, and $I_j^{(k)}$ are disjoint open intervals.

By Theorem 2 we know that

$$v_{*p'}(\Omega_k^*) \leqslant C v_{*p'}(\Omega_k).$$

In fact, since $v_{*p'} \in (A_1)$, we have [1]

$$\int_{\boldsymbol{R}} \chi^{*2}_{\Omega_k}(x)\, v_{*p'}(x)\, dx \leqslant C \int_{\boldsymbol{R}} \chi^2_{\Omega_k}(x)\, v_{*p'}(x)\, dx$$

thus

$$v_{*p'}(\Omega_k^*) = \int_{\Omega_k^*} v_{*p'}(x)\, dx \leqslant 4 \int_{\boldsymbol{R}} \chi^{*2}_{\Omega_k}(x)\, v_{*p'}(x)\, dx$$
$$\leqslant C \int_{\boldsymbol{R}} \chi^2_{\Omega_k}(x)\, v_{*p'}(x)\, dx = C \int_{\Omega_k} v_{*p'}(x)\, dx = C v_{*p'}(\Omega_k).$$

By the Hölder inequality we have

$$
\begin{aligned}
\Big|\iint_{\mathbf{R}_+^2} F(x,t)\,v(x,t)\,dx\,dt\Big|
&\leqslant \Big|\sum_{k=-\infty}^{+\infty} \iint_{\hat{\Omega}_k^*-\hat{\Omega}_{k+1}^*} F(x,t)\,v(x,t)\,dx\,dt\Big|\\
&\leqslant \Big|\sum_{k=-\infty}^{\infty}\sum_j \iint_{\hat{I}_j^{(k)}-\bigcup_l \hat{I}_l^{(k+1)}} F(x,t)\,v(x,t)\,dx\,dt\Big|\\
&\leqslant \sum_{k=-\infty}^{+\infty}\sum_j \Big(\iint_{\hat{I}_j^{(k)}-\bigcup_l \hat{I}_l^{(k+1)}} |F(x,t)|^p\,dx\,dt\Big)^{1/p}\Big(\iint_{\hat{I}_j^{(k)}-\bigcup_l \hat{I}_l^{(k+1)}} |v(x,t)|^{p'}\,dx\,dt\Big)^{1/p'}.
\end{aligned}
$$

Now we need an inequality

$$
\iint_{\hat{I}_j^{(k)}-\bigcup_s \hat{I}_l^{(k+1)}} |F(x,t)|^p\,dx\,dt \leqslant C \int_{I_j^{(k)}-\bigcup_s J_s^{(k+1)}} (A_p(F)(x))^p\,dx. \tag{6}
$$

If (6) is true and we observe

$$
\int_{I_j^{(k)}} v_{*p'}(x)\,dx \geqslant \Big(\frac{1}{|I_j^{(k)}|}\iint_{\hat{I}_j^{(k)}} |v(y,t)|^{p'}\,dy\,dt\Big)^{1/p'} |I_j^{(k)}|,
$$

then

$$
\begin{aligned}
\Big|\iint_{\mathbf{R}_+^2} F(x,t)\,v(x,t)\,dx\,dt\Big|
&\leqslant C\sum_{k=-\infty}^{+\infty}\sum_j \Big(\int_{I_j^{(k)}-\bigcup_s J_s^{(k+1)}} (A_p(F)(x))^p\,dx\Big)^{1/p}\Big(\iint_{\hat{I}_j^{(k)}} |v(x,t)|^{p'}\,dx\,dt\Big)^{1/p'},\\
&\leqslant C\sum_{k=-\infty}^{+\infty}\sum_j 2^{k+1}|I_j^{(k)}|\Big(\frac{1}{|I_j^{(k)}|}\iint_{\hat{I}_j^{(k)}} |v(x,t)|^{p'}\,dx\,dt\Big)^{1/p'}\\
&\leqslant C\sum_{k=-\infty}^{+\infty} 2^k\sum_j \int_{I_j^{(k)}} v_{*p'}(x)\,dx\\
&= C\sum_{k=-\infty}^{+\infty} 2^k v_{*p'}(\Omega_k^*) \leqslant C\sum_{k=-\infty}^{\infty} 2^k v_{*p'}(\Omega_k)\\
&\leqslant C\int_{\mathbf{R}} A_p(F)(x)\,v_{*p'}(x)\,dx.
\end{aligned}
$$

This is the desired inequality.

Now let us go back to the proof of (6). We introduce a characteristic function $\chi(s)=\chi_{[-1,1]}(s)$. We start from the right-hand side of (6),

$$
\begin{aligned}
\int_{I^{(k)}-\bigcup_s J_s^{(k+1)}} (A_p(F)(x))^p\,dx &= \int_{I^{(k)}-\bigcup_s J_s^{(k+1)}} dx \iint_{\Gamma(x)} |F(y,t)|^p\,\frac{dy\,dt}{t}\\
&= \iint_{(I^{(k)}-\bigcup J_s^{(k+1)})\times \mathbf{R}_+^2} |F(y,t)|^p\,\chi\Big(\frac{x-y}{t}\Big)\frac{dx\,dy\,dt}{t}\\
&\geqslant \int_{\hat{I}^{(k)}-\bigcup_l \hat{I}_l^{(k+1)}} |F(y,t)|^p\,\frac{dy\,dt}{t}\int_{I^{(k)}-\bigcup J_s^{(k+1)}} \chi\Big(\frac{x-y}{t}\Big)dx.
\end{aligned} \tag{7}
$$

For any fixed $(y,t)\in \hat{I}^{(k)}-\bigcup \hat{I}_l^{(k+1)}$ we clearly know that

$$
I\cap \Omega_{k+1}^{*C}\neq \emptyset,
$$

where $I=(y-t,\,y+t)$, Ω_{k+1}^{*C} is the complement of Ω_{k+1}^*. It means that there exists a point $x_0\in I$ such that

$$
\chi_{\Omega_{k+1}}^*(x_0)\leqslant \tfrac{1}{2},
$$

which implies

$$
\frac{1}{|I|}\int_I \chi_{\Omega_{k+1}}(x)\,dx\leqslant \tfrac{1}{2}.
$$

Thus

$$
\begin{aligned}
\frac{1}{|I|}\int_{I^{(k)}-\bigcup J_s^{(k+1)}} \chi\Big(\frac{x-y}{t}\Big)dx &= \frac{1}{|I|}\int_{I-\bigcup J_s^{(k+1)}} \chi\Big(\frac{x-y}{t}\Big)dx\\
&= \frac{1}{|I|}\int_I \{1-\chi_{\Omega_{k+1}\cap I}(x)\}\,dx \geqslant \tfrac{1}{2},
\end{aligned}
$$

i.e.,

$$
\int_{I^{(k)}-\bigcup J_s^{(k+1)}} \chi\Big(\frac{x-y}{t}\Big)dx \geqslant t.
$$

Substituting this into (7), we prove (6).

When $p=\infty$, the proof is easy. In fact,

$$
\begin{aligned}
\Big|\iint_{\mathbf{R}_+^2} F(x,t)\,v(x,t)\,dx\,dt\Big| &\leqslant \sum_{k=-\infty}^{\infty}\iint_{\hat{\Omega}_k-\hat{\Omega}_{k+1}} |F(x,t)\,v(x,t)|\,dx\,dt\\
&\leqslant \sum_{k=-\infty}^{\infty}\sum_j 2^{k+1}\int_{\hat{I}_j^{(k)}} v(x,t)\,dx\,dt \leqslant C\sum_{k=-\infty}^{+\infty} 2^k \sum_j \int_{I_j^{(k)}} v_{*1}(x)\,dx\\
&\leqslant C\sum_{k=-\infty}^{+\infty} 2^k v_{*1}(\Omega_k) \leqslant C\int A_\infty(F)(x)\,v_{*1}(x)\,dx.
\end{aligned}
$$

Modulo Theorem 2, the proof of Theorem 1 is complete.

3. Proof of Theorem 2. At first we prove that the operator

$$T\colon u(y,t) = |v(y,t)|^p \to u^*(x) = \sup_{x\in I} \frac{1}{|I|}\int_{\hat I} u(y,t)\,dy\,dt$$

is of weak type (1.1). In fact, let $\Omega = \{x\colon u^*(x) > \lambda\}$. Thus, for every $x\in\Omega$, there exists $I_x \supset x$ such that

$$\frac{1}{|I_x|}\int_{\hat I_x} u(y,t)\,dy\,dt > \lambda .$$

Then all $\{I_x\colon x\in\Omega\}$ constitute a cover of Ω. By a cover lemma ([5], p. 9) there exist $\{I_k\}$, $I_i\cap I_j = \emptyset$ $(i\neq j)$, such that

$$|\Omega| \leqslant C\sum |I_k| .$$

Then

$$|\Omega| \leqslant (C/\lambda)\sum_k \int_{\hat I_k} u(y,t)\,dy\,dt \leqslant (C/\lambda)\iint_{\boldsymbol{R}^2_+} u(y,t)\,dy\,dt .$$

Secondly, we prove that

$$v_{*p}(x) = u^*(x)^{1/p} = u^*(x)^\delta \in (A_1) \qquad (\delta = 1/p < 1).$$

For any I, decompose

$$u(y,t) = u_1(y,t) + u_2(y,t),$$

where

$$u_1(y,t) = u(y,t)\chi_{\widehat{3I}}(y,t).$$

Since T is of weak type (1, 1), by the Kolmogorov inequality we have

$$\int_I u_1^{*\delta} \leqslant C|I|^{1-\delta}\Big(\iint_{\boldsymbol{R}^2_+} u_1(y,t)\,dt\Big)^\delta ,$$

i.e.,

$$\frac{1}{|I|}\int_I u_1^{*\delta} \leqslant C\left(\frac{1}{|I|}\iint_{\boldsymbol{R}^2_+} u_1(y,t)\,dy\,dt\right)^\delta \leqslant C\left(\frac{1}{|3I|}\iint_{\widehat{3I}} u(y,t)\,dy\,dt\right)^\delta \leqslant Cu^*(y)^\delta$$

for any $y\in I$. Thus

$$\frac{1}{|I|}\int_I u_1^{*\delta} \leqslant C\inf_{y\in I} u^*(y)^\delta .$$

On the other hand, for any $x, z\in I$ we have

$$u_2^*(x) \leqslant Cu_2^*(z).$$

In fact, suppose that

$$u_2^*(x) = \frac{1}{|J|}\iint_{\hat J} u_2(y,t)\,dy\,dt .$$

If $u_2^*(x) \neq 0$, then clearly $z\in 3J$. So

$$u_2^*(x) \leqslant \frac{1}{|J|}\iint_{\widehat{3J}} u_2(y,t)\,dy\,dt \leqslant Cu_2^*(z).$$

Thus

$$\frac{1}{|I|}\int_I u_2^{*\delta} \leqslant C\inf_{z\in I} u_2^*(z)^\delta \leqslant C\inf_{z\in I} u^*(z)^\delta .$$

Since

$$u^*(x)^\delta \leqslant C\big(u_1^*(x)^\delta + u_2^*(x)^\delta\big),$$

we obtain

$$\frac{1}{|I|}\int_I u^{*\delta} \leqslant C\left(\frac{1}{|I|}\int_I u_1^{*\delta} + \frac{1}{|I|}\int_I u_2^{*\delta}\right) \leqslant C\inf_{y\in I} u^*(y)^\delta ,$$

and thus we end the proof of Theorem 2.

References

[1] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250.

[2] R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque 57, Société Mathématique de France, 1978.

[3] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[4] L. Hörmander, *L^p estimates for (pluri) subharmonic functions*, Math. Scand. 20 (1967), 65–78.

[5] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton 1970.

PEKING UNIVERSITY

Received May 10, 1982 (1759)