Proof. We prove first that if $Mf(\log^+ Mf)^\alpha$ is integrable on B, then it is also integrable on $2B$ (ball with center at the origin and radius two).

Considering the region $G = \{x \in R^n\colon 1 < |x| < 2\}$, it is enough to prove that the function under consideration is integrable on G. Now, the transformation $x \to y$ given by $y = x/|x|^2$ (inversion with respect to the unit sphere $|x| = 1$) maps G bijectively onto the region $G_0 = \{x\colon 1/2 < |x| < 1\}$ and a simple geometrical consideration shows that for each $x \in G$ we have $Mf(x) \leqslant Mf(y)$.

Taking into account that the Jacobian determinant $\partial x/\partial y$ is bounded on G_0, our assertion follows from the formula for changing variables.

By repeated application of the preceding argument, we see that the function $Mf(\log^+ Mf)^\alpha$ is locally integrable. Moreover, since $Mf(x)$ tends to zero as $|x|$ tends to infinity, the set $\{Mf > \lambda\}$ is bounded and the integral

$$\int\limits_{Mf>\lambda} \frac{Mf}{\lambda}\left(\log\frac{Mf}{\lambda}\right)^\alpha dx$$

is finite for each positive number λ. Hence $Mf \in R_\alpha$ and from Theorem 2 it follows that $f \in R_{\alpha+1}$. In particular, f belongs to $L(\log^+ L)^{\alpha+1}$. The proof of the converse is straightforward.

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Two problems in prediction theory*

by

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Abstract. We give an expression in terms of w of the quantities

$$\tau_n(w) = \inf_f \int_0^{2\pi} |1+f|^2 w\, d\theta/2\pi \quad (n = 0, 1, 2, \ldots),$$

where f ranges over the trigonometric polynomials with frequences in the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots\}$. This solves the first prediction problem due to G. Szegö for $n = 0$ and the second prediction problem due to A. Kolmogorov for $n = \infty$. In case $n = 1$, the expression is

$$\tau_1(w) = \exp \int_0^{2\pi} \log w\, d\theta/2\pi \Big(1+\Big|\int_0^{2\pi} e^{-i\theta} \log w\, d\theta/2\pi\Big|^2\Big)^{-1}.$$

1. Introduction. For $n = 0, 1, 2, \ldots$ let S_n be the manifold of trigonometric polynomials whose frequences are in the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots\}$. Let $d\theta$ be Lebesgue measure on $[0, 2\pi)$, and let $w \in L^1(d\theta/2\pi)$ and $w \geqslant 0$. The main result in this paper is a formula giving the distance $\tau_n(w)^{1/2}$ from 1 to S_n in $L^2(w\, d\theta/2\pi)$, that is,

$$\tau_n(w) = \inf\Big\{\int |1+f|^2 w\, d\theta/2\pi;\ f \in S_n\Big\}.$$

Szegö (cf. [3], p. 44) showed that

$$\tau_0(w) = \exp \int \log w\, d\theta/2\pi$$

and Kolmogorov (cf. [3], p. 208) showed that

$$\tau_\infty(w) = \Big(\int w^{-1}\, d\theta/2\pi\Big)^{-1}.$$

Let P_n be the manifold of trigonometric polynomials whose frequences are in the set $\{n+1, n+2, \ldots\}$ and $n \geqslant 0$. The author and K. Takahashi [7] got a formula giving the distance $\varrho_n(w, 2)^{1/2}$ from 1 to P_n in $L^2(w\, d\theta/2\pi)$. In this paper we prove $\tau_n(w) = \varrho_n(w^{-1}, 2)^{-1}$ in case $w^{-1} \in L^1(d\theta/2\pi)$. Then a formula giving $\tau_n(w)$ follows from the expression of $\varrho_n(w^{-1}, 2)$. Moreover, we generalize the expression of $\varrho_n(w, 2)^{1/2}$ to

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$\varrho_n(w, p)^{1/p}$ $(1 \leqslant p < \infty)$ which denotes the distance from 1 to P_n in $L^p(w\,d\theta/2\pi)$, that is,

$$\varrho_n(w, p) = \inf\{\int |1+f|^p w\,d\theta/2\pi;\ f \in P_n\}.$$

This result reduces the prediction problem to an extension problem previously studied by W. W. Rogosinski and H. S. Shapiro (cf. [1], pp. 139–142). It is known as Szegő's theorem (cf. [2], p. 136) that

$$\varrho_0(w, p) = \exp \int \log w\,d\theta/2\pi.$$

In Section 2 we prove an abstract prediction theorem. In Section 3 we get a formula giving $\varrho_n(w, p)$. In Section 4 we get a formula giving $\tau_n(w, 2)$.

2. The abstract prediction theorem. In this section we shall prove an abstract prediction theorem which is an extension of the second prediction theorem. However, the first prediction theorem follows immediately from it. Let (X, m) be a probability measure space and $w \in L^1(m)$ with $w > 0$ a.e. Suppose $w^{-1} \in L^1(m)$ through the section. If $f \in L^2(w\,dm)$ and $g \in L^2(w^{-1}\,dm)$, by the Schwarz lemma we have

$$\int |fg|\,dm \leqslant (\int |f|^2 w\,dm)^{1/2} (\int |g|^2 w^{-1}\,dm)^{1/2}$$

and so $fg \in L^1(dm)$. Hence $L^2(w\,dm) \subset L^1(dm)$ since 1 belongs to $L^2(w^{-1}\,dm)$ and similarly $L^2(w^{-1}\,dm) \subset L^1(dm)$

PROPOSITION 1. *Let M be a subspace of $\{f \in L^2(w\,dm);\ \int f\,dm = 0\}$ and set*

$$N = \{g \in L^2(w^{-1}\,dm);\ \int g\,dm = 0 \text{ and } \int f\bar{g}\,dm = 0 \text{ for all } f \in M\}.$$

Then

$$\inf_{f \in M} \int |1+f|^2 w\,dm = (\inf_{g \in N} \int |1+g|^2 w^{-1}\,dm)^{-1}.$$

Proof. If $f \in M$ and $g \in N$, by the Schwarz inequality we have

$$(\int |1+f|^2 w\,dm)^{1/2} (\int |1+g|^2 w^{-1}\,dm)^{1/2} \geqslant \int |(1+f)(1+g)|\,dm \geqslant 1.$$

So

$$\inf_{f \in M} \int |1+f|^2 w\,dm \geqslant (\inf_{g \in N} \int |1+g|^2 w^{-1}\,dm)^{-1}.$$

To prove the equality choose a unique f_0 in the closure of M in $L^2(w\,dm)$ such that

$$\inf_{f \in M} \int |1+f|^2 w\,dm = \int |1+f_0|^2 w\,dm.$$

Then, by the minimum property of $\int |1+f_0|^2 w\,dm$, $1+f_0$ is orthogonal to M in $L^2(w\,dm)$. Set $g_0 = (1+f_0)w$; then $g_0 \in L^2(w^{-1}\,dm)$. Since $w^{-1} \in L^1(m)$, the infimum taken over M is positive and so

$$\int |1+f_0|^2 w\,dm = \int (1+\bar{f}_0)(1+f_0) w\,dm = \int (1+f_0) w\,dm = \int g_0\,dm$$

and $\int g_0\,dm > 0$. Set $1+h_0 = g_0/\int g_0\,dm$; then

$$(\int g_0\,dm)^2 \int |1+h_0|^2 w^{-1}\,dm = \int |g_0|^2 w^{-1}\,dm = \int |1+f_0|^2 w\,dm = \int g_0\,dm$$

and so

$$\int |1+f_0|^2 w\,dm = \int g_0\,dm = (\int |1+h_0|^2 w^{-1}\,dm)^{-1}.$$

Since $1+f_0$ is orthogonal to M in $L^2(w\,dm)$, h_0 belongs to N. Thus the equality in proposition follows.

Let B and D be subspaces of $\{f \in L^\infty(m);\ \int f\,dm = 0\}$. Suppose B is in the orthogonal complement of D in $L^2(m)$ and $B+D+\{1\}$ is dense in $L^2(m)$. For any subset $S \subseteq L^2(w^{-1}\,dm)$, denote by $[S]$ the closed linear span of S. Then $N = L^2(w^{-1}\,dm) \ominus [w(B+\{1\})]$, that is, N is the orthogonal complement of $[w(B+\{1\})]$ in $L^2(w^{-1}\,dm)$.

LEMMA 1. *If $w \in L^\infty(m)$, then $D \subseteq N$ and N is contained in the closure of D in $L^2(m)$. If $N_0 = N \ominus [D]$, then*

$$w^{-1} N_0 \subseteq \{f \in L^1(m);\ \int f\bar{g}\,dm = 0 \text{ for all } g \in D\}.$$

Proof. $D \subseteq N$ is clear. Since $w \in L^\infty(m)$, $L^2(w^{-1}\,dm) \subset L^2(m)$ and so N is contained in the closure of D in $L^2(m)$. If $f \in L^2(w^{-1}\,dm)$, then

$$\int |w^{-1} f|\,dm \leqslant (\int |f|^2 w^{-1}\,dm)^{1/2} (\int w^{-1}\,dm)^{1/2}$$

by the Schwarz inequality and hence $w^{-1} L^2(w^{-1}\,dm) \subset L^1(m)$ and so $w^{-1} N_0 \subset \{f \in L^1(m);\ \int f\bar{g}\,dm = 0 \text{ for all } g \in D\}$.

THEOREM 1. *If $w^{-1} \in L^1(m)$, then*

$$\inf_{f \in B} \int |1+f|^2 w\,dm \geqslant (\inf_{g \in D} \int |1+g|^2 w^{-1}\,dm)^{-1}.$$

If D is dense in $N = L^2(w^{-1}\,dm) \ominus [w(B+\{1\})]$, then the equality holds. So if w, $w^{-1} \in L^\infty(m)$, then the equality holds.

Proof. Apply Proposition 1 with $M = B$; then $D \subset N$ and so the inequality follows. If D is dense in N, then the equality holds obviously. If w, $w^{-1} \in L^\infty(m)$, then $[D]$ is the closure of D in $L^2(m)$ and so $[D] = N$ by Lemma 1.

3. The prediction n units of time ahead. In this section, we shall give an expression in terms of w of the distance $\varrho_n(w, p)^{1/p}$ from 1 to P_n in $L^p(w\,d\theta/2\pi)$ $(0 < p < \infty)$. We may assume $\log w$ is summable because $\varrho_n(w, p) = 0$ for all n in case $\log w$ is not summable (cf. [2], p. 136). If $\log w \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ and $0 < p < \infty$ for each $j \geqslant 0$, set

$$A_{p,j}(w) = \sum{}' \left(\frac{2}{p}\right)^{m_1+\ldots+m_j} \frac{a_1^{m_1} \ldots a_j^{m_j}}{m_1! \ldots m_j!},$$

where $\sum'$ is the summation over all permutations of non-negative integers $m_1, m_2, \ldots, m_j$ with $m_1+2m_2+\ldots+jm_j = j$. Then

$$A_{p,0}(w) = 1, \quad A_{p,1}(w) = \left(\frac{2}{p}\right) a_1,$$

$$A_{p,2}(w) = \left(\frac{2}{p}\right)^2 \frac{a_1^2}{2} + \left(\frac{2}{p}\right) a_2.$$

THEOREM 2. *Suppose* $w \in L^1(d\theta/2\pi)$, $w \geqslant 0$ *and* $\log w \in L^1(d\theta/2\pi)$. *If* $0 < p < \infty$, *then*

$$\varrho_n(w, p) = \exp\int \log w\, d\theta/2\pi \inf\left\{\int\left|\sum_{j=0}^{n} A_{p,j}(w)\, e^{ij\theta} + f\right|^p d\theta/2\pi;\ f \in P_n\right\}.$$

For $0 < p < \infty$, the *Hardy space* H^p is the $L^p(d\theta/2\pi)$-closure of $\{1\}+P_0$, and H^∞ is defined to be the weak-* closure of $\{1\}+P_0$ in $L^\infty(d\theta/2\pi)$. In the theorem above, it is called the *minimal interpolation problem* to estimate

$$\inf\int\left|\sum_{j=0}^{n} A_{p,j}(w)\, e^{ij\theta} + f\right|^p d\theta/2\pi.$$

This problem was solved by Rogosinski and Shapiro (cf. [1], pp. 139–142). Hence our theorem solves the prediction problem above. By the duality relation (cf. [1], p. 130), the next corollary follows immediately from Theorem 2.

COROLLARY 1. *With regard to Theorem* 2, *if* $1 \leqslant p < \infty$, *then*

$$\varrho_n(w, p) = \exp\int \log w\, d\theta/2\pi \sup\left|\sum_{j=0}^{n} c_j A_{p,j}(w)\right|^p,$$

where the supremum is taken over all $g \in H^q$ *with* $\int |g|^q d\theta/2\pi \leqslant 1$ *and* $g(z) = \sum_{j=0}^{\infty} c_j z_j$ $(|z| < 1)$, *and* $1/p+1/q = 1$.

COROLLARY 2 (Nakazi and Takahashi [7]). *With regard to Theorem* 3, *we have*

$$\varrho_n(w, 2) = \exp\int \log w\, d\theta/2\pi \sum_{j=0}^{n} |A_{2,j}|^2.$$

COROLLARY 3. *If* $0 < p < \infty$, *then*

$$\varrho_1(w, p) = \exp\int \log w\, d\theta/2\pi \times$$
$$\times \inf\left\{\int\left|1+\left(\frac{2}{p}\int e^{-i\theta}\log w\, d\theta/2\pi\right)e^{i\theta}+f\right|^p d\theta/2\pi;\ f \in P_1\right\}$$

and if $p \geqslant 1$, *then for* $1/p+1/q = 1$

$$\varrho_1(w, p) = \exp\int \log w\, d\theta/2\pi \sup\left\{\left|c_0+\left(c_1\frac{2}{p}\int e^{-i\theta}\log w\, d\theta/2\pi\right)e^{i\theta}\right|\right\}$$

with regard to Corollary 1.

Proof of Theorem 2. Suppose $\log w \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. Set

$$h_1(z) = a_0/p + \sum_{l=1}^{n} (2a_l/p)\, z^l \quad \text{and} \quad h_2(z) = \sum_{l \geqslant n+1}^{\infty} (2a_l/p)\, z^l \quad (|z| < 1);$$

then their radial limits satisfy

$$w(e^{i\theta}) = |\exp h_1(e^{i\theta})|^p |\exp h_2(e^{i\theta})|^p$$

a.e. θ and $\exp(h_1+h_2)$ is an outer function (cf. [6], p. 62). Since Beurling's theorem for $0 < p < \infty$ (cf. [2], p. 63) implies that there exist $g_n \in P$ such that $g_n \exp(h_1+h_2) \to 1$ in $L^p(d\theta)$ and $\exp h_2 = 1+k$ for some $k \in e^{i(n+1)\theta} H^p$,

$$\begin{aligned}\varrho_n(w, p) &= \inf\left\{\int |\exp h_1 \exp h_2 + f|^p d\theta/2\pi;\ f \in P_n\right\}\\ &= \inf\left\{\int |\exp h_1 + k\exp h_1 + f|^p d\theta/2\pi;\ f \in P_n\right\}\\ &= \inf\left\{\int |\exp h_1 + f|^p d\theta/2\pi;\ f \in P_n\right\}.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\exp h_1(e^{i\theta}) &= \exp\frac{a_0}{p}\exp\left(\sum_{l=1}^{n}\frac{2a_l}{p}e^{il\theta}\right)\\ &= \exp\frac{a_0}{p}\prod_{l=1}^{n}\exp\left(\frac{2a_l}{p}e^{il\theta}\right)\\ &= \exp\frac{a_0}{p}\sum\frac{(2a_1 e^{i\theta}/p)^{m_1}\ldots(2a_n e^{in\theta}/p)^{m_n}}{m_1!\ldots m_n!},\end{aligned}$$

where the m_j ranges independently over non-negative integers. Thus

$$\varrho_n(w, p) = e^{a_0}\inf\left\{\int\left|\sum_{j=0}^{n} A_{p,j}\, e^{ij\theta}+f\right|^p d\theta/2\pi;\ f \in P_n\right\}.$$

4. The general prediction theorem. In this section, we shall study a general prediction problem which connects the first prediction problem and the second one. That is, we shall determine $\tau_n(w)$, using the results in Sections 2 and 3. In this section we shall assume that the probability measure space (X, m) in Section 2 is $([0, 2\pi), d\theta/2\pi)$. The manifold $\bar{P}_n+\{1\}+S_n$ is the set of all trigonometric polynomials and $S_0 = P_0$. Let $\mathfrak{P}_n$ be the manifold of trigonometric polynomials whose frequences are in the set

$\{1, 2, \ldots, n\}$. The proof of Corollary 4 is an elegant alternative proof based on Theorem 2.

COROLLARY 4 (Szegö). *If* $w \in L^1(d\theta/2\pi)$ *and* $w \geq 0$, *then*

$$\tau_0(w) = \exp \int \log w \, d\theta/2\pi.$$

If $\log w$ *is not summable*, $\tau_0(w)$ *is zero.*

Proof. Apply Theorem 2 with $B = S_0$ and $D = \bar{P}_0$; then

$$\inf_{f \in S_0} \int |1+f|^2 w \, d\theta/2\pi = \big(\inf_{g \in S_0} \int |1+g|^2 w^{-1} d\theta/2\pi\big)^{-1}$$

in case $w, w^{-1} \in L^1(d\theta/2\pi)$ because $S_0 = P_0$. By the inequality of arithmetic and geometric means and Jensen's inequality (cf. [1], p. 23), for $f, g \in S_0$

$$\int |1+f|^2 w \, d\theta/2\pi \geq \exp \int \log w \, d\theta/2\pi$$

and

$$\int |1+f|^2 w^{-1} d\theta/2\pi \geq \exp \int \log w^{-1} d\theta/2\pi.$$

Hence if $w, w^{-1} \in L^\infty(d\theta/2\pi)$, then $\tau_0(w) = \exp \int \log w \, d\theta/2\pi$. For any $w \in L^1(d\theta/2\pi)$, we can prove the theorem by a well-known argument.

COROLLARY 5 (Kolmogorov). *If* $w \in L^1(d\theta/2\pi)$ *and* $w \geq 0$, *then*

$$\tau_\infty(w) = \big(\int w^{-1} d\theta/2\pi\big)^{-1}.$$

If w^{-1} *is not summable*, $\tau_\infty(w)$ *is zero.*

Proof. Apply Theorem 2 with $B = S_\infty$ and $D = \{0\}$. Then the corollary follows since $N = L^2(w^{-1} d\theta/2\pi) \ominus [w(S_\infty + \{1\})] = 0$.

LEMMA 2. *Suppose* $w \in L^\infty(d\theta/2\pi)$ *and* $w^{-1} \in L^1(d\theta/2\pi)$ *and*

$$N = L^2(w^{-1} d\theta/2\pi) \ominus [w(P_n + \{1\})],$$

then $\bar{S}_n$ *is dense in* N.

Proof. Apply Lemma 1 with $B = P_n$ and $D = \bar{S}_n$; then $\bar{S}_n \subset N$ and $N \subset e^{-i\theta} \bar{H}^2 + \mathfrak{P}_n$ and $w^{-1} N_0 \subset e^{i(n+1)\theta} H^1 + \{1\}$. So if $f \in N_0$, then

$$w^{-1}|f|^2 \in e^{i\theta} H^{1/2} + \mathfrak{P}_n$$

and $w^{-1}|f|^2 \geq 0$. By [5], p. 11, $w^{-1}|f|^2$ is a trigonometric polynomial and $\int w^{-1}|f|^2 d\theta/2\pi = 0$. Thus $N_0 = \{0\}$ and so $\bar{S}_n$ is dense in N.

PROPOSITION 2. *Suppose* $w \in L^1(d\theta/2\pi)$ *with* $w \geq 0$ *and* $w^{-1} \in L^1(d\theta/2\pi)$. *Then*

$$\tau_n(w) = \varrho_n(w^{-1}, 2)^{-1}$$

and

$$\begin{aligned}\big(\int w^{-1} d\theta/2\pi\big)^{-1} &= \tau_\infty(w) \leq \tau_n(w) \\ &\leq \tau_0(w) = \varrho_0(w, 2) = \exp \int \log w \, d\theta/2\pi \\ &\leq \varrho_m(w, 2) \leq \varrho_\infty(w) = \int w \, d\theta/2\pi.\end{aligned}$$

Moreover, $\varrho_m(w, 2) \uparrow \varrho_\infty(w, 2)$ *as* $m \to \infty$ *and* $\tau_n(w) \uparrow \tau_\infty(w)$ *as* $n \to \infty$.

Proof. Since $(w+\varepsilon)^{-1} \in L^\infty(d\theta/2\pi)$ for any $\varepsilon > 0$, by Theorem 1 and Lemma 2, $\varrho_n((w+\varepsilon)^{-1}, 2) = \tau_n(w+\varepsilon)^{-1} \leq \tau_n(w)^{-1} \leq \varrho_n(w^{-1}, 2)$. By Corollary 2, $\varrho_n((w+\varepsilon)^{-1}, 2) \to \varrho_n(w^{-1}, 2)$ as $\varepsilon \to 0$ and hence $\tau_n(w) = \varrho_n(w^{-1}, 2)^{-1}$. It is known ([4], p. 22) that $\varrho_m(w, 2) \uparrow \varrho_\infty(w, 2)$ as $m \to \infty$ and so $\tau_n(w) \downarrow \tau_\infty(w)$ as $n \to \infty$. Now Corollaries 4 and 5 imply this theorem.

Now we shall prove the main theorem which contains Corollaries 4 and 5. If $\log w \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, for each j

$$B_j(w) = \sum{}' (-1)^{m_1+\ldots+m_j} \frac{a_1^{m_1} \ldots a_j^{m_j}}{m_1! \ldots m_j!},$$

where $\sum'$ is the summation over all permutation of non-negative integers $m_1, m_2, \ldots, m_j$ with $m_1 + 2m_2 + \ldots + jm_j = j$. Then $B_0(w) = 1$, $B_1(w) = -a_1$, $B_2(w) = a_1^2/2 - a_2$.

THEOREM 3. *Suppose* $w \in L^1(d\theta/2\pi)$ *with* $w \geq 0$ *and* $w^{-1} \in L^1(d\theta/2\pi)$. *Then*

$$\tau_n(w) = \exp \int \log w \, d\theta/2\pi \big(\sum_{j=0}^{n} |B_j|^2\big)^{-1}.$$

Proof is clear by Theorem 2 and Proposition 2.

COROLLARY 6. *With regard to Theorem 3, we have*

$$\tau_1(w) = \exp \int \log w \, d\theta/2\pi \big(1 + \big|\int e^{-i\theta} \log w \, d\theta/2\pi\big|^2\big)^{-1}.$$

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Random integrals of Banach space valued functions

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Abstract. In this paper we study random integrals of the form $\int f dM$, where f is a deterministic Banach space valued function and M an independently scattered random measure. Random integrals of this type are a natural generalization of random series with Banach space valued coefficients. We prove an analogon of the Ito–Nisio theorem for random integrals, a comparison theorem and some contraction principles. Results are applied for stable measures on Banach spaces.

1. Introduction. The present paper is devoted to a study of random integrals of the form $\int f\, dM$, where f is a deterministic function taking values in a Banach space E and M is an independently scattered random measure. Random integrals of this type are a natural generalization of random series with Banach space valued coefficients. It is well known that the assymptotic behaviour of such series depends also on some geometric properties of the Banach space. Analogously, the existence of certain bounded linear operators on appropriate function spaces which we call random integrals, depends in general on a geometric structure of E. Hoffmann-Jørgensen and Pisier [7] defined Gaussian random integrals for spaces of type 2. Marcus and Woyczyński [14] and Okazaki [15] considered p-stable random integrals assuming that E is of stable type p. Woyczyński [26] investigated Poissonian random integrals for spaces of Rademacher type p.

In this article we define and study random integrals without any restrictions on a geometry of E, per an analogy to the theory of random series with Banach space valued coefficients. Such approach for Gaussian random integrals was presented in [6] and [19] and for stable random integrals in [18]. It permits to have a non-trivial class of integrable functions in each Banach space which was impossible under the classical approach (see [27]). A general theory of bilinear random integrals is developed in [17].

In Section 2 we consider preliminary facts concerning a random integral. An analogon of the well-known Ito–Nisio theorem for symmetric summands (see f.e. [3], Chap. 3, Th. 2.10) is proved in Section 3 (Theorem 3.4). Namely, if for every $x^* \in E^*$ the real random integral $\int \langle x^*, f\rangle dM$ exists and there exists a Radon probability measure μ on E such that

$$\hat{\mu}(x^*) = E\exp\{i\int \langle x^*, f\rangle dM\}, \qquad x^* \in E^*,$$