The generalization of Cellina's Fixed Point Theorem

by

ANDRZEJ FRYSZKOWSKI (Warsaw)\(^*\)

Abstract. Let \( L(T, Z) \) be the Banach space of integrable functions from a compact space \( T \) into a Banach space \( Z \). A set \( K \subseteq L(T, Z) \) is called decomposable if, for every \( u, v \in K \) and measurable \( A \subseteq T, u \cdot \chi_A + v \cdot \chi_{Z \setminus A} \in K \). In this note we prove that each compact mapping from a closed and decomposable subset \( K \subseteq L(T, Z) \) into itself has a fixed point.

§1. Introduction. In paper [2] Cellina proved that the set \( K_p \) of all functions integrable on a closed interval \([a, b]\) whose values belong to a fixed closed subset \( P \) of a Euclidean space \( R^m \) has a fixed point property; this means that each compact mapping from \( K_p \) into itself has a fixed point. The set \( K_p \) can be nonconvex; thus the result of Cellina is interesting when confronted with Schauder's Fixed Point Theorem, where the assumption of convexity is essential (see [3], [8]).

In this note we generalize the above result to an arbitrary closed and decomposable subset \( K \) of the space of integrable functions. The decomposability of a set \( K \) means that for each \( u, v \in K \) and \( A \) measurable \( u \cdot \chi_A + v \cdot \chi_{Z \setminus A} \in K \), where \( \chi_A \) stands for the characteristic function of \( A \).

Obviously, the set \( K_p \) in the theorem of Cellina is decomposable. This generalization is quite easy to obtain if we apply a certain theorem on continuous selections proved by the author in [5]. The theorem is an abstract version of Antosiewicz and Cellina's Selection Theorem [1] and can also be applied to the problem of the existence of solutions for the functional-differential inclusion \( x(t) \in F(t, x(\cdot)) \) (see [6]). The required facts about the selections are given in §3. We formulate the main results in §2 and prove it in §4.

§2. The main result. Let \( T \) be a compact topological space with a \( \sigma \)-field \( \mathcal{M} \) of measurable subsets of \( T \) given by a nonnegative, regular Borel measure \( dt \) and let \( Z \) be a separable Banach space with norm \( |\cdot| \). By \( L(T, Z) \) we denote the Banach space of functions \( u: T \to Z \), integrable in the Bochner sense, with norm \( ||u||_B = \int |u(t)| \, dt \).

\(^*\) Current address: Institute of Mathematics, Technical University of Warsaw, 00-661 Warsaw, Pl. Jana Kochana 1, Poland.
We call a set $K \subseteq L(T, Z)$ decomposable if $u \cdot \lambda + v \cdot \phi \in K$ for every $u, v \in K$ and $\lambda \in \mathbb{R}$. The family of all nonempty closed and decomposable subsets of $L(T, Z)$ we denote by $d(L)$. From this moment let $K$ be a fixed set from $d(L)$. The main result is the following:

**Theorem.** Let $\varphi : K \to K$ be a compact mapping. Then $\varphi$ has a fixed point.

**Corollary.** Let $\Omega$ be an abstract space with a $\sigma$-field $\Sigma$ and let $\varphi : \Omega \times K \to K$ be a function measurable in the first variable and compact in the second. Then there exists a $\Sigma$-measurable function $s : \Omega \to K$ such that, for each $\omega \in \Omega$, $\varphi(\omega, s(\omega)) = s(\omega)$. This function $s$ is a $\Sigma$-measurable selection of the map $P$ from $\Omega$ into closed subsets of $K$ given by $P(\omega) = \{ s \in K : \forall \omega \in K \} = s$ which is $\Sigma$-measurable (see [4], [7]).

§3. Selection Theorem. Let $S$ and $X$ be topological spaces. Denote by $\text{cl}(X)$ the family of all nonempty and closed subsets of $X$ and let $P : S \to \text{cl}(X)$ be the multivalued map. The function $p : S \times X$ is a selection of $P$ if, for each $s \in S$, we have $p(s) \in P(s)$.

The map $P : S \to \text{cl}(X)$ is called lower semicontinuous (l.s.c.), if the set $P^{-1}(U) = \{ s \in S : P(s) \subseteq U \}$ is open for each open $U \subseteq X$.

The following selection theorem was proved in [5]:

**Selection Theorem.** Assume that $S$ is a compact topological space and the map $L : S \to d(L)$ is l.s.c. Then $L$ admits a continuous selection.

We apply this theorem to the maps $L_\varepsilon$ defined on the set $G = \text{cl}(\text{co} \varphi(K))$.

(1)

$$S = \text{cl}(\text{co} \varphi(K))$$

for each $\varepsilon > 0$ by the formulas

(2)

$$L_\varepsilon(s) = \text{cl}(\{ u \in K : |u(t) - s(t)| < \text{ess inf}_t |u(t) - s(t)| + \varepsilon \})$$

almost everywhere in $T$, where $\text{ess inf}$ stands for the essential infimum and $\underline{\varphi}$ and $K$ are as in the Theorem.

If the sets $L_\varepsilon(s)$ are nonempty follows from the observation that for each $s \in S$ there exists an element $u_s \in K$ such that $|u_s(t) - s(t)| = \text{ess inf}_t |u(t) - s(t)|$ a.e. in $T$ (see [5], Prop. 2.1). The lower semicontinuity and the decomposability of $L_\varepsilon$ given by (2) can easily be deduced from Proposition 2.3 in [5] if we observe that the map $\psi$ defined by $\psi(s) = \text{ess inf}_t |u(t) - s(t)|$ is a Lipschitz function in $L$-norm. For this purpose fix $s_1$ and $s_2$ from $S$ and let $u_t \in K$ be such an element that $|u_t(t) - s_1(t)| = \psi(s_1)(t)$ a.e. in $T$.

Then the Lipschitz condition follows from the inequalities

$$|u_t(t) - s_2(t)| \leq |u_t(t) - s_1(t)| + |s_1(t) - s_2(t)|$$

a.e. in $T$.

§4. Proof of the Theorem. Let $S$ be defined by (1). Obviously $S$ is a convex and compact subset of $L(T, Z)$. Consider the map $L_\varepsilon$ given by (2) and let $L_\varepsilon : S \to K$ be a continuous selection of $L_\varepsilon$. From the definition of $L_\varepsilon$ it follows that for every $s \in \varphi(K)$ the inequality

(3)

$$||L_\varepsilon(s) - s|| \leq \varepsilon \cdot ||x||$$

holds.

Consider the continuous maps $\varphi \circ L_\varepsilon : S \to \varphi(K)$. The Schauder Fixed Point Theorem implies that for each $\varepsilon > 0$ there exist points $s_\varepsilon$ such that

(4)

$$\varphi[L_\varepsilon(s_\varepsilon)] = s_\varepsilon.$$

Those points belongs to $\varphi(K)$ and from (3) it follows that for each $\varepsilon > 0$ we have

(5)

$$||L_\varepsilon(s_\varepsilon) - s_\varepsilon|| \leq \varepsilon \cdot ||x||.$$

Obviously the net $\{ s_\varepsilon \}$ is totally bounded and we may assume that it converges. Let $s_0 = \lim_{\varepsilon \to 0} s_\varepsilon$. Then also $\lim_{\varepsilon \to 0} L_\varepsilon(s_\varepsilon) = s_0$ because of (5). Taking the limits in (4) we notice that $s_0$ is the fixed point of $\varphi$, which completes the proof.

**References**


