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Mapping properties of maximal operators

by

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Abstract. The concept of a maximal operator is formed by abstraction of the fundamental properties of both the Hardy–Littlewood and the ergodic maximal operator. We study maximal operators in connection with a family of spaces of measurable functions which is naturally associated with such operators.

1. Introduction. As a motivation, let us consider the Hardy–Littlewood maximal operator M defined for each measurable function f on R^n by means of the formula

$$(1) \qquad Mf(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x, r)$ denotes the ball of radius r centered at x and vertical bars outside the integral stand for Lebesgue measure.

Assuming that f has support in the unit ball $B = \{x \in R^n: |x| < 1\}$, a well-known theorem of Hardy and Littlewood [2] which dates back to 1930, asserts that if f belongs to $L(\log^+ L)$, then $Mf \in L^1(B)$.

In more recent times E. M. Stein [4] proved that the converse of the above-mentioned theorem is also true.

The present note, which is a postscript to our paper [1], contains a generalization of those two theorems and, what is perhaps more relevant, it describes the way in which maximal operators map a certain class of spaces of measurable functions which is naturally associated with such operators. These properties were overlooked at the time when [1] was written.

Moreover, we have eliminated an unnecessary restriction concerning positivity from the definition of a maximal operator as originally given in [1].

2. Classes R_α. Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. For each $\alpha \geqslant 0$, we shall denote by R_α the class of all functions f measurable on X such that the integral

$$(2) \qquad \int_{|f|>\lambda} \frac{|f|}{\lambda} \left(\log \frac{|f|}{\lambda}\right)^{\alpha} d\mu$$

is finite for every positive number λ. These classes were introduced in [1] in connection with the ergodic theorem and the strong differentiability of integrals. In particular, R_0 consists of all functions f such that f is integrable over the set where $|f| > \lambda$ for each positive λ.

When α is positive, the integral (2) may be extended over the whole space X without changing its value provided that in the integrand we write $\log^+$ instead of log.

Denoting, as usual, by $L(\log^+ L)^\alpha$ the class of all functions f such that (2) is finite for some positive λ, we see that $R_\alpha \subset L(\log^+ L)^\alpha$ and that both classes coincide if and only if $\mu(X) < \infty$.

We refer to [1] for the easy proof of the following facts:

(i) each class R_α is a linear subspace of $L(\log^+ L)^\alpha$;

(ii) among these classes, we have the relations

$$R_\beta \subset R_\alpha \quad \text{if} \quad \alpha < \beta; \tag{3}$$

(iii) if for the sake of brevity we write simply L^p in place of $L^p(\mu)$, then $L^1 \subset R_0 \subset L^1 + L^\infty$, so that R_0 coincides with L^1 when $\mu(X)$ is finite;

(iv) if $1 < p < \infty$, then $L^p \subset R_\alpha$ for each $\alpha \geqslant 0$.

When $\alpha \geqslant 1$, the class $L(\log^+ L)^\alpha$ becomes a Banach space by defining the norm of f as the infimum of all positive numbers λ such that integral (2) is less than or equal to one. In this circumstance it can be shown that R_α is a closed subspace of the Banach space $L(\log^+ L)^\alpha$.

3. Maximal operators. An operator T mapping R_0 into the class of all measurable functions on X will be called a *maximal operator* if it satisfies the following properties:

(A) $|Tf| \leqslant T|f|$;

(B) $0 \leqslant f \leqslant g$ implies $Tf \leqslant Tg$;

(C) T is sublinear, that is, $|T(f+g)| \leqslant |Tf| + |Tg|$ and $|T(\lambda f)| = |\lambda|\,|Tf|$;

(D) $\|Tf\|_\infty \leqslant \|f\|_\infty$;

(E) T is of weak type 1-1; in other words, there exists a constant C depending only on T, such that

$$\mu\{|Tf| > \lambda\} \leqslant \frac{C}{\lambda}\|f\|_1$$

for every positive number λ.

All order or equality relations among functions in conditions (A), (B) and (C) are to be understood as holding almost everywhere with respect to μ. Condition (A) implies that T is a positive operator.

A maximal operator T satisfies *Wiener's inequality*:

$$\mu\{|Tf| > t\} \leqslant \frac{2C}{t} \int_{2|f| > t} |f|\, d\mu \quad (t > 0) \tag{4}$$

which is of the utmost importance for the study of such operators and whose proof is also given in [1], p. 276.

From Marcinkiewicz's interpolation theorem or else directly from Wiener's inequality and Lemma 1 below, it follows that every maximal operator maps L^p into L^p for each $p > 1$.

The Hardy–Littlewood maximal operator (1), the ergodic maximal operator defined in [1], p. 272, and the "partial" maximal operators M_i $(i = 1, \ldots, n)$ defined for each function f in $L^1(R^n)$ by means of the formulae

$$M_i f(x) = \sup_{a < x_i < b} \frac{1}{b-a} \int_a^b |f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)|\, dt \tag{5}$$

are the most important examples of maximal operators. The proof that the M_i's defined by (5) actually are maximal operators only requires the use of Fubini's theorem.

In the sequel we shall repeatedly use the following basic lemma, whose elementary proof is given in [3].

Lemma 1. *Let φ be a non decreasing function on the real interval $0 \leqslant t < \infty$, such that $\varphi(0) = 0$ and φ is absolutely continuous on every finite subinterval of the given half line. Then for any non negative measurable function f on $(X, \mathscr{A}, \mu)$, we have the formula*

$$\int_X \varphi \circ f\, d\mu = \int_0^\infty \mu\{f > t\}\, \varphi'(t)\, dt. \tag{6}$$

Our first aim in this note is to prove that, in general, we have the following theorem.

Theorem 1. *If T is a maximal operator, then for every $\alpha \geqslant 0$, T maps $R_{\alpha+1}$ into R_α.*

Proof. Assuming that $f \in R_{\alpha+1}$, we must prove that the integral

$$\int_X \frac{|Tf|}{\lambda}\left(\log^+ \frac{|Tf|}{\lambda}\right)^\alpha d\mu \tag{7}$$

is finite for every positive number λ (we give the proof assuming that α is positive, the remaining case being quite similar and, in fact, simpler).

Considering the function

$$\varphi(t) = \frac{t}{\lambda}\left(\log^+ \frac{t}{\lambda}\right)^\alpha \tag{8}$$

and using formula (6) of Lemma 1, we may write (7) in the form

$$\int_0^\infty \mu\{|Tf| > t\}\, \varphi'(t)\, dt. \tag{9}$$

On the other hand, by virtue of Wiener's inequality (4), we can dominate (9) by

$$\int_0^\infty \left[\frac{2C}{t}\int_{2|f|>t} |f|\,d\mu\right]\varphi'(t)\,dt$$

$$= 2C\int_X d\mu\,|f|\int_0^{2|f|}\frac{\varphi'(t)}{t}\,dt$$

$$= 2C\int_X d\mu\,|f|\int_\lambda^{2|f|}\left\{\frac{1}{\lambda t}\left(\log\frac{t}{\lambda}\right)^\alpha+\frac{\alpha}{\lambda t}\left(\log\frac{t}{\lambda}\right)^{\alpha-1}\right\}dt$$

$$= A_\alpha\int_X \frac{2|f|}{\lambda}\left(\log^+\frac{2|f|}{\lambda}\right)^{\alpha+1}d\mu + C\int_X\frac{2|f|}{\lambda}\left(\log^+\frac{2|f|}{\lambda}\right)^{\alpha}d\mu,$$

where $A_\alpha = C(\alpha+1)^{-1}$.

Since $R_{\alpha+1}\subset R_\alpha$, our assumption about f implies that the last two integrals are finite for every positive λ and consequently, so is (7) as we wanted to prove.

From Theorem 1 we see at once that if $T_1,\ldots,T_k$ are maximal operators, then the product $T_k T_{k-1}\ldots T_1$ is well defined on R_{k-1}. Moreover, if f belongs to R_k, then the function $T_k T_{k-1}\ldots T_1 f$ is in R_0; in particular, it is integrable over every set of finite measure.

The last assertions explain by themselves why we have chosen R_0 as the natural domain of general maximal operators while the spaces R_k, k an integer, are the natural domains of products of such operators. At this point we wish to remark that the main result in [1] is a weak type estimate concerning precisely the product $T_k T_{k-1}\ldots T_1$ of k maximal operators. Namely, therein it is proved that for every function f in R_{k-1} and every positive number λ, we have the inequality

(10) $$\mu\{|T_k\ldots T_1 f|>4\lambda\}\leqslant C\int_X\frac{|f|}{\lambda}\left(\log^+\frac{|f|}{\lambda}\right)^{k-1}d\mu,$$

where C is a constant depending on k and the constants of weak type corresponding to the operators involved.

In the light of Theorem 1, the proof of (10) becomes simpler by removing the unnecessary step of verifying that the operation $T_k T_{k-1}\ldots T_1 f$ is well defined for any function f in R_{k-1}.

Finally, we remark that (10) can be used to obtain an alternative proof of a theorem of Zygmund [5] concerning the differentiability of integrals with respect to families of intervals having a certain number of sides with equal size.

4. Stein's inequality. In the case of the Hardy–Littlewood operator (1), E.M. Stein has proved in [4] that Wiener's inequality (4) may be reversed in a certain sense. More precisely, he proved that for every measurable function f on R^n and every $t>0$, we have the inequality

(11) $$|\{Mf>t\}|\geqslant\frac{2^{-n}c^{-1}}{t}\int_{|f|>ct}|f|\,dx,$$

where c is a positive constant depending only on the dimension n.

On the basis of Stein's inequality (11) we prove the following theorem.

THEOREM 2. *If for some $\alpha\geqslant 0$ we have $Mf\in R_\alpha$, then necessarily $f\in R_{\alpha+1}$.*

Proof. If $Mf\in R_\alpha$ and λ is a positive number, we have

$$\infty>\int_{R^n}\frac{Mf}{\lambda}\left(\log^+\frac{Mf}{\lambda}\right)^\alpha dx.$$

By means of function (8), we may write the last integral in the form

(12) $$\int_0^\infty |\{Mf>t\}|\,\varphi'(t)\,dt.$$

On the other hand, by virtue of (11), the integral (12) is not less than

$$\int_0^\infty\left[\frac{2^{-n}c^{-1}}{t}\int_{|f|>ct}|f|\,dx\right]\varphi'(t)\,dt\geqslant 2^{-n}c^{-1}\int_{R^n}dx\,|f|\int_\lambda^{c^{-1}|f|}\frac{1}{\lambda t}\left(\log\frac{t}{\lambda}\right)^\alpha dt$$

$$=2^{-n}(\alpha+1)^{-1}\int_{R^n}\frac{|f|}{c\lambda}\left(\log^+\frac{|f|}{c\lambda}\right)^{\alpha+1}dx.$$

So that $f\in R_{\alpha+1}$ as we wanted to prove.

Hence, $Mf\in R_\alpha$ if and only if $f\in R_{\alpha+1}$. Moreover, from the preceding theorems, we derive the following corollary:

COROLLARY. *Let f be a function on R^n with support in the unit ball B and let α be a non negative number. Then $Mf(\log^+ Mf)^\alpha$ is integrable on B if and only if f is in $L(\log^+ L)^{\alpha+1}$.*

Proof. We prove first that if $Mf(\log^+ Mf)^\alpha$ is integrable on B, then it is also integrable on $2B$ (ball with center at the origin and radius two).

Considering the region $G = \{x \in R^n: 1 < |x| < 2\}$, it is enough to prove that the function under consideration is integrable on G. Now, the transformation $x \to y$ given by $y = x/|x|^2$ (inversion with respect to the unit sphere $|x| = 1$) maps G bijectively onto the region $G_0 = \{x: 1/2 < |x| < 1\}$ and a simple geometrical consideration shows that for each $x \in G$ we have $Mf(x) \leqslant Mf(y)$.

Taking into account that the Jacobian determinant $\partial x/\partial y$ is bounded on G_0, our assertion follows from the formula for changing variables.

By repeated application of the preceding argument, we see that the function $Mf(\log^+ Mf)^\alpha$ is locally integrable. Moreover, since $Mf(x)$ tends to zero as $|x|$ tends to infinity, the set $\{Mf > \lambda\}$ is bounded and the integral

$$\int_{Mf>\lambda} \frac{Mf}{\lambda}\left(\log\frac{Mf}{\lambda}\right)^\alpha dx$$

is finite for each positive number λ. Hence $Mf \in R_\alpha$ and from Theorem 2 it follows that $f \in R_{\alpha+1}$. In particular, f belongs to $L(\log^+ L)^{\alpha+1}$. The proof of the converse is straightforward.

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Two problems in prediction theory*

by

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Abstract. We give an expression in terms of w of the quantities

$$\tau_n(w) = \inf_f \int_0^{2\pi} |1+f|^2 w\, d\theta/2\pi \quad (n = 0, 1, 2, \ldots),$$

where f ranges over the trigonometric polynomials with frequences in the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots\}$. This solves the first prediction problem due to G. Szegö for $n = 0$ and the second prediction problem due to A. Kolmogorov for $n = \infty$. In case $n = 1$, the expression is

$$\tau_1(w) = \exp \int_0^{2\pi} \log w\, d\theta/2\pi \big(1 + \big|\int_0^{2\pi} e^{-i\theta} \log w\, d\theta/2\pi\big|^2\big)^{-1}.$$

1. Introduction. For $n = 0, 1, 2, \ldots$ let S_n be the manifold of trigonometric polynomials whose frequences are in the set $\{-n, -n+1, \ldots, -1, 1, 2, \ldots\}$. Let $d\theta$ be Lebesgue measure on $[0, 2\pi)$, and let $w \in L^1(d\theta/2\pi)$ and $w \geqslant 0$. The main result in this paper is a formula giving the distance $\tau_n(w)^{1/2}$ from 1 to S_n in $L^2(w\, d\theta/2\pi)$, that is,

$$\tau_n(w) = \inf\{\textstyle\int |1+f|^2 w\, d\theta/2\pi;\ f \in S_n\}.$$

Szegö (cf. [3], p. 44) showed that

$$\tau_0(w) = \exp \textstyle\int \log w\, d\theta/2\pi$$

and Kolmogorov (cf. [3], p. 208) showed that

$$\tau_\infty(w) = (\textstyle\int w^{-1}\, d\theta/2\pi)^{-1}.$$

Let P_n be the manifold of trigonometric polynomials whose frequences are in the set $\{n+1, n+2, \ldots\}$ and $n \geqslant 0$. The author and K. Takahashi [7] got a formula giving the distance $\varrho_n(w, 2)^{1/2}$ from 1 to P_n in $L^2(w\, d\theta/2\pi)$. In this paper we prove $\tau_n(w) = \varrho_n(w^{-1}, 2)^{-1}$ in case $w^{-1} \in L^1(d\theta/2\pi)$. Then a formula giving $\tau_n(w)$ follows from the expression of $\varrho_n(w^{-1}, 2)$. Moreover, we generalize the expression of $\varrho_n(w, 2)^{1/2}$ to

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