[4] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[5] K. Floret, J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lect. Notes in Math. 56, Springer, Berlin–Heidelberg–New York 1968.

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A simple complement to Mikusiński's operational calculus

by

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Abstract. According to Mikusiński's operational calculus, any solution $y = \{y(t)\}$ of the Cauchy problem for the nth order linear ordinary differential equation with complex coefficients and with inhomogeneous term $f = \{f(t)\} \in C[0, \infty)$ must satisfy

$$(*) \qquad (\alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_0) y = f + \beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \ldots + \beta_0,$$

where α_j's and β_k's are complex numbers with $\alpha_n \neq 0$. The entitled complement proves that $y = \{y(t)\}$ given by $(*)$ is n-times continuously differentiable in t so that $y(t)$ is in truth the unique solution of the original Cauchy problem. Cf. the subsequent paper by S. Okamoto (this volume, pp. 99–101).

The entitled "complement" will be stated in § 2. For the sake of the reader's convenience, I shall begin with a brief prerequisite from the operational calculus of J. Mikusiński [1] as exposed in a joint paper [2] of the present author.

§ 1. The prerequisite. Let $\mathscr{C}'$ denote the totality of complex-valued continuous functions defined on $[0, \infty)$. We denote such a function by $\{f(t)\}$ or simply by f, while $f(t)$ means the value at t of the function f. For $f, g \in \mathscr{C}$ and $\alpha, \beta \in K$ (= the complex number field) we define

$$(1) \qquad \alpha f + \beta g = \{\alpha f(t) + \beta g(t)\} \quad \text{and} \quad fg = \Big\{\int_0^t f(t-\tau) g(\tau)\, d\tau\Big\}.$$

Then $\mathscr{C}$ is a *commutative ring* with respect to the above *addition* and *multiplication* over the coefficient field K.

We shall denote by h the *constant function* $\{1\} \in \mathscr{C}$ so that we have

$$(2) \qquad hf = \Big\{\int_0^t f(\tau)\, d\tau\Big\} \quad \text{for} \quad f \in \mathscr{C} \quad \text{and} \quad h^n = \Big\{\frac{t^{n-1}}{(n-1)!}\Big\}.$$

For any integer $n \geqslant 1$ and $f \in \mathscr{C}$, we have, by (2),

$$(3) \qquad h^n f = 0 \quad \text{implies} \quad f = 0,$$

where 0 denotes $\{0\} \in \mathscr{C}$. Therefore we can define the commutative super-ring $\mathscr{C}_H$ of $\mathscr{C}$ by

(4) $$\mathscr{C}_H = \{f/h^n;\ f\in\mathscr{C} \text{ and } n=1,2,\ldots\},$$

where the *equality* means

(5) $$\frac{f}{h^n} = \frac{f'}{h^{n'}} \quad\text{if and only if}\quad f h^{n'} = f' h^n$$

and the *addition* and *multiplication* are defined by

(6) $$\frac{f}{h^n}+\frac{f'}{h^{n'}} = \frac{fh^{n'}+f'h^n}{h^n h^{n'}}, \qquad \frac{f}{h^n}\,\frac{f'}{h^{n'}} = \frac{ff'}{h^n h^{n'}}.$$

We introduce

(7) $$I = \frac{h^n}{h^n}\in\mathscr{C}_H \quad (n=1,2,\ldots) \quad\text{and}\quad s = \frac{h^n}{h^{n+1}}\in\mathscr{C}_H \quad (n=1,2,\ldots)$$

so that we have

(8) $sh = hs = I$, and I is the *multiplicative unit* of the ring $\mathscr{C}_H$.

Then, if both f and its derivative f' belong to $\mathscr{C}$, we have

(9) $$f' = sf - [f(0)], \quad\text{where}\quad [f(0)] = s\{f(0)\},$$

because of the *Newton formula*

(9)′ $$hf' = \Big\{\int_0^t f'(\tau)\,d\tau\Big\} = \{f(t)-f(0)\} = f - \{f(0)\}.$$

Formula (9) is generalized as follows:

If f is n-times continuously differentiable, we have

(9)″ $$f^{(n)} = s^n f - s^{n-1}[f(0)] - \ldots - [f^{(n-1)}(0)].$$

Hereafter, we shall write $f^{(j)}(0)$ for $[f^{(j)}(0)]$ in case there be no confusion of identifying $f^{(j)}(0)$ with $\{f^{(j)}(0)\}$. We have then

PROPOSITION. *For any $\alpha\in K$ and for any positive integer n, we have the result that*

$$(s-[\alpha])^n = (s-\alpha)^n = \frac{(h-[\alpha]h^2)^n}{h^{2n}}\in\mathscr{C}_H$$

admits a uniquely determined multiplicative inverse in $\mathscr{C}_H$ given by

(10) $$\frac{I}{(s-[\alpha])^n} = (s-\alpha)^{-n} = \Big\{\frac{t^{n-1}}{(n-1)!}e^{\alpha t}\Big\}\in\mathscr{C}\subseteq\mathscr{C}_H,$$

because $(s-[\alpha])\{e^{\alpha t}\} = I$ *by* (9).

§ 2. The complement. Consider the following *Cauchy problem* for linear ordinary differential equation with coefficients belonging to K:

(11) $$\begin{gathered} a_n y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = f\in\mathscr{C} \qquad (a_n\neq 0),\\ y(0) = \gamma_1,\quad y'(0) = \gamma_2,\ \ldots,\ y^{(n-1)}(0) = \gamma_{n-1}. \end{gathered}$$

By virtue of (9)″, this problem shall be converted into the equation in $\mathscr{C}_H$:

(11)′ $$\begin{gathered} (a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0)y = f + \beta_{n-1}s^{n-1} + \beta_{n-2}s^{n-2} + \ldots + \beta_0,\\ \beta_m = a_{m+1}\gamma_0 + a_{m+2}\gamma_1 + \ldots + a_n\gamma_{n-m-1} \qquad (m = 0,1,2,\ldots,n-1). \end{gathered}$$

Since the *polynomial ring of polynomials in s with coefficients belonging to K is free from zero factors*, we can define rational functions in s:

(12) $$F_1 = \frac{I}{a_n s^n + \ldots + a_0} \quad\text{and}\quad F_2 = \frac{\beta_{n-1}s^{n-1}+\ldots+\beta_0}{a_n s^n+\ldots+a_0},$$

and obtain their partial fraction decompositions

(12)′ $$F_1 = \sum_j\sum_{k=1}^{m_j} c_{jk}(s-r_j)^{-k} \quad\text{and}\quad F_2 = \sum_j\sum_{k=1}^{m_j} d_{jk}(s-r_j)^{-k},$$

where r_j's are distinct roots of the algebraic polynomial

(13) $$p(z) = a_n z^n + \ldots + a_0 = a_n\prod_j (z-r_j)^{m_j} \qquad \Big(\sum_j m_j = n\Big).$$

By virtue of (10), we have

(12)″ $$F_1 = \sum_j\sum_{k=1}^{m_j} c_{jk}\Big\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\Big\}\in\mathscr{C}, \qquad F_2 = \sum_j\sum_{k=1}^{m_j} d_{jk}\Big\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\Big\}\in\mathscr{C}$$

so that we obtain the solution y of equation (11)′ given by a function of $\mathscr{C}$:

(14) $$\begin{aligned} y &= \frac{I}{p(s)}f + \frac{\beta_{n-1}s^{n-1}+\ldots+\beta_0}{p(s)}\\ &= \sum_j\sum_{k=1}^{m_j} c_{jk}\Big\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\Big\}\{f(t)\} + \sum_j\sum_{k=1}^{m_j} d_{jk}\Big\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\Big\}. \end{aligned}$$

However, *it is not apparent that this function y of t is precisely the solution of* (11), *because it is not apparent that y is n-times continuously differentiable in t.* In fact, if f is not differentiable in t, then, e.g.,

$$w = \{e^{rt}\}\{f(t)\} = \Big\{\int_0^t e^{r(t-\tau)}f(\tau)\,d\tau\Big\}$$

is not twice continuously differentiable.

Our *complement* says that, in spite of the above example, we can prove that *the function y given by* (14) *is n-times continuously differentiable so that it is the unique solution of* (11).

Proof. Multiplying both sides of (11)′ by h^n we obtain

$$a_n y + a_{n-1} hy + \ldots + a_0 h^n y = h^n f + \beta_{n-1} h + \ldots + \beta_0 h^n.$$

Then $F(t) = h^n f + \beta_{n-1} h + \ldots + \beta_0 h^n$ is surely n-times continuously differentiable. Thus, by $y \in \mathscr{C}'$ and by (2), we have the result:

$$y = -a_n^{-1}(a_{n-1} hy + a_{n-2} h^2 y + \ldots + a_0 h^n y) + a_n^{-1}\{F(t)\}$$

is once continuously differentiable and its derivative satisfies

$$(15) \quad y' = -a_n^{-1}(a_{n-1} hy' + a_{n-2} h^2 y' + \ldots + a_0 h^n y') + a_n^{-1}\{F'(t)\} + \text{ a polynomial in } t,$$

because, e.g.,

$$(h^3 y)' = h^2 y = h^2(hy' + y(0)) = h^3 y' + h^2 y(0)$$

by (9). Thus y' given by (15) is continuously differentiable in t and satisfies

$$y'' = -a_n^{-1}(a_{n-1} hy'' + a_{n-2} h^2 y'' + \ldots + a_0 h^n y'') + a_n^{-1}\{F''(t)\} + \text{ a polynomial in } t$$

and so forth.

References

[1] J. Mikusiński, *Sur les fondaments de calcul opératoire*, Studia Math. 11 (1949), 41–70.
[2] K. Yosida and S. Okamoto, *A note on Mikusiński's operational calculus*, Proc. Japan Acad. 56 A (1) (1980), 1–3.

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A remark on Yosida's complement to Mikusiński's operational calculus

by

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Dedicated to Professor J. Mikusiński on his 70th birthday

Abstract. According to the Mikusiński theory of operational calculus, the Cauchy problem for the nth order ordinary differential equation with complex coefficiants and with inhomogeneous term $f \in C[0, \infty)$ is transformed into the operational equation:

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) y = f + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \ldots + c_0.$$

As a complement to the theory, Prof. K. Yosida showed the fact which states that the solution y of the above operational equation is n-times continuously differentiable so that y is the true solution of the original equation. In this paper, a remark on the above complement is made by giving a direct proof.

It is well known, in the Mikusiński theory of operational calculus, that the Cauchy problem:

$$(1) \quad \begin{aligned} & a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = f, \\ & y(0) = b_0, \quad y'(0) = b_1, \ldots, y^{(n-1)}(0) = b_{n-1}, \\ & a_i \in C, \quad i = 0, \ldots, n, \quad b_j \in C, \ j = 0, \ldots, n-1 \quad \text{and} \quad f \in C[0, \infty) \end{aligned}$$

is transformed into the operational equation:

$$(2) \quad \begin{aligned} & (a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) y = f + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \ldots + c_0, \\ & c_m = a_{m+1} b_0 + a_{m+2} b_1 + \ldots + a_n b_{n-m-1}, \quad m = 0, 1, \ldots, n-1, \end{aligned}$$

where $s = 1/h$ $(= 1/\{1\})$ (cf. [1], [2] and [3]). Therefore we have

$$(3) \quad y = \frac{f}{p(s)} + \frac{q(s)}{p(s)}$$

with $p(s) = a_n s^n + \ldots + a_0 = a_n (s - \alpha_1)(s - \alpha_2) \ldots (s - \alpha_n)$

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