in the initial interval of length k_2 and leave intact the blanks in the following k_2-blocks up to $k_3 = n_2 k_2$ $(n_2 > 1)$ where the fraction of the filled positions is less than ε. The next blocks of length k_3 are repeated periodically. We continue this process by induction by filling up the blanks with 1's and 0's at odd and even steps, respectively. The fraction of the filled positions in the initial interval of length k_n is less than ε at the nth step. In the resulting sequence z the initial blocks of lengths $k_1, k_2, \ldots$ are repeated at intervals of lengths $k_2, k_3, \ldots$, respectively. Since $k_n \to \infty$, (i) is clearly satisfied. The density of 1's in the initial block of length k_n is less than ε for n odd and greater than $1-\varepsilon$ for n even, whence (ii) is also satisfied.

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY, WROCŁAW

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Some convergence properties of convolutions

by

KLAUS KELLER (Dortmund)

Abstract. For certain spaces X of test functions the following reversed form of continuity is shown to hold for convolutions: If $T_j \in X'$ and $T_j * \varphi \to 0$ in X' for all $\varphi \in X$, then $T_j \to 0$ in X'. The proofs are based on theorems of Grothendieck and Raikov–e Silva on inductive limit spaces.

Considering the operation of convolution on $\mathscr{S}' \times \mathscr{S}$ ($\mathscr{S}$ the space of rapidly decreasing C^∞-functions, $\mathscr{S}'$ its strong dual), the following property is easily proved: If $T_j \to 0$ in $\mathscr{S}'$, then $T_j * \varphi \to 0$ in $\mathscr{S}'$ for all $\varphi \in \mathscr{S}$. In the sequential approach of Mikusiński [1] and [2] a reversed problem is of interest: If for $T_j \in \mathscr{S}'$ we have $T_j * \varphi \to 0$ in $\mathscr{S}'$ for all $\varphi \in \mathscr{S}$, is it true that $T_j \to 0$ in $\mathscr{S}'$? Similarly does $T_j * \varphi \to 0$ in $\mathscr{D}'$ for all $\varphi \in \mathscr{D}$ imply $T_j \to 0$ in $\mathscr{D}'$? In both cases we infer from

$$(T_j * \varphi, \psi) = (T_j, \varphi^\vee * \psi), \qquad \varphi^\vee(x) := \varphi(-x),$$

(weak) convergence on the subspaces $[\mathscr{S} * \mathscr{S}] \subset \mathscr{S}$ or $[\mathscr{D} * \mathscr{D}] \subset \mathscr{D}$, which are built of finite sums of convolution products. If $[\mathscr{S} * \mathscr{S}]$ ($[\mathscr{D} * \mathscr{D}]$) is equal to $\mathscr{S}$ ($\mathscr{D}$) or at least of second category in $\mathscr{S}$ ($\mathscr{D}$), then (T_j) is certainly bounded and convergent. Performing Fourier transformation we may ask whether $[\mathscr{S} \cdot \mathscr{S}] = \mathscr{S}$ and $[\mathscr{Z} \cdot \mathscr{Z}] = \mathscr{Z}$? Whereas $[\mathscr{Z} \cdot \mathscr{Z}] = \mathscr{Z}$ can be verified by rather deep and lengthy methods [3], a systematic treatment for the case of other test function spaces is not known.*

In this note we give a direct solution for the original problem. It utilizes the following "Theorem B" of Grothendieck [4] and can be generalized to various other test function spaces.

THEOREM B. *Let E be a locally convex Hausdorff space, F and F_i ($i \in N$) Fréchet spaces. Let u be a continuous mapping $F \to E$ and u_i continuous mappings $F_i \to E$. If $u(F) \subset \bigcup_i u_i(F_i)$, then there exists some index i_0 such that $u(F) \subset \bigcup_{i < i_0} u_i(F_i)$.*

This theorem will imply the convergence of $(T_j * \varphi)_j$ on a larger space, so factorizations of type $\psi = \psi_1 * \psi_2$ become possible.

* In the mean time J. Voigt has announced a proof of $\mathscr{S} \cdot \mathscr{S} = \mathscr{S}$.

We first study convergence on $\mathscr{S}*\mathscr{S}$, then on $\mathscr{D}(K')*\mathscr{D}(K)$ and $\mathscr{D}*\mathscr{D}$. Finally the problem is solved for $\mathscr{E}*\mathscr{D}$ where Fourier transformation is not applicable.

THEOREM 1. *Let* $(T_j)_j \subset \mathscr{S}'(\boldsymbol{R}^N)$ *and* $T_j*\varphi\to 0$ *in* $\mathscr{S}'(\boldsymbol{R}^N)$ *for all* $\varphi \in S(\boldsymbol{R}^N)$. *Then* $T_j\to 0$ *in* $\mathscr{S}'(\boldsymbol{R}^N)$.

Proof. We consider the equivalent problem $T_j\cdot\varphi\to 0$ in $\mathscr{S}'$. If we have succeeded in showing that $(T_j)_j$ is bounded in $\mathscr{S}'$, the theorem is proved: Suppose $T_j \not\to 0$ in the topology of $\mathscr{S}'$. Then for some $\varphi\in\mathscr{S}$ and some $\varepsilon>0$ we may select a subsequence $(T_{n_j})_j$ with $|(T_{n_j},\varphi)|>\varepsilon$. Since $\mathscr{S}'$ is a Montel space, we can find a subsequence $(T_{n_j'})_j$ of $(T_{n_j})_j$ with $T_{n_j'}\to T$ in $\mathscr{S}'$. But for all $\psi\in D$ we have

$$(T_{n_j'},\psi)=(T_{n_j'},\chi\cdot\psi)=(T_{n_j'}\cdot\chi,\psi)\to 0$$

choosing $\chi\in\mathscr{D}$ with $\chi\equiv 1$ in a neighbourhood of $\operatorname{supp}\psi$. Now the density of $\mathscr{D}$ in $\mathscr{S}$ implies $T=0$ which contradicts $|(T_{n_j'},\varphi)|>\varepsilon$.

So let us assume that (T_j) is unbounded in $\mathscr{S}'$. Then there exists some $\varphi_0\in\mathscr{S}$ such that $(T_j,\varphi_0)_j$ is unbounded. Passing over to a suitable subsequence which again will be denoted by (T_j) we attain $|(T_j,\varphi_0)|>j$. Next we introduce for $k\in\boldsymbol{N}$ the Banach spaces

$$E_k:=\Big\{a\equiv(a_{jq})_{jq}\ \Big|\ j\in\boldsymbol{N},\ q\in\boldsymbol{Z}^N,\ a_{jq}\in\boldsymbol{C},\quad \|a\|_k:=\sup_{j,q}\big|a_{jq}(1+|q|)^{-k}\big(\ln(j+1)\big)^{-k}\big|<\infty\Big\}.$$

As $E_k\subset E_{k+1}$ we can define $E:=\operatorname{limind}E_k$ which is a locally convex Hausdorff space. Moreover, it is an (LS)-space [5] since all embeddings are compact. This can be derived from the fact that the embedding $l^\infty(c)\to l^\infty(c')$ is compact if $\lim_{n\to\infty}c_n'/c_n=0$ holds true for the weight factors $c=(c_n)_n$, $c'=(c_n')$.

Now let a positive $e\in C^\infty$ be fixed with $e(x)=1$ for $|x|\leqslant 1$ and $\operatorname{supp}e\subset\{x\mid |x|<3/2\}$. Further, let $e_q(x):=e(x+q)$ for all $q\in\boldsymbol{Z}^N$. We want to show that by $\varphi\mapsto(a_{jq})_{jq}=a$, $a_{jq}:=(T_je_q,\varphi)$, a continuous mapping $u\colon\mathscr{S}\to E$ is defined.

First, since $T_j\cdot\varphi\to 0$ in $\mathscr{S}'$ implies boundedness in some $(\mathscr{S}^k)'$, we get

$$\begin{aligned}|a_{jq}|&=|(T_j\cdot\varphi,e_q)|\leqslant\|T_j\cdot\varphi\|_{-k}\cdot\|e_q\|_k\\&\leqslant A\cdot\sup_{x,|\alpha|\leqslant k}|(1+|x|)^kD^\alpha e_q(x)|\\&\leqslant A\cdot\sup_{x,|\alpha|\leqslant k}|(1+|x-q|)^k|\,|e^{(\alpha)}(x)|\leqslant A'\cdot(1+|q|)^k.\end{aligned}$$

Hence $\|a\|_k<\infty$. It remains to verify the continuity of u. As E is an LF-space we may invoke the graph theorem: Let $\varphi_v\to 0$ in $\mathscr{S}$ and $u(\varphi_v)\to a^*$ in E. Since E is an (LS)-space, there exists some k such that $\{u(\varphi_v)\}\subset E_k$ and $u(\varphi_v)\to a^*$ in the topology of E_k. On the other hand, $a_{jq}(\varphi_v)=(T_je_q,\varphi_v)$

$\to 0$ for all j,q because of $T_je_q\in\mathscr{S}'$. As pointwise convergence is weaker than convergence in the norm of E_k, we come to the conclusion $a^*=0$.

Now Grothendieck's theorem tells that $u(\mathscr{S})\subset E_k$ for some $k\in\boldsymbol{N}$. Let $k':=k+N+1$ and

$$f:=\sum_{q\in\boldsymbol{Z}^N}(1+|q|)^{-k'}e_q.$$

Clearly $f\in C^\infty$. Moreover, from

$$|f^{(\alpha)}(x)|\leqslant\sum_q(1+|q|)^{-k'}|e_q^{(\alpha)}(x)|\leqslant C_N\sup_x|e^{(\alpha)}(x)|\leqslant C_N'$$

we know $f\in\mathscr{O}_M$. Because of

$$|f(x)|\geqslant(1+|q_0|)^{-k'}|e_{q_0}|\geqslant(2+|x|)^{-k'}$$

where q_0 satisfies $|x+q_0|\leqslant 1$ and

$$(1/f)^{(\alpha)}=\sum_{\substack{\alpha_1\ldots\alpha_r\\ r\leqslant|\alpha|}}C_{\alpha_1\ldots\alpha_r}(1/f)^{r+1}\underbrace{f^{(\alpha_1)}\ldots f^{(\alpha_r)}}_{\in\mathscr{O}_M}\qquad\Big(\sum_r\alpha_r=\alpha\Big),$$

$1/f\in\mathscr{O}_M$ must be true, too.

We next prove that $\{(T_jf)(\ln(j+1))^{-k}\mid j\in\boldsymbol{N}\}$ is bounded in $\mathscr{S}'$: Indeed, for any $\varphi\in\mathscr{S}$ one has

$$\begin{aligned}\Big|\big((\ln(j+1))^{-k}T_j\cdot f,\varphi\big)\Big|&\leqslant(\ln(j+1))^{-k}|(T_j,f\cdot\varphi)|\\&\leqslant\sum_q(1+|q|)^{-k'}(\ln(j+1))^{-k}\cdot|(T_je_q,\varphi)|\\&\leqslant\|u(\varphi)\|_k\cdot\sum_q(1+|q|)^{-N-1}\leqslant C\|u(\varphi)\|_k.\end{aligned}$$

But this leads to a contradiction:

$$\begin{aligned}\Big|\big((\ln(j+1))^{-k}T_j,\varphi_0\big)\Big|&=\Big|\big((\ln(j+1))^{-k}T_j,f\cdot(1/f)\varphi_0\big)\Big|\\&=\Big|\big((\ln(j+1))^{-k}T_j\cdot f,\underbrace{(1/f)\varphi_0}_{\in\mathscr{S}}\big)\Big|,\end{aligned}$$

thus

$$|(T_j,\varphi_0)|\leqslant C(\ln(j+1))^k.\ ∎$$

A direct application of Grothendieck's theorem to the case $\mathscr{D}*\mathscr{D}$ is impossible since $\mathscr{D}$ is not a Fréchet space. However we can handle this problem by passing over to the Fréchet space $\mathscr{D}(K)$:

THEOREM 2. *Let* $(T_j)\subset\mathscr{D}'$, $T_j*\varphi\to 0$ *in* $\mathscr{D}'(K')$ *for all* $\varphi\in\mathscr{D}(K)$. *Then* $T_j\to 0$ *in* $\mathscr{D}'(\mathring{K}'-\mathring{K})$, $\mathring{K}'$, $\mathring{K}$ *being the interior of the compacta* K', $K\subset\boldsymbol{R}^N$.

Proof. Suppose we can show that for each $x_0 \in \mathring{K}' - \mathring{K}$ there exists some neighbourhood $U(x_0)$ such that $\{(T_j, \varphi) |\ j \in N\}$ is bounded for all $\varphi \in \mathscr{D}(U(x_0))$, then—by using a partition of unity—$(T_j)_j$ is bounded on $\mathscr{D}(\mathring{K}' - \mathring{K})$. In like manner as in the proof of Theorem 1 this property implies $T_j \to 0$ in $\mathscr{D}'(\mathring{K}' - \mathring{K})$:

Let $T_j \to T$ in $\mathscr{D}'(\mathring{K}' - \mathring{K})$. Then $(T*\varphi, \psi) = \lim (T_j, \psi * \varphi^\vee) = 0$ for all $\varphi \in \mathscr{D}(K)$ and $\psi \in \mathscr{D}(\mathring{K}')$. As $T*\varphi \in C^\infty$, it follows that $(T*\varphi)(y) = 0$ for all $y \in \mathring{K}'$. Hence $(T, \tilde{\varphi}) = 0$ for all $\tilde{\varphi} \in \mathscr{D}(y - K)$.

Thus let $x_0 = y' - y$ $(y' \in \mathring{K}', y \in \mathring{K})$ be a point where our assumption does not hold. We choose some (open) neighbourhood $U(x_0)$ in $\mathring{K}' - \mathring{K}$ satisfying $y' - U(x_0) \subset \mathring{K}$ and $y + U(x_0) \subset \mathring{K}'$. Fixing some closed cube $Q = x_0 + [-L, L]^N \subset U(x_0)$ we can find an element $\psi_0 \in \mathscr{D}(Q)$ such that $\{(T_j, \psi_0)\}$ is unbounded. By passing over to a suitable subsequence we may again assume $|(T_j, \psi_0)| > j$. Let E_k and E be as before. For fixed $e_0 \in \mathscr{D}(y + U(x_0))$ with $e_0 = 1$ on $y + Q$ and $e_q(x) := e_0 \cdot \exp i\pi q x/L$ $(q \in Z^N)$ we consider the assignment

$$\varphi \mapsto a_{jq}(\varphi) := (T_j * \varphi, e_q), \qquad \varphi \in \mathscr{D}(K).$$

Because of the boundedness of $T_j * \varphi$ in some $(\mathscr{D}^m(K'))'$ we get the estimate

$$|a_{jq}(\varphi)| \leqslant \|T_j * \varphi\|_{(\mathscr{D}^m(K'))'} \cdot \|e_q\|_{\mathscr{D}^m(K')}$$
$$\leqslant A \sup_{\substack{x \in K' \\ |\alpha| \leqslant m}} |D^\alpha (e_0 \cdot \exp i\pi q x/L)| \leqslant A' \cdot (1 + |q|)^m$$

and so $(a_{jq}(\varphi))_{jq} \in E$.

The closed graph theorem combined with the continuity of $(T_j * e_q^\vee, \varphi^\vee)$ in $\varphi \in \mathscr{D}(K)$ implies that $u: \varphi \mapsto u(\varphi) := (a_{jq}(\varphi))_{jq}$ is a continuous mapping $\mathscr{D}(K) \to E$.

Now Grothendieck's theorem tells that $u(\mathscr{D}(K)) \subset E_k$ for some k. From this fact will follow that $\{(\ln(j+1))^{-k} T_j * \varphi |\ j \in N\}$ is bounded on $\mathscr{D}^{k'}(y + Q)$ for $k' := k + N + 1$. Indeed, by utilizing the Fourier expansion,

$$\psi(x) = e_0(x) \sum_q c_q \exp i\pi q x/L = \sum_q c_q e_q(x)$$

for $\psi \in \mathscr{D}^{k'}(y + Q)$ where

$$q^\alpha |c_q| \leqslant d_\alpha \int_{y+Q} d^N x\, |\psi^{(\alpha)}(x)| \leqslant M_\alpha$$

holds for $|\alpha| \leqslant k'$, we obtain

$$\big|(\ln(j+1))^{-k} (T_j * \varphi, \psi)\big| = (\ln(j+1))^{-k} \Big|\Big(T_j * \varphi, \sum c_q e_q\Big)\Big|$$
$$\leqslant (\ln(j+1))^{-k} \sum_q |c_q|\ |(T_j * \varphi, e_q)|$$
$$\leqslant \sum_q |c_q|\ |a_{jq}(\varphi)|\ (\ln(j+1))^{-k} \leqslant C \sum_q |c_q| (1 + |q|)^k$$
$$\leqslant C \cdot \sum_q (1 + |q|)^{-N-1} < C'.$$

Now it is not difficult to produce a contradiction:

$$(T_j, \psi_0) = (T_j, \psi_0 * \delta) = \big(T_j, \psi_0 * (\chi P(D) G)\big)$$
$$= \big(T_j, \tau_{y'} \psi_0 * \tau_{-y'} (\chi P(D) G)\big).$$

Here $(\tau_h f)(x) := f(x+h)$ and χ is any element in $\mathscr{D}([-L/2, L/2]^N)$ with $\chi(0) = 1$. By $G \in C^{k'}$ we denote Green's function

$$G(x) := (2\pi)^{-N} \int e^{-ix\xi} (1 + \xi^2)^{-k'-N} d\xi$$

for $P(D) := (1 - \Delta_N)^{k'+N}$.

Exploiting the generalized Leibniz rule

$$f P(D) g = \sum_{\alpha < \infty} \frac{1}{\alpha!} D^\alpha \big(g P^{(\alpha)}(-D) f\big)$$

we arrive at

$$(T_j, \psi_0) = \Big(T_j, \tau_{y'} \psi_0 * \tau_{-y'} \sum_\alpha \frac{1}{\alpha!} D^\alpha \big(G \cdot P^{(\alpha)}(-D) \chi\big)\Big)$$
$$= \sum_\alpha \frac{1}{\alpha!} \big(T_j, \tau_{y'} \psi_0^{(\alpha)} * \tau_{-y'} (G \cdot P^{(\alpha)}(-D) \chi)\big)$$
$$= \sum_\alpha \frac{1}{\alpha!} (T_j * \varphi_\alpha, \psi_\alpha)$$

with

$$\varphi_\alpha := (\tau_{y'} \psi_0^{(\alpha)})^\vee \in \mathscr{D}(y' - Q) \subset D(\mathring{K})$$

and

$$\psi_\alpha := \tau_{-y'} \big(G P^{(\alpha)}(-D)_\chi\big) \in \mathscr{D}^{k'}(y' + [-L/2, L/2]^N) \subset \mathscr{D}^{k'}(y + Q).$$

The boundedness of $(T_j * \varphi_a)\big((\ln(j+1))^{-k}\big)_j$ on $\mathcal{D}^{k'}(y+Q)$ then implies

$$|(T_j, \psi_0)| \leqslant M(\ln(j+1))^k. \blacksquare$$

By the definition of $\mathcal{D}(\Omega), \mathcal{D}'(\Omega')$ for open sets $\Omega, \Omega' \subset \mathbf{R}^N$ the next theorem is a consequence of Theorem 2:

THEOREM 3. *Let* $(T_j) \subset \mathcal{D}'$, $T_j * \varphi \to 0$ *in* $\mathcal{D}'(\Omega')$ *for all* $\varphi \in \mathcal{D}(\Omega)$. *Then* $T_j \to 0$ *in* $\mathcal{D}'(\Omega' - \Omega)$.

Proof. As weak convergence agrees with strong convergence in $\mathcal{D}'$ for sequences it suffices to show that $(T_j, \psi) \to 0$ for all $\psi \in \mathcal{D}(\Omega' - \Omega)$. But since $\operatorname{supp}\psi \Subset \Omega' - \Omega$, we also have $\operatorname{supp}\psi \subset K' - \overset{\circ}{K}$ for some suitable compact sets $K' \subset \Omega'$, $K \subset \Omega$. Now the preceding theorem applies. ■

Again in a different manner Grothendieck's theorem is used in the proof of our last theorem.

THEOREM 4. *Let* $T_j \in \mathcal{E}'$. *If* $T_j * \varphi \to 0$ *in* $\mathcal{E}'$ (*or* $\mathcal{D}'$) *for all* $\varphi \in \mathcal{D}$ (*or* $\varphi \in \mathcal{E}$), *then* $T_j \to 0$ *in* $\mathcal{E}'$.

Proof. In virtue of $(T_j * \varphi, \psi) = (T_j * \psi^{\vee}, \varphi^{\vee})$, the second assertion follows from the first. To prove the first we note that by Theorem 3 $T_j \to 0$ is surely true in the topology of $\mathcal{D}'$. Thus it suffices to show that $\bigcup_j \operatorname{supp} T_j$ is a compact set. Let $\chi \in \mathcal{D}(|x| < 3)$ be identical one for $|x| \leqslant 2$ and let further functions $\chi_{pq} \in \mathcal{D}$ be defined for $p, q \in \mathbf{Z}^N$ by

$$\chi_{pq}(x) := \chi(p+x)\exp i\pi qx/2.$$

We now introduce the Banach spaces

$$\hat{E}_k := \{(a_{jpq})_{jpq} \equiv a \mid a_{jpq} \in \mathbf{C},\ j \in \mathbf{N},\ p, q \in \mathbf{Z}^N,\ a_{jpq} = 0 \text{ if } |p| \geqslant k,$$
$$|a|_k := \sup_{j,p,q} |a_{jpq}|(1+|q|)^{-k}(\ln(j+1))^{-k} < \infty\}.$$

As the embedding $\pi_k \colon \hat{E}_k \to \hat{E}_{k+1}$ may be regarded as the embedding of finitely many copies of E_k into finitely many copies of E_{k+1}, the mapping π_k is compact according to foregoing arguments. Therefore $\hat{E} := \operatorname{limind} \hat{E}_k$ is a separated (LS)-space. Now let $a_{jpq}(\varphi) := (T_j * \varphi, \chi_{pq})$ for $\varphi \in \mathcal{D}(B)$ with the ball $B := \{x \mid |x| \leqslant 1\} = -B$. We want to show that by $\varphi \mapsto (a_{jpq}(\varphi))_{jpq}$ a continuous mapping $u \colon \mathcal{D}(B) \to \hat{E}$ is defined:

The family $\{T_j * \varphi \mid j \in \mathbf{N}\}$ being bounded in $\mathcal{E}'$, a compact set K_φ exists such that

$$\bigcup_j \operatorname{supp}(T_j * \varphi) \subset K_\varphi.$$

Consequently $a_{jpq}(\varphi) = 0$ for all j, q if $|p|$ is sufficiently large. Furthermore,

$$\sup_j |a_{jpq}(\varphi)| \leqslant \sup_j \|T_j * \varphi\|_{(\mathcal{E}^r(K_\varphi))'} \cdot \|\chi_{pq}\|_{\mathcal{E}^r(K_\varphi)}$$
$$\leqslant C_\varphi \sup_{\substack{x \in K_\varphi \\ |\alpha| \leqslant r}} |D^\alpha \chi_{pq}(x)| \leqslant C'_\varphi \sup_{\substack{x \in K_\varphi \\ |\alpha| \leqslant r}} |\chi^{(\alpha)}(x)| \cdot \sup_{|\alpha| \leqslant r} |q|^\alpha$$
$$\leqslant C''_\varphi (1+|q|)^r.$$

Therefore $u(\varphi) \subset \hat{E}$. Continuity is proved by using again the closed graph theorem and the fact that for fixed j, p, q the expression $a_{jpq}(\varphi) = (T_j * * \chi_{pq}^{\vee}, \varphi^{\vee})$ is continuous in $\varphi \in \mathcal{D}(B)$. Grothendieck's theorem now implies $u(\mathcal{D}(B)) \subset \hat{E}_k$ for some k, thus $(T_j * \varphi, \chi_{pq}) = 0$ for all j, q and $\varphi \in \mathcal{D}(B)$ provided $|p| > k$ is the case.

Since any function in $\mathcal{D}(p+Q)$, $Q := \{x \mid |x| \leqslant 3/2\}$ can be approximated in the topology of $\mathcal{D}$ by terms of the kind $\sum_{q<\infty} c_q \chi_{pq}$, we get $(T_j * \varphi, \psi) = 0$ for all $\psi \in \mathcal{D}(p+Q)$ with $|p| > k$. But then

$$\operatorname{supp}(T_j * \varphi) \subset K' := \mathbf{R}^N \setminus \bigcup_{|p|>k} (p+Q).$$

Because of $T_j * \varphi \in C^\infty$ this inclusion implies that $(T_j * \varphi)(y) = (T_j, \tau_{-y}\varphi^{\vee})$ vanishes for all $\varphi \in \mathcal{D}(B)$ if $y \notin K'$. Hence $(T_j, \psi) = (T_j, \tau_{-y}(\tau_y \psi)) = 0$ for all $\psi \in \mathcal{D}(y+B)$ with $y \notin K'$. But this means

$$\operatorname{supp} T_j \subset K' \quad \text{for all } j \in \mathbf{N}. \blacksquare$$

It is clear that the methods of this paper may be carried over to other spaces of distributions (e.g., $K[M_p]$-spaces or S_α^β-spaces).

Finally let us point out that Theorems 1 to 4 are extendable to nets: $T_\alpha * \varphi \to 0$ implies $T_\alpha \to 0$. Indeed, if $T_\alpha \to 0$ would not hold true, we could find a zero neighbourhood V and an infinite sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ such that $T_{\alpha_r} \notin V$, and $T_{\alpha_r} * \varphi \to 0$. But since $T_{\alpha_r} * \varphi \to 0$, our theorems for sequences imply $T_{\alpha_r} \to 0$.

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MATHEMATISCHES INSTITUTE, LEHRSTUHL I
UNIVERSITÄT DORTMUND
D-4600 Dortmund, FRD

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A simple complement to Mikusiński's operational calculus

by

KÔSAKU YOSIDA (Tokyo)

Abstract. According to Mikusiński's operational calculus, any solution $y = \{y(t)\}$ of the Cauchy problem for the nth order linear ordinary differential equation with complex coefficients and with inhomogeneous term $f = \{f(t)\} \in C[0, \infty)$ must satisfy

(*) $$(\alpha_n s^n + \alpha_{n-1} s^{n-1} + \ldots + \alpha_0) y = f + \beta_{n-1} s^{n-1} + \beta_{n-2} s^{n-2} + \ldots + \beta_0,$$

where α_j's and β_k's are complex numbers with $\alpha_n \neq 0$. The entitled complement proves that $y = \{y(t)\}$ given by (*) is n-times continuously differentiable in t so that $y(t)$ is in truth the unique solution of the original Cauchy problem. Cf. the subsequent paper by S. Okamoto (this volume, pp. 99–101).

The entitled "complement" will be stated in § 2. For the sake of the reader's convenience, I shall begin with a brief prerequisite from the operational calculus of J. Mikusiński [1] as exposed in a joint paper [2] of the present author.

§ 1. The prerequisite. Let $\mathscr{C}'$ denote the totality of complex-valued continuous functions defined on $[0, \infty)$. We denote such a function by $\{f(t)\}$ or simply by f, while $f(t)$ means the value at t of the function f. For $f, g \in \mathscr{C}$ and $\alpha, \beta \in K$ (= the complex number field) we define

(1) $$\alpha f + \beta g = \{\alpha f(t) + \beta g(t)\} \quad \text{and} \quad fg = \Big\{\int_0^t f(t-\tau) g(\tau)\, d\tau\Big\}.$$

Then $\mathscr{C}$ is a *commutative ring* with respect to the above *addition* and *multiplication* over the coefficient field K.

We shall denote by h the *constant function* $\{1\} \in \mathscr{C}$ so that we have

(2) $$hf = \Big\{\int_0^t f(\tau)\, d\tau\Big\} \quad \text{for} \quad f \in \mathscr{C} \quad \text{and} \quad h^n = \Big\{\frac{t^{n-1}}{(n-1)!}\Big\}.$$

For any integer $n \geqslant 1$ and $f \in \mathscr{C}$, we have, by (2),

(3) $$h^n f = 0 \quad \text{implies} \quad f = 0,$$

where 0 denotes $\{0\} \in \mathscr{C}$. Therefore we can define the commutative superring $\mathscr{C}_H$ of $\mathscr{C}$ by