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Unique ergodicity of irreducible Markov operators on $C(X)$

by

ANZELM IWANIK (Wrocław)

To Professor J. Mikusiński on the 70th birthday

Abstract. An irreducible Markov operator on $C(X)$ (X compact Hausdorff) is uniquely ergodic iff there exists a sequence R_n of affine combinations of iterates of T such that R_n' converge for the weak* operator topology to an operator Q with $QT' = Q$ (a semigroup zero). Other "weak" conditions implying unique ergodicity are also found.

1. Introduction. Let X be a compact Hausdorff space. A linear operator T acting on $C(X)$, the space of real-valued continuous functions, is said to be *Markov* if $T1 = 1$ and $Tf \geqslant 0$ whenever $f \geqslant 0$. A Borel subset B of X is *invariant* if $T'\delta_x(B) = 1$ for every $x \in B$. T is called *irreducible* if there are no proper closed invariant sets.

Schaefer [10] proved that an irreducible Markov operator is *uniquely ergodic*, i.e., there is a unique invariant probability (Radon) measure for T' iff the Cesaro means

$$A_n f = n^{-1}(f + Tf + \ldots + T^{n-1}f)$$

converge uniformly for every $f \in C(X)$ (see also Sine [11]). The strong operator convergence of A_n can be replaced by convergence of other averages and in fact is equivalent to the existence of an operator Q_0 satisfying $TQ_0 = Q_0$ and contained in the strong operator closed convex hull of the cyclic semigroup $\{I, T, T^2, \ldots\} \subset L(C(X))$. It was shown in [3] that this assumption can also be replaced by pointwise convergence of $A_n f$ for every $f \in C(X)$ (this extends an earlier result of Oxtoby [6] on continuous transformations). In this paper the assumption is further relaxed by allowing more general averaging methods (Theorem 1).

For certain compact spaces all irreducible Markov operators are uniquely ergodic. This is clearly the case for finite spaces. Ando ([1], Proposition 2) proved that if X is σ-Stonian and there exists an order continuous probability measure on X then every irreducible Markov operator on $C(X)$ is uniquely ergodic (more recently Lotz [5] proved that the Cesaro means A_n must then converge in the operator norm). In Section 3 we prove that unique ergodicity occurs whenever $A_n f$ converge on sufficiently large sets (Theorem 2). This is in particular the case if X has isolated points.

An example of a deterministic irreducible Markov operator with many invariant measures was given by Raimi [7]. The example relies on topological properties (discovered by Rudin [9]) of the space $\beta N \setminus N$. It was subsequently extended by Ando ([1], Proposition 1 and Theorem 1) to more general σ-Stonian spaces and by Lotz ([5], Example 5 and Corollary 4). In all these examples the underlying space is nonmetrizable. In Section 4 we observe that the familiar unilateral shift on 0-1 sequences restricted to certain invariant sets may result in non-uniquely ergodic irreducible systems. A similar example for the bilateral shift can be found in [6], § 10.

2. Weak* mean ergodicity. Let T be a Markov operator on $C(X)$. There exists a Borel measurable (Radon) transition probability $P(x, B)$ such that

$$Tf(x) = \int f(y)P(x, dy)$$

for every $f \in C(X)$ (see Rosenblatt [8]). The same formula extends T to an operator acting on all bounded Borel functions. By the Ionescu Tulcea theorem, to every initial probability μ on X corresponds a unique probability measure P_μ on the space of trajectories X^N defining the canonical Markov process ξ_n.

The following proof is a modified version of the argument in [3], Theorem 5.

LEMMA. *Suppose we are given an irreducible Markov operator T on $C(X)$, an invariant Borel set B, and a bounded Borel function h. If the equality $Th(x) = h(x)$ holds on B and if $h|B$ has a continuity point x_0 then h is constant on B.*

Proof. Let $x \in B$. The trajectory ξ_n stays forever in B P_x-almost surely. Since the restricted function $h|B$ is invariant (for the restricted transition probability), it follows that $h(\xi_n)$ is a bounded martingale and converges P_x-almost surely to a bounded random variable g. We have

$$|h(\xi_n) - h(x_0)| < \varepsilon$$

nfinitely often with probability one for every $\varepsilon > 0$ since, by irreducibility, every neighbourhood of x_0 is visited infinitely many times (Lemma 2 in [3]; metrizability is unnecessary for this assertion). Therefore $g = h(x_0)$ holds almost surely and we have $h(x_0) = \int g\,dP_x = \int h(\xi_0)\,dP_x = h(x)$ by the martingale property. This clearly implies $h = \text{const}$ on B.

The next result extends the corollary in [3] and should be compared with [6], (5.3).

THEOREM 1. *Let T be an irreducible Markov operator on $C(X)$. Then the following conditions are equivalent:*

(1) *T is uniquely ergodic.*

(2) *$A_n f$ converge uniformly for every $f \in C(X)$.*

(3) *$A_n f$ converge pointwise for every $f \in C(X)$.*

(4) *There exists an infinite matrix (a_{ni}) $(n \geqslant 1, i \geqslant 0)$ satisfying $a_{n0} \geqslant a_{n1} \geqslant \ldots \geqslant 0$, $\sum_i a_{ni} = 1$, and $\lim_n a_{n0} = 0$ such that the averages $\sum_i a_{ni}T^i f$ converge pointwise for every $f \in C(X)$.*

(5) *There exists a sequence R_n of affine combinations of iterates of T such that R_n' converge for the weak* operator topology in $L(C(X)')$ to an operator Q such that $QT' = Q$.*

Proof. (1) ⇒ (2) is well known (see e.g. [11]) and (2) ⇒ (3) ⇒ (4) are trivial. (4) ⇒ (5) follows from the Lebesgue dominated convergence theorem and from the following observation:

$$\Big\|(I-T)\sum_i a_{ni}T^i\Big\| \leqslant a_{n0} + \sum_i (a_{ni} - a_{n,i+1}) = 2a_{n0} \to 0.$$

We prove (5) ⇒ (1). Suppose $Q = \lim R_n'$ where R_n are affine combinations of $I, T, T^2, \ldots$ For every $f \in C(X)$ and $x \in X$ the sequence $R_n f(x) = (f, R_n'\delta_x)$ converges to $\bar f(x) = (f, Q\delta_x)$. The function $\bar f$ is bounded by the Uniform Boundedness Principle and is T-invariant since $QT' = Q$. As a limit of a sequence of continuous functions, $\bar f$ has a continuity point. By the Lemma, $\bar f$ must be a constant function. To prove unique ergodicity assume μ and ν are invariant probabilities. The Lebesgue dominated convergence theorem implies

$$(f, \mu) = (R_n f, \mu) = (\bar f, \mu) = (\bar f, \nu) = (f, \nu),$$

which implies $\mu = \nu$ because f was arbitrary in $C(X)$.

3. Points of convergence. If X contains isolated points then every irreducible Markov operator is uniquely ergodic. Indeed, if μ and ν are extreme (hence ergodic) invariant probabilities then, by irreducibility, every isolated point x_0 is contained in both $\operatorname{supp}\mu$ and $\operatorname{sup}\nu$ whence $\mu(\{x_0\}) > 0$ and $\nu(\{x_0\}) > 0$. This implies $\mu \wedge \nu \neq 0$, and so $\mu = \nu$ since ergodic measures form an orthogonal family (this fact is well known and follows from the individual ergodic theorem). Now invoke the Krein–Milman theorem to obtain unique ergodicity.

The scope of the above remark is rather limited, because for deterministic irreducible Markov operators isolated points only exist in the case of finite X. Indeed, assume that an irreducible operator is given by $T_\varphi f(x)$

$=f\varphi(x)$ where $\varphi\colon X\to X$ is continuous. If x_0 is an isolated point then letting $X_0=\{x_0\}$ and $X_n=\varphi^{-1}(X_{n-1})$ for $n\geqslant 1$ we obtain

$$Y=\bigcap_k\Big(\bigcup_{n\geqslant k}X_n\Big)^-\neq\emptyset$$

and $\varphi(Y)\subset Y$. This implies $Y=X$ and $x_0\in X_n$ for some $n\geqslant 1$. It follows that the finite invariant set $\{x_0,\varphi(x_0),\ldots,\varphi^{n-1}(x_0)\}$ coincides with X.

The following result extends the remark to isolated points and gives another equivalent condition for unique ergodicity.

THEOREM 2. *Let T be an irreducible Markov operator on $C(X)$. Then T is uniquely ergodic iff*

(6) *For every $f\in C(X)$ there exists an open set $V\neq\emptyset$ such that $\lim A_nf(x)=\bar f(x)$ exists on V.*

Proof. It suffices to prove the "if" part. The proof will be probabilistic. Fix $f\in C(X)$ and $x_0\in X$. By Lemma 2 in [3], the trajectory ξ_n visits V P_{x_0}-almost surely. Let τ denote the Markov time of the first visit to V after time 0. We have $\tau<\infty$ almost surely and

$$n^{-1}\big(f(\xi_0)+\ldots+f(\xi_{n-1})\big)=n^{-1}\big(f(\xi_\tau)+\ldots+f(\xi_{\tau+n-1})\big)+$$
$$+n^{-1}\big(f(\xi_0)+\ldots+f(\xi_{\tau-1})-f(\xi_n)-\ldots-f(\xi_{n+\tau-1})\big)$$

for $n\geqslant 1$. The second term on the right is bounded (since the first is bounded and so is the left-hand side) and converges to zero as $n\to\infty$. By the strong Markov property we get

$$E_{x_0}n^{-1}\big(f(\xi_\tau)+\ldots+f(\xi_{\tau+n-1})\big)=E_{x_0}E_{\xi_\tau}n^{-1}\big(f(\xi_0)+\ldots+f(\xi_{n-1})\big)$$
$$=E_{x_0}A_nf(\xi_\tau)\to E_{x_0}\bar f(\xi_\tau)$$

because $\xi_\tau\in V$. Therefore

$$A_nf(x_0)=E_{x_0}n^{-1}\big(f(\xi_0)+\ldots+f(\xi_{n-1})\big)\to E_{x_0}\bar f(\xi_\tau)$$

by the Lebesgue dominated convergence theorem. Since x_0 was arbitrary, A_nf converges pointwise. Now apply Theorem 1.

The next theorem uses the lemma in its full strength and gives one more equivalent condition for unique ergodicity. Note that for every $f\in C(X)$ the set of all points x for which the limit of $A_nf(x)$ exists is an $F_{\sigma\delta}$ (by the Cauchy condition) and of invariant measure one (by the individual ergodic theorem).

THEOREM 3. *Let T be an irreducible Markov operator on $C(X)$. Then T is uniquely ergodic iff*

(7) *For every $f\in C(X)$ there exists a G_δ set G_0 of invariant measure one such that $A_nf(x)$ converges on G_0.*

Proof. It suffices to show unique ergodicity assuming (7). There exists a decreasing sequence of open sets V_n such that $\lim\chi_{V_n}=\chi_{G_0}$.

Since T is given by a transition probability, we have $T\chi_{V_n}\downarrow T\chi_{G_0}$. Therefore

$$\{x\colon T\chi_{G_0}(x)=1\}=\bigcap_n\bigcap_k\{x\colon T\chi_{V_n}(x)>1-1/k\}$$

is a G_δ set (the functions $T\chi_{V_n}$ are lower semi-continuous since the V_n are open and the transition probabilities are Radon measures). This implies that the set

$$G_1=\{x\in G_0\colon T\chi_{G_0}=1\}$$

and similarly the sets

$$G_n=\{x\in G_{n-1}\colon T\chi_{G_{n-1}}=1\}$$

for $n>1$ are G_δ's of invariant measure one. Clearly

$$G=\bigcap_n G_n$$

is a G_δ set of invariant measure one. It is easy to see that G is invariante By assumption the function $\bar f=\lim A_nf$ is well defined on G and is of the first Baire class. The space G is topologically complete, hence $\bar f$ has a continuity point. By the lemma, $\bar f$ must be constant on G. For any two invariant probabilities μ and ν we have

$$(f,\mu)=\int_G\bar f\,d\mu=\int_G\bar f\,d\nu=(f,\nu).$$

This implies that T is uniquely ergodic.

It should be noted that instead of Cesaro means in (6) and (7) other means can be used provided they satisfy the conditions of (a_{nt}) in (4).

4. Example. We present an example of a non-uniquely ergodic irreducible component of the unilateral shift φ defined on the Cantor set $\{0,1\}^{\mathbf{N}}$.

We shall construct a 0-1 sequence z such that

(i) every initial block is repeated at equal intervals,

(ii) the density of 1's does not exist.

By (i) the set $X=\{\varphi^n(z)\colon n\geqslant 0\}^-$ will be minimal invariant (see Proposition 2.4 in Furstenberg and Weiss [2]). The second condition implies that T_φ is not uniquely ergodic since $A_nf(z)$ diverges for $f(x_1,x_2,\ldots)\equiv x_1$.

Let $0<\varepsilon<1/2$. In the first step of the construction we place 1's at the positions nk_1+1 $(n\geqslant 0)$ where $1/k_1<\varepsilon$. All the remaining positions are left blank. In the second step we fill up the blanks $2,\ldots,k_1$ with 0's; in the subsequent blocks of length k_1 the blanks are left intact. We continue up to a point $k_2=n_1k_1$ $(n_1>1)$ where the fraction of the filled positions in the initial interval of length k_2 is less than ε. The next blocks of length k_2 are repeated periodically. In the third step we fill up with 1's all the blanks in the initial interval of length k_2 and leave intact the blanks in the following k_2-blocks up to $k_3 = n_2 k_2$ $(n_2 > 1)$ where the fraction of the filled positions is less than ε. The next blocks of length k_3 are repeated periodically. We continue this process by induction by filling up the blanks with 1's and 0's at odd and even steps, respectively. The fraction of the filled positions in the initial interval of length k_n is less than ε at the nth step. In the resulting sequence z the initial blocks of lengths $k_1, k_2, \ldots$ are repeated at intervals of lengths $k_2, k_3, \ldots$, respectively. Since $k_n \to \infty$, (i) is clearly satisfied. The density of 1's in the initial block of length k_n is less than ε for n odd and greater than $1-\varepsilon$ for n even, whence (ii) is also satisfied.

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY, WROCŁAW

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Some convergence properties of convolutions

by

KLAUS KELLER (Dortmund)

Abstract. For certain spaces X of test functions the following reversed form of continuity is shown to hold for convolutions: If $T_j \in X'$ and $T_j * \varphi \to 0$ in X' for all $\varphi \in X$, then $T_j \to 0$ in X'. The proofs are based on theorems of Grothendieck and Raikov–e Silva on inductive limit spaces.

Considering the operation of convolution on $\mathcal{S}' \times \mathcal{S}$ ($\mathcal{S}$ the space of rapidly decreasing C^∞-functions, $\mathcal{S}'$ its strong dual), the following property is easily proved: If $T_j \to 0$ in $\mathcal{S}'$, then $T_j * \varphi \to 0$ in $\mathcal{S}'$ for all $\varphi \in \mathcal{S}$. In the sequential approach of Mikusiński [1] and [2] a reversed problem is of interest: If for $T_j \in \mathcal{S}'$ we have $T_j * \varphi \to 0$ in $\mathcal{S}'$ for all $\varphi \in \mathcal{S}$, is it true that $T_j \to 0$ in $\mathcal{S}'$? Similarly does $T_j * \varphi \to 0$ in $\mathcal{D}'$ for all $\varphi \in \mathcal{D}$ imply $T_j \to 0$ in $\mathcal{D}'$? In both cases we infer from

$$(T_j * \varphi, \psi) = (T_j, \varphi^\vee * \psi), \quad \varphi^\vee(x) := \varphi(-x),$$

(weak) convergence on the subspaces $[\mathcal{S} * \mathcal{S}] \subset \mathcal{S}$ or $[\mathcal{D} * \mathcal{D}] \subset \mathcal{D}$, which are built of finite sums of convolution products. If $[\mathcal{S} * \mathcal{S}]$ ($[\mathcal{D} * \mathcal{D}]$) is equal to $\mathcal{S}$ ($\mathcal{D}$) or at least of second category in $\mathcal{S}$ ($\mathcal{D}$), then (T_j) is certainly bounded and convergent. Performing Fourier transformation we may ask whether $[\mathcal{S} \cdot \mathcal{S}] = \mathcal{S}$ and $[\mathcal{Z} \cdot \mathcal{Z}] = \mathcal{Z}$? Whereas $[\mathcal{Z} \cdot \mathcal{Z}] = \mathcal{Z}$ can be verified by rather deep and lengthy methods [3], a systematic treatment for the case of other test function spaces is not known.*

In this note we give a direct solution for the original problem. It utilizes the following "Theorem B" of Grothendieck [4] and can be generalized to various other test function spaces.

THEOREM B. *Let E be a locally convex Hausdorff space, F and F_i $(i \in N)$ Fréchet spaces. Let u be a continuous mapping $F \to E$ and u_i continuous mappings $F_i \to E$. If $u(F) \subset \bigcup_i u_i(F_i)$, then there exists some index i_0 such that $u(F) \subset \bigcup_{i \le i_0} u_i(F_i)$.*

This theorem will imply the convergence of $(T_j * \varphi)_j$ on a larger space, so factorizations of type $\psi = \psi_1 * \psi_2$ become possible.

* In the mean time J. Voigt has announced a proof of $\mathcal{S} \cdot \mathcal{S} = \mathcal{S}$.