for every pair a, b of positive numbers linearly independent over the field generated by A, i.e. the field of rational numbers. Suppose now that $b \to a$, a, b being linearly independent. Then, by (i) and (15), the family $\{\nu_{ab}\}$ is conditionally compact. Let ν be its cluster point. Then by (15)

$$(\delta_a \mu) * (\delta_a \mu) = \mu\nu \tag{29}$$

and, by (28),

$$\nu(\{0\}) \geqslant \mu(\{1\}).$$

Since $A = \{-1, 1\}$, we get, by virtue of (29), the relation

$$2\mu(\{1\})^2 = (\delta_a \mu) * (\delta_a \mu)(\{0\}) = (\mu\nu)(\{0\}) = \nu(\{0\}) \geqslant \mu(\{1\}).$$

Thus $\mu(\{1\}) \geqslant 1/2$. Since $\mu(\{-1\}) = \mu(\{1\})$ and $\mu(\{-1\}) + \mu(\{1\}) \leqslant 1$, we have $\mu(\{1\}) = \mu(\{-1\}) = 1/2$ and, consequently, $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$ which completes the proof.

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Non-Leibniz algebras

by

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Dedicated to Professor J. Mikusiński on the 70th birthday

Abstract. We consider algebras with right invertible operators in the case when the Leibniz condition

$$D(xy) = xDy + yDx \quad \text{for} \quad x, y \in \operatorname{dom} D \tag{L}$$

(provided that $xy \in \operatorname{dom} D$) is not satisfied. In particular, it is shown that in a large class of non-Leibniz algebras all initial operators are averaging.

We shall consider algebras with right invertible operators in the non-Leibniz case, i.e., in the case where the condition

$$D(xy) = xDy + yDx \quad \text{for} \quad x, y \in \operatorname{dom} D \tag{L}$$

is not satisfied (provided that $xy \in \operatorname{dom} D$).

Some particular cases have been studied by Dudek [1] and by the author and von Trotha (cf. [3], [4], [9]).

In [5], [6], [7] we have shown that the Green formula, the Euler–Lagrange equation and the Picone identity hold in the general non-Leibniz case. In [8] there was given a classification of non-Leibniz algebras. A large class of algebras which are in a sense "close" to Leibniz case has been distinguished. Properties of right invertible operators and their inverses in these algebras, in particular, Wroński theorems have been studied.

1. Preliminaries. Let X be a linear space over a field $\mathscr{F}$ of scalars. Let $L(X)$ be the set of all linear operators A such that the domain of A (denoted by $\operatorname{dom} A$) is a linear subset of X and $AX \subset X$. In particular, we write: $L_0(X) = \{A \in L(X)\colon \operatorname{dom} A = X\}$. Let $R(X)$ be the set of all right invertible operators belonging to $L(X)$. For a given $D \in R(X)$ we denote by $\mathscr{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$ the set of all right inverses of D. We shall assume that $R_\gamma \in L_0(X)$ for $\gamma \in \Gamma$. Here and in the sequel we shall assume also that $\dim \ker D > 0$, i.e., D is right invertible but not left invertible. Any element of $\ker D$ is a *constant* for D. By definition, F is an initial operator for D if is a projection *onto* $\ker D$ such that $FR = 0$ for a right inverse R.

This implies that F is an initial operator if and only if there is a right inverse R such that $F = I - RD$ on $\operatorname{dom} D$.

One can also prove (cf. for instance [3]) that *any* projection F onto $\ker D$ is an initial operator for D corresponding to a right inverse $R = R_0 - FR_0$ and that the definition of R does not depend on the choice of the right inverse R_0.

2. Integration in D-algebras. Let X be a commutative algebra over $\mathscr{F}$ (i.e., a commutative linear ring over $\mathscr{F}$) and let $D \in R(X)$. X is said to be a *D-algebra* if the following condition is satisfied:

$$(2.1)\qquad x \in \operatorname{dom} D \quad\text{and}\quad y \in \operatorname{dom} D \quad\text{implies}\quad xy \in \operatorname{dom} D$$

i.e., if the domain of D is a sub-algebra of X. Here and in the sequel we shall assume that X is a D-algebra. Write:

$$(2.2)\qquad f_D(x, y) = D(xy) - c_D(xDy + yDx) \quad \text{for all } x, y \in \operatorname{dom} D$$

where

(1) c_D is a scalar dependent on D only;

(2) $f_D\colon \operatorname{dom} D \times \operatorname{dom} D \to \operatorname{dom} D$ is a bilinear and symmetric mapping dependent on D only, i.e.,

$$(2.3)\qquad f_D(y, x) = f_D(x, y) \quad \text{for all } x, y \in \operatorname{dom} D.$$

Using the notation (2.2) we can write

$$(2.4)\qquad D(xy) = c_D(xDy + yDx) + f_D(x, y) \quad \text{for } x, y \in \operatorname{dom} D.$$

The bilinear operator f_D will be called a *non-Leibniz* component. Non-Lebniz components for powers of D are determined by the following recursion formulae (proof by induction):

$$f_D^{(0)} = 0, \quad f_D^{(1)} = f_D \quad \text{and for } k = 2, 3, \ldots,\ x, y \in \operatorname{dom} D^k$$

$$(2.5)\qquad f_D^{(k)}(x, y) = c_D^k[(Dx)(D^{k-1}y) + (D^{k-1}x)(Dy)] + c_D^{k-1}[f_D(x, D^{k-1}y) + f_D(D^{k-1}x, y)] + Df_D^{(k-1)}(x, y)$$

or in another (equivalent) form:

$$(2.5')\qquad f_{D)}^{(k}(x, y) = c_D^k[(Dx)(D^{k-1}y) + (D^{k-1}x)(Dy)] + c_D[f^{(k-1)}(x, Dy) + f^{(k-1)}(Dx, y)] + D^{k-1}f_D(x, y).$$

Similar formulae hold for a superposition of right invertible operators. For an arbitrary scalar $p \neq 0$ we have $c_{pD} = c_D$, $f_{pD} = pf_D$.

Other properties of non-Leibniz components and several examples of D-algebras can be found in [6] (cf. also [5]). Without loss of generality we can assume here and in the sequel that $c_D \neq 0$ (cf. Example 1.8 in [6]). An extension of a D-algebra over $\boldsymbol{R}$ to a D-algebra over the field C of complexes can be made in a standard way (cf. [5]).

THEOREM 2.1 (Generalized integration by parts formula*). *Let X be a D-algebra and let F be an initial operator for D corresponding to a right inverse R. Then for all $x, y \in \operatorname{dom} D$ and for any positive integer n the following formula holds:*

$$(2.6)\qquad R^n(xy) = c_D^{-n}xR^ny - \sum_{j=1}^{n} c_D^{-j}R^{n-j}\{c_DR[(Dx)R^jy] + F(xR^jy) + Rf_D(x, R^jy)\}.$$

Proof. Let $x, y \in \operatorname{dom} D$. Write $u = Ry$. Then $u \in \operatorname{dom} D$, and formula (2.4) implies that $xDu = c_D^{-1}D(xu) - uDx - c_D^{-1}f_D(x, u)$. Since $RD = I - F$ and $y = Du$, acting on both sides of this equality by the operator R we obtain

$$(2.7)\qquad \begin{aligned} R(xy) &= R(xDu) = c_D^{-1}RD(xu) - R(uDx) - c_D^{-1}Rf_D(x, u) \\ &= c_D^{-1}[xu - R(uDx) - F(xu) - Rf_D(x, u)] \\ &= c_D^{-1}xRy - R[(Dx)(Ry)] - c_D^{-1}F(xRy) - c_D^{-1}Rf_D(x, Ry), \end{aligned}$$

which proves formula (2.6) for $n = 1$.

Suppose now formula (2.6) to be true for an arbitrary fixed $n > 1$. Then

$$(2.8)\qquad \begin{aligned} R^{n+1}(xy) &= R[R^n(xy)] \\ &= c_D^{-n}R(xR^ny) - \sum_{j=1}^{n} c_D^{-j}R^{n+1-j}\{c_DR[(Dx)R^jy] + F(xR^jy) + Rf_D(x, R^jy)\} \\ &= c_D^{-(n+1)}xR^{n+1}y - c_D^{-n}R[(Dx)(R^{n+1}y)] - c_D^{-(n+1)}F(xR^{n+1}y) - \\ &\quad - c_D^{-(n+1)}Rf_D(x, R^{n+1}y) - \sum_{j=1}^{n} c_D^{-j}R^{n+1-j}\{c_DR[(Dx)R^jy] + F(xR^jy) + Rf_D(x, R^jy)\} \\ &= c_D^{-(n+1)}xR^{n+1}y - \sum_{j=1}^{n+1} c_D^{-j}R^{n+1-j}\{c_DR[(Dx)(R^jy)] + F(xR^jy) + Rf_D(x, R^jy)]\}, \end{aligned}$$

which proves formula (2.6) for $n+1$.

COROLLARY 2.1. *Suppose that all assumptions of Theorem 2.1 are satisfied.*

(a) *If $z \in \ker D$, $x \in \operatorname{dom} D$ then*

$$(2.9)\qquad R^n(zx) = c_D^{-n}zR^nx - \sum_{j=1}^{n} c_D^{-j}R^{n-j}[F(zR^jx) + Rf_D(z, R^jx)].$$

* This formula and formulae (2.9), (2.10), (2.11), (2.12) in the Leibniz case have been proved by H. von Trotha.

(b) *If X has a unit e, $z \in \ker D$ and $x \in \operatorname{dom} D$ then*

$$(2.10)\quad R^n x = c_D^{-n} x R^n e - \sum_{j=1}^{n} c_D^{-j} R^{n-j} \{c_D R[(Dx) R^j e] + F(xR^j e) + f_D(x, R^j e)\},$$

$$(2.11)\quad R^n z = c_D^{-n} z R^n e - \sum_{j=1}^{n} c_D^{-j} R^{n-j} [F(zR^j e) + R f_D(z, R^j e)].$$

(c) *If X has a unit e then the remainder in the Taylor Formula is of the form*

$$(2.12)\quad R^n D^n x = c_D^{-n} (D^n x) R^n e - \\ - \sum_{i=1}^{n} c_D^{-j} R^{n-j} \{c_D R[(D^{n+1} x) R^j e] + F[(D^n x) R^j y] + y + R f_D(D^n x, R^j e)\}$$

provided that $x \in \operatorname{dom} D^{n+1}$.

(d) *If x has a unit e and F is multiplicative then for $x \in \operatorname{dom} D$, $z \in \ker D$*

$$(2.13)\quad R^n x = c_D^{-n} x R^n e - \sum_{j=1}^{n} c_D^{-j} R^{n-j} \{c_D R[(De) R^j e] + R f_D(x, R^j e)\},$$

$$(2.14)\quad R^n z = c_D^{-n} z R^n e - \sum_{j=1}^{n} c_D^{j} R^{n+1-j} f_D(z, R^j e).$$

Proof. In order to prove formula (2.9) observe that $Dz = 0$ and apply formula (2.6) for $y = z$. If X has a unit e then applying formula (2.6) for $y = e$ we obtain (2.10) for $xe = x$. Applying formula (2.10) for $x = z \in \ker D$ we obtain (2.11). Write: $w = D^n x$, $y = e$. Then $Dw = D^{n+1} x$. Applying formula (2.10) to the element $w \in \operatorname{dom} D$, we obtain the required formula (2.12).

If F is multiplicative, we have $FR = 0$ by definition. Then $F(xR^j e) = (Fx)(FR^j e) = 0$ for $j = 1, \ldots, n$. This and formula (2.10) together imply (2.13). Applying (2.13) for $z \in \ker D$, we obtain (2.14).

Formulae (2.6)–(2.14) are, as a matter of fact, formulae of *integration* in D-algebras.

THEOREM 2.2. *If X is a D-algebra, $x \in \operatorname{dom} D$ and $n \geqslant 2$ is an arbitrary positive integer then $x^n \in \operatorname{dom} D$ and*

$$(2.15)\quad Dx^n = d_n x^{n-1} + \sum_{j=0}^{n-2} c_D^j x^j f_D(x, x^{n-1-j})$$

where

$$(2.16)\quad d_1 = 1; \quad d_2 = 2c_D; \quad d_n = 2c_D^{n-1} + \sum_{j=1}^{n-2} c_D^j \quad \textit{for} \quad n > 2.$$

Proof. It is by induction (cf. [8]).

COROLLARY 2.2. *If X is a D-algebra then, for all $x \in \operatorname{dom} D^n$, $z \in \ker D$, $n \in N$,*

$$(2.17)\quad D^n(xz) = c_D^n z D^n x + f_D^{(n)}(z, D^n x)$$

where $f_D^{(n)}$ are defined by formulae (2.5);

$$(2.18)\quad Dz^n = \sum_{j=1}^{n-2} c_D^j z^j f_D(z, z^{n-1-j}).$$

Indeed, since $Dz = 0$, we have $D^n(xz) = c_D^n(xD^n z + zD^n x) + f_D^{(n)}(z, D^n x) = c_D^n zD^n x + f_D^{(n)}(z, Dx)$. Also formula (2.15) implies (2.18).

COROLLARY 2.3. *If X is a D-algebra with unit e then*

$$(2.19)\quad (1 - 2c_D) De = f_D(e, e),$$

$$(2.20)\quad zD^n e = e_D^{-n} f_D^{(n)}(z, D^n e) \quad \textit{for all} \quad z \in \ker D, \; n \in N.$$

Indeed, if we put $n = 2$, $x = e$ in (2.15), then we have $De = De^2 = D(e \cdot e) = 2c_D eDe + f_D(e, e) = 2c_D De + f_D(e, e)$, which implies (2.19). If we put $x = e$ in (2.17), we find $0 = D^n z = D^n(ez) = c_D^n zD^n e + f_D^{(n)}(z, D^n e)$ since $Dz = 0$.

Formula (2.19) implies that the unit e is a constant if $f_D(e, e) = 0$.

3. Initial operators in D-algebras. In order to study some properties of initial operators in D-algebras we have to introduce some definitions.

Suppose that X is a commutative algebra. An operator $A \in L(X)$ is said to be *averaging* if

$$(3.1)\quad A(xAy) = (Ax)(Ay) \quad \text{for} \quad x, y \in \operatorname{dom} A$$

$A \in L(X)$ is said to be a *Reynolds operator* if

$$(3.2)\quad A(xy) = A(xAy) + A[(x - Ax)(y - Ay)] \quad \text{for} \quad x, y \in \operatorname{dom} A$$

(cf. [11]).

Suppose that X is a D-algebra and F is an initial operator for D. Then F is said to be *almost averaging* if

$$(3.3)\quad F(zx) = zFx \quad \text{for all } x \in X, \; z \in \ker D.$$

The last property is very useful and has been used in several applications (cf. [2]–[9]).

By definition, any multiplicative initial operator is almost averaging. Indeed, since $Fz = z$ for $z \in \ker D$, we find that $F(zx) = (Fz)(Fx) = zFx$ for $x \in X$.

The converse statement is not true, as several examples show.

THEOREM 3.1. *Suppose that X is a D-algebra and F is an initial operator for D. Then the following conditions are equivalent:*

(i) *F is almost averaging,*

(ii) *F is averaging.*

Proof. Suppose that F is averaging and $x \in X$, $z \in \ker D$ are arbitrary. Since F is a projection onto $\ker D$, there exist $y \in X$ such that $Fy = z$. Then condition (3.1) implies

$$F(zx) = F(xFy) = (Fx)(Fy) = zFx,$$

i.e., F is almost averaging. Conversely, suppose that F is almost averaging and x, y are arbitrary. Since $z = Fy \in \ker D$ and condition (3.3) holds, we have

$$F(xFy) = F(zx) = zFx = (Fx)(Fy),$$

i.e., F is averaging.

THEOREM 3.2. *Suppose that X is a D-algebra with unit $e \in \ker D$ (over a field $\mathscr{F}$ of characteristic zero) and F is an almost averaging initial operator. Then*

(a) *F is a Reynolds operator;*

(b) *we have*

$$Fz^n = z^n \quad \text{for all } z \in \ker D,\ n \in N; \tag{3.4}$$

(c) *any power of a constant is again a constant;*

(d) *we have*

$$f_D(z, z) = 0 \quad \text{for all } z \in \ker D. \tag{3.5}$$

Proof. By our assumption $Fe = e$. By Theorem 3.1 F is averaging. Any averaging operator in a commutative algebra with unit such that $Ae = e$ is a Reynolds operator (cf. Rota, [11]). Since F is a Reynolds operator in a commutative algebra with unit (over a field of characteristic zero), the following identity holds:

$$nF[x(Fx)^{n-1}] = (n-1)F[(Fx)^n] + (Fx)^n \quad \text{for all } x \in X,\ n \in N \tag{3.6}$$

(cf. also Rota, [11]). But $z = Fx \in \ker D$ and F is almost averaging. Then identity (3.6) can be rewritten as follows:

$$nF(xz^{n-1}) = (n-1)Fz^n + z^n \quad \text{for} \quad n \in N. \tag{3.7}$$

For $n = 2$ we have $2z^2 = 2zFx = 2F(xz) = Fz^2 + z^2$, which implies that $Fz^2 = z^2$ and z^2 is a constant for $Fz^2 \in \ker D$. Suppose that $Fz^k = z^k$ for an arbitrary fixed $2 < k \in N$. Then $z_k = Fz^k \in \ker D$

Since F is averaging and (3.7) holds, we find

$$(k+1)z^{k+1} = (k+1)z^kFx = (k+1)F(xz^k) = kFz^{k+1} + z^{k+1},$$

which implies $Fz^{k+1} = z^{k+1}$. Therefore $Fz^n = z^n$ for all $n \in N$. Since $Fz^n \in \ker D$ for $n \in N$, we conclude that any power of a constant is again a constant. Since $z^2 \in \ker D$ for any $z \in \ker D$, we conclude that

$$0 = Dz^2 = 2c_D zDz + f_D(z, z) = f_D(z, z),$$

i.e., the non-Leibniz component f_D vanishes on constants.

4. Almost Leibniz algebras. A D-algebra X is said to be *almost Leibniz* if

$$f_D(x, z) = 0 \quad \text{for all } x \in \operatorname{dom} D,\ z \in \ker D. \tag{4.1}$$

The following D-algebras are almost Leibniz:

(1) *Leibniz algebras* since we have

$$D(xy) = xDy + yDx \quad \text{for} \quad x, y \in \operatorname{dom} D.$$

Here $f_D = 1$.

(2) *Quasi-Leibniz algebras* since we have

$$D(xy) = xDy + yDx + d(Dx)(Dy) \quad \text{for} \quad x, y \in \operatorname{dom} D$$

(where $d \neq 0$ is a given scalar dependent on D only). Hence $f_D(x, z) = d(Dx)(Dz) = 0$ for $z \in \ker D$.

(3) *Simple Duhamel algebras* since we have

$$D(xy) = xDy \quad \text{for} \quad x, y \in \operatorname{dom} D.$$

Indeed, the commutativity of multiplication in X implies that $D(xy) = yDx$. We therefore conclude that $D(xy) = \frac{1}{2}(xDy + yDx)$ for $x, y \in \operatorname{dom} D$, which implies $f_D = 0$.

(4) Suppose that X is a *Leibniz D-algebra*.

Then, for an arbitrary positive integer $n > 1$, X is an almost-Leibniz D^n-algebra. Indeed, the Leibniz condition implies that for $x, y \in \operatorname{dom} D$

$$D^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (D^k x)(D^{n-k} y) = xD^n y + yD^n x + \sum_{k=1}^{n-1} \binom{n}{k} (D^k x)(D^{n-k} y).$$

Hence

$$f_{D^n}(x, z) = \sum_{k=1}^{n-1} \binom{n}{k} (D^k x)(D^{n-k} z) = 0 \quad \text{for} \quad z \in \ker D$$

(cf. also Examples 1.1–1.8 in [6]).

THEOREM 4.1. *Suppose that X is an almost Leibniz D-algebra. Then*

(4.2) $f_D^{(n)}(x, z) = 0$ *for all* $x \in \operatorname{dom} D$, $z \in \ker D$ $(n = 1, 2, \ldots)$.

Proof. It is by induction (cf. [8]).

COROLLARY 4.1. *Suppose that X is an almost Leibniz D-algebra. Then*

(4.3) $D^n(xz) = c_D^n\, zD^n x$ *for all* $x \in \operatorname{dom} D^n$, $z \in \ker D$ $(n = 1, 2, \ldots)$,

(4.4) $z \in \ker D$ *implies* $z^n \in \ker D$ *for* $n = 1, 2, \ldots$,

i.e., a power of a constant is again a constant. If X has a unit e and

(4.5) $zD^n e = 0$ *for all* $z \in \ker D$ $(n = 1, 2, \ldots)$.

Indeed, formulae (4.2) and (2.17) imply (4.3). Formula (4.2) and condition (4.1) imply that $Dz^n = 0$. Hence $z^n \in \ker D$ $(n = 1, 2, \ldots)$ and z^n is a constant. Formulae (2.19) and (4.2) imply (4.5).

COROLLARY 4.2. *If X is an almost Leibniz D-algebra with unit e and constants are not zero divisors then $e \in \ker D$, i.e., e is a constant and $c_D = 1$.*

THEOREM 4.2. *Suppose that X is an almost Leibniz D-algebra and F is an initial operator for D corresponding to a right inverse R. Then for $x, y \in \operatorname{dom} D$, $z \in \ker D$ and $n \in N$*

$$(4.6)\quad \begin{aligned} R^n(xy) &= c_D^{-n} xR^n y - \sum_{j=1}^{n} c_D^{-j} R^{n-j}\{c_D R[(Dx)R^j y] + \\ &\qquad + F(xR^j y) - Rf_D(x, R^j y)\}, \\ R^n(zx) &= c_D^{-n} zR^n x - \sum_{j=1}^{n} c_D^{-j} R^{n-j} F(zR^j x). \end{aligned}$$

If X has a unit then

$$(4.7)\quad R^n z = c_D^{-n} zR^n e - \sum_{j=1}^{n} c_D^{-j} R^{n-j} F(zR^j e).$$

The proof immediately follows from our assumptions, Theorem 2.1 and Corollary 2.1.

THEOREM 4.3. *Suppose that X is an almost Leibniz D-algebra. Then all initial operators for D are averaging.*

Proof. Let F be an arbitrary fixed initial operator. Let $x \in X$ and $z \in \ker D$ be also arbitrary. By our assumption, $f_D(x, z) = 0$. By formula 4.7, since $RD = I - F$, we find

$$(4.8)\quad \begin{aligned} F(xz) &= xz - RD(xz) = xz - R[c_D(xDz + zDx) + f_D(xz)] \\ &= xz - c_D R(zDx) = xz - zRDx - F(zRDx) \\ &= xz - z(I - F)x - F[z(I - F)x] \\ &= xz - xz + zFx - F(zx) + F(zFx), \end{aligned}$$

which implies

$$(4.9)\quad 2F(xz) = zFx + F(zFx) \quad \text{for} \quad x \in X,\ z \in \ker D.$$

Since $F^2 = F$, acting on both sides of this equality by the operator F we obtain

$$2F(xz) = 2F^2(xz) = F(zFx) + F^2(zFx) = 2F(zFx),$$

i.e.,

$$(4.10)\quad F(xz) = F(zFx) \quad \text{for} \quad x \in X,\ z \in \ker D.$$

If we apply the last equality in (4.9) we get

$$2F(zFx) = 2F(xz) = zFx + F(zFx),$$

i.e.,

$$(4.11)\quad F(zFx) = zFx \quad \text{for} \quad x \in X,\ z \in \ker D.$$

Equalities (4.10) and (4.11) together imply that

$$(4.12)\quad F(xz) = F(zFx) = zFx \quad \text{for} \quad x \in X,\ z \in \ker D,$$

i.e., F is almost averaging. By Theorem 3.1 we conclude that F is averaging.

Note that Theorems 4.3 and 3.2 together imply that in almost Leibniz D-algebras a power of a constant is again a constant, which gives another proof of formula (4.4).

COROLLARY 4.3. *Suppose that X is an almost Leibniz D-algebra with unit e and F is an initial operator for D corresponding to a right inverse R. Then*

$$\begin{aligned} &(4.13)\quad F(zR^n e) = 0 \\ &(4.14)\quad R^n z = c_D^{-n} zR^n e \end{aligned} \quad \text{for} \quad z \in \ker D \quad (n = 1, 2, \ldots),$$

Indeed, by our assumptions F satisfies (4.12) and $FR = 0$. Hence for $z \in \ker D$ and $n = 1, 2, \ldots$ we have

$$F(zR^n e) = zFR^n e = 0$$

and

$$R^n z = c_D^{-n} zR^n e - \sum_{j=1}^{n} c_D^{-j} R^{n-j} F(zR^j e) = c_D^{-n} zR^n e.$$

THEOREM 4.4 (Integration of unit formula). *Suppose that X is an almost Leibniz D-algebra with unit e such that constants are not zero divisors. Suppose that F is an initial operator for D corresponding to a right inverse R. Then*

$$(4.15)\quad R^n e = \frac{(Re)^n}{d(n)} - \sum_{k=2}^{n} \frac{1}{d(k)} R^{n-k} F[(Re)^k] \quad (n = 1, 2, \ldots)$$

where we assume that

$$d(n) = d_1 \ldots d_n \neq 0 \quad (n = 1, 2, \ldots) \tag{4.16}$$

and $d_1, \ldots, d_n$ *are defined by formulae* (2.16), *i.e.*,

$$d_1 = 1, \quad d_2 = 2c_D, \quad d_n = 2c_D^{n-1} + \sum_{j=1}^{n-2} c_D^j \quad \text{for} \quad n \geqslant 3. \tag{4.17}$$

Proof. Observe that by our assumption $d_1 \neq 0$, $d_2 \neq 0$ and

$$d_{n+1} = c_D(d_n+1), \quad d(n+1) = d_{n+1}d(n), \; d(n) \neq 0 \quad (n = 1, 2, \ldots).$$

Our assumptions and Corollary 4.2 together imply that $De = 0$. Then e is a constant. Write: $g = Re$. Then $Dg = DRe = e$. Theorem 2.2 and our assumptions together imply that $Dg^n = d_n g^{n-1} Dg = d_n g^{n-1} e = d_n g^{n-1}$ $(n = 1, 2, \ldots)$. Hence

$$g^n - Fg^n = (I-F)g^n = RDg^n = d_n R g^{n-1} \quad (n = 2, 3, \ldots).$$

Observe that $d(2)R^2 e = d_2 Rg = g^2 - Fg^2 = g^2 - d_1 Fg^2$. Suppose that for an arbitrarily fixed $n > 2$ we have

$$R^n e = \frac{g^n}{d(n)} - \sum_{k=2}^{n} \frac{1}{d(k)} R^{n-k} Fg^k.$$

Then

$$R^{n+1}e = R(R^n e) = \frac{1}{d(n)} Rg^n - \sum_{k=2}^{n} \frac{1}{d(k)} R^{n+1-k} Fg^k$$

$$= \frac{1}{d_{n+1}d(n)} (g^{n+1} - Fg^{n+1}) - \sum_{k=2}^{n} \frac{1}{d(k)} R^{n+1-k} Fg^k$$

$$= \frac{1}{d(n+1)} g^{n+1} - \frac{1}{d(n+1)} Fg^{n+1} - \sum_{k=2}^{n} \frac{1}{d(k)} R^{n+1-k} Fg^k$$

$$= \frac{g^{n+1}}{d(n+1)} - \sum_{k=2}^{n+1} \frac{1}{d(k)} R^{n+1-k} Fg^k,$$

which was to be proved.

In the Leibniz case this theorem was proved in [9]. The present proof is simpler. Note that in the Leibniz case $c_D = 1$. Hence $d_n = n$ and

$$d(n) = n! \quad (n = 1, 2, \ldots). \tag{4.18}$$

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