Since $\lambda > 0$ is arbitrary, it follows that

$$\overline{\int_a^b} f(u)\,du \leqslant \underline{\int_a^b} f(u)\,du.$$

But the opposite inequality is trivial, hence f is integrable on $[a, b]$.

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Walsh equiconvergence for best l_2-approximates

by

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Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. We obtain equiconvergence results for polynomials interpolating at a subsed of the roots of unity and best approximating f in the l_2-sense on the complementary set.

1. Introduction. Let A_ϱ $(1 < \varrho < \infty)$ denote the class of functions $f(z)$ analytic in $|z| < \varrho$ but not in $|z| \leqslant \varrho$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then let $p_{m-1}(z; f)$ denote the Lagrange polynomial which interpolates f in the m roots of unity. If $S_{m-1}(z; f) = \sum_{j=0}^{m-1} a_j z^j$, then a beautiful theorem of Walsh [5] states that

$$\lim_{m\to\infty} \{p_{m-1}(z; f) - S_{m-1}(z; f)\} = 0 \quad \text{for} \quad |z| < \varrho^2, \tag{1.1}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^2$. Moreover the result is best possible in the sense that for every z with $|z| = \varrho^2$, there is an $f \in A_\varrho$ for which (1.1) fails.

Recently extensions of this theorem have been made in various directions. We refer the reader to a survey article by R. S. Varga [4] for further references. Here we generalize a result of Rivlin [3] which extends Walsh's theorem in the l_2-sense. If $m = nq + c$ and if $p_{n,m}(z; f) \in \pi_n$ minimizes

$$\sum_{k=0}^{m-1} |P_n(\omega^k; f) - f(\omega^k)|^2, \qquad \omega^m = 1 \tag{1.2}$$

over all polynomials $P_n \in \pi_n$, then Rivlin [3] showed that

$$\lim_{n\to\infty} \{P_{n,m}(z; f) - S_n(z; f)\} = 0 \quad \text{for} \quad |z| < \varrho^{1+q}, \tag{1.3}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^{1+q}$. Moreover, the result is best possible in the same sense as described above.

In Section 2, we obtain equiconvergence results for polynomials interpolating at a subset of the roots of unity and best approximating f in the l_2-sense on the complementary subset.

2. Mixed interpolation and l_2-approximation. Let m be an integer with $m = nrs$, $s > 1$ and let $\omega^m = 1$. Let U_s be the subset of the m roots of unity defined by

$$U_s = \{\omega^\nu;\ \nu = 0, 1, \ldots, nrs-1,\ \nu \not\equiv 0 (\mathrm{mod}\, s)\}, \quad s > 1. \tag{2.1}$$

Note that the complement of the set U_s is the set $\{\exp(2\pi ik/nr);\ k = 0, 1, \ldots, nr-1\}$, i.e., the set of zeros of $z^{nr}-1$. Thus U_s consists of the zeros of $W(z) = (z^{nrs}-1)/(z^{nr}-1)$.

Let $\mathscr{L}(f;\ U_s) \in \pi_{N+n-1}$, $N = nr(s-1)$ denote the class of polynomials of degree $\leqslant N+n-1$ interpolating f at the set U_s. Let $P_{N+n-1}(z) \in \mathscr{L}(f;\ U_s)$ be the solution of the minimization problem

$$\min \Big\{\sum_{\nu=0}^{nr-1} |f(\omega^{\nu s}) - Q(\omega^{\nu s})|^2,\ Q \in \mathscr{L}(f;\ U_s)\Big\}. \tag{2.2}$$

We shall prove

THEOREM 1. *Let $f(z) \in A_\varrho$ $(\varrho > 1)$ and let $P_{N+n-1}(z;f)$ be the polynomial $\subset \mathscr{L}(f;\ U_s)$ which solves the minimization problem* (2.2). *Then*

$$\lim_{n\to\infty} \{P_{N+n-1}(z;f) - S_{N+n-1}(z;f)\} = 0 \quad \text{for} \quad |z| < \varrho^{s/(s-1)}, \tag{2.3}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^{s/(s-1)}$. Moreover, the result is best possible.

Remark 1. When $s = 1$, U_s is a null set and we get Rivlin's problem (1.2). The result follows from the derivation below. For $j > 1$, we introduce the polynomials $S_{N+n-1,j}(z;f)$ derived from the power series for $f(z)$. Set

$$S_{N+n-1,j}(z;f) = \sum_{\nu=0}^{N+n-1} a_{jnrs+\nu} z^\nu - z^n \Big(\sum_{\lambda=0}^{s-2} z^{\lambda nr}\Big) \sum_{\nu=0}^{nr-n-1} a_{jnrs-nr+n+\nu} z^\nu \qquad (j = 1, 2, \ldots).$$

Then we can show that

$$\lim_{n\to\infty} \Big[P_{N+n-1}(z;f) - S_{N+n-1}(z;f) - \sum_{j=1}^{l-1} S_{N+n-1,j}(z;f)\Big] = 0 \tag{2.4}$$

for $|z| < \varrho^{ls/(s-1)}$.

Proof. Since $Q(z) \in \mathscr{L}(f;\ U_s)$ can be written as

$$Q(z) = L_{N-1}(z;f) + W(z) Q_{n-1}(z), \qquad Q_{n-1} \in \pi_{n-1},$$

where $L_{N-1}(z;f)$ is the Lagrange interpolant to f on U_s, we can reduce problem (2.2) to the following:

$$\min \Big\{\sum_{\nu=0}^{nr-1} |g(\omega^{\nu s}) - Q_{n-1}(\omega^{\nu s})|^2;\ Q_{n-1} \in \pi_{n-1}\Big\} \tag{2.5}$$

where $g(z) = s^{-1}[f(z) - L_{N-1}(z;f)]$.

Appealing to Rivlin's result [3], we see that the solution $p_{n-1}(z;f)$ to (2.5) is given by

$$\begin{aligned} p_{n-1}(z;f) &= S_{n-1}[z;\ L_{nr-1}(z;\ g)] \\ &= \frac{1}{s} S_{n-1}[z;\ L_{nr-1}(z;f)] - \frac{1}{s} S_{n-1}\{z;\ L_{nr-1}[z;\ L_{N-1}(z;\ f)]\} \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{f(t)\, t^{nr-n}(t^n - z^n)}{W(t)(t^{nr}-1)(t-z)}\, dt \end{aligned}$$

where Γ is the circle $|t| = R$, $R < \varrho$. Thus we now get

$$P_{N+n-1}(z;f) - S_{N+n-1}(z;f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} K(t, z)\, dt$$

where

$$\begin{aligned} K(t, z) &= \frac{W(t) - W(z)}{W(t)} - \frac{W(z)}{W(t)} \frac{t^{nr-n}(t^n - z^n)}{t^{nr}-1} - \frac{t^{N+n} - z^{N+n}}{t^{N+n}} \\ &= \frac{z^{N+n}}{t^{N+n}} - \frac{z^{nrs}-1}{t^{nrs}-1} \cdot \frac{(t^{nr-n} z^n - 1)}{z^{nr}-1}. \end{aligned}$$

When $s = 1$, we have $N = 0$ and

$$K(t, z) = \frac{t^n - z^n}{t^n (t^{nr}-1)}$$

which tends to zero for $|z| < \varrho^{1+r}$. When $s > 1$, we can rewrite

$$K(t, z) = \frac{t^{N+n} - z^{N+n} - t^{N+n}(t^{nr-n} - z^{nr-n}) z^n \big(\sum_{\lambda=0}^{s-2} z^{\lambda nr}\big)}{t^{N+n}(t^{nrs}-1)}. \tag{2.6}$$

From this we can see that $K(t, z)$ tends to zero for $|z| < R^{s/(s-1)}$, $R < \varrho$, which gives the result.

Moreover, from (2.6), we can write $K(t, z)/(t-z)$ as a polynomial in z and expanding $(t^{nrs}-1)^{-1}$ in powers of t^{-nrs} we can see that

$$P_{N+n-1}(z;f) - S_{N+n-1}(z;f) - \sum_{j=1}^{l-1} S_{N+n-1,j}(z;f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z} \cdot \frac{K(t,z)}{t^{(l-1)nrs}}\, dt$$

from which we can get (2.4).

We can see that the results are best possible on taking $f(z) = (\varrho - z)^{-1}$.

Remark 2. Other variations of the problem (2.2) can be handled as above. We shall now consider the case of Hermite interpolation on the set U_s, defined as above, where the values and derivatives up to order $p-1$ are prescribed. Let $\mathscr{H}_p(f; U_s)$ be the set of polynomials in π_{pN+n-1} satisfying the Hermite interpolating conditions on U_s. We note that any $Q \in \mathscr{H}_p(f; U_s)$ can be written as

$$Q(z) = H_{pN-1}(z; f) + W^p(z) Q_{n-1}(z) \tag{2.7}$$

where H_{pN-1} is the unique Hermite interpolant of degree $pN-1$. If we now seek to solve the minimization problem

$$\min\Big\{\sum_{\nu=0}^{nr-1} |Q(\omega^{\nu s}) - f(\omega^{\nu s})|^2;\ Q \in \mathscr{H}_p(f; U_s)\Big\}, \tag{2.8}$$

then from (2.7), we see that (2.8) is equivalent to

$$\min\Big\{\sum_{\nu=0}^{nr-1} |g(\omega^{\nu s}) - Q_{n-1}(\omega^{\nu s})|^2,\ Q_{n-1} \in \pi_{n-1}\Big\} \tag{2.9}$$

where

$$g(z) = \{f(z) - H_{pN-1}(z/f)\}/s^p.$$

As seen above, the solution $P_{n-1}(z)$ of (2.9) is given by

$$\begin{aligned} P_{n-1}(z) &= S_{n-1}\big(z; L_{nr-1}(z; g)\big) \\ &= s^{-p} S_{n-1}\{z; L_{nr-1}[z; f - H_{pN-1}(z; f)]\}. \end{aligned} \tag{2.10}$$

We observe that

$$H_{pN-1}(z; f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,\frac{W^p(z) - W^p(z)}{W^p(t)}\,dt \tag{2.11}$$

so that

$$H_{pN-1}(\omega^{\nu s}; f) = f(\omega^{\nu s}) - \frac{s^p}{2\pi i}\int_\Gamma \frac{f(t)}{t-\omega^{\nu s}}\cdot\frac{1}{W^p(t)}\,dt. \tag{2.12}$$

Combining (2.10) and (2.12), we have

$$\begin{aligned} P_{n-1}(z) &= S_{n-1}\Big[\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)}\,\frac{t^{nr} - z^{nr}}{(t^{nr}-1)W^p(t)}\,dt\Big] \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\,t^{nr-n}(t^n - z^n)}{(t-z)(t^{nr}-1)W^p(t)}\,dt. \end{aligned}$$

If $P_{pN+n-1}(z; f)$ is the solution of (2.8), then (2.7) now yields

$$P_{pN+n-1}(z; f) - S_{pN+n-1}(z; f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,K_1(t, z)\,dt \tag{2.13}$$

where

$$\begin{aligned} K_1(t, z) &= \frac{W^p(t) - W^p(z)}{W^p(t)} + \frac{W^p(z)}{W^p(t)}\,\frac{t^{nr-n}(t^n - z^n)}{t^{nr}-1} - \frac{t^{pN+n} - z^{pN+n}}{t^{pN+n}} \\ &= \frac{z^{pN+n}}{t^{pN+n}} - \frac{(z^{nrs}-1)^p}{(t^{nrs}-1)^p}\cdot\frac{(t^{nr}-1)^{p-1}}{(z^{nr}-1)^p}\,(t^{nr-n}z^n - 1), \end{aligned}$$

$$N = nrs - nr.$$

If $s = 1$, we return to the Rivlin case since U_s is empty. If $s > 1$, we can rewrite

$$\begin{aligned} K_1(t, z) = {}& [z^{pN+n}(t^N + \ldots + 1)^p (t^{nr}-1) - t^{nrs} z^n (z^N + \ldots + 1)^p + \\ & + t^{pN+n}(z^N + \ldots + 1)^p] / t^{pN+n}(t^N + \ldots + 1)^p (t^{nr}-1) \end{aligned}$$

and a careful observation shows that this tends to zero for $|z| < \varrho^{1+1/p(s-1)}$. We have thus proved

THEOREM 2. *Let $f(z) \in A_\varrho$ ($\varrho > 1$) and let $P_{pN+n-1}(z; f)$ be the polynomial $\in \mathscr{H}_p(f; U_s)$ which minimizes* (2.8). *Then*

$$\lim_{n\to\infty} [P_{pN+n-1}(z; f) - S_{pN+n-1}(z; f)] = 0, \qquad |z| < \varrho^{1+1/p(s-1)} \tag{2.14}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^{1+1/p(s-1)}$. Moreover, the result is best possible.

An analogue of fomula (2.4) can be given, but it gets very unwieldy and is left out.

Remark. It would be interesting to extend the equiconvergence results to polynomials of best approximation in a weighted l_2-norm computed at values which are not roots of unity.

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