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Zygmund's Lemma and Riemann integrability

by

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Dedicated to Jan Mikusiński

Abstract. The use of an elementary property of Dini derivates and a simple lemma of Zygmund results in a "real variable" proof of the existence of the Riemann integral of a bounded function that goes through even when the classical hypothesis of a.e. continuity is weakened to the mere existence a.e. of a limit from either side.

§1. The real variable approach to integrability questions. One appealing way to establish the existence of the Riemann integral of a continuous function f on a finite interval $[a, b]$ is by introducing the function

$$F(x) = \overline{\int_a^x} f(u)\,du - \underline{\int_a^x} f(u)\,du \qquad (a \leqslant x \leqslant b).$$

Direct estimation of the difference quotients reveals that the derivative of F exists throughout (a, b) and vanishes identically there. Since F is continuous on the closed interval, the mean-value theorem of differential calculus implies that F must be constant. Hence

$$0 = F(a) = F(b) = \overline{\int_a^b} f(u)\,du - \underline{\int_a^b} f(u)\,du,$$

and the integrability of f follows.

An attractive feature of this proof is that it shows how a basic theorem of the *integral* calculus can be proved by using techniques of the *differential* calculus. Another desirable aspect of the proof is that it avoids any need to invoke the uniform continuity of the function f, since only the local behavior of f comes into play.

This last feature caught the attention of Hans Rademacher and led him to examine the possibility of proving along these lines the more general existence theorem of Lebesgue, where f is only assumed to be bounded and continuous a.e., by using more refined real variable tools. For this purpose, in his paper [5], he employed the Dini derivates of F, together with a mean-value theorem for them that had been developed by G. C. and W. H. Young. He found, however, that his method would not yield directly the existence of the integral under Lebesgue's condition, and he was obliged to interpose an equivalent, but more complicated form of the integrability condition, due to du Bois-Ryemond. Having established the existence theorem under this hypothesis, he then deduced the Lebesgue result, just as Lebesgue had done, by use of the Heine-Borel theorem.

Of course, Rademacher recognized that this last step was "essentially equivalent" to the use of the uniform continuity of f, which he had aimed to avoid. His paper thus left open the question as to whether, by some variant of his method, it might be possible to arrive at a "real variable" proof of Lebesgue's theorem which makes no appeal to the Heine-Borel theorem.

The goal of the present paper is to give just such a proof. While we continue to employ Dini derivates, we abandon the mean-value theorem and use in its place a result of Zygmund which has become basic in the study of differential inequalities. Pursuing this approach, we have been led to an unexpected discovery: the technique can be made to yield an existence theorem more general than Lebesgue's! By considering the relationship that must hold between two opposite Dini derivates, we have found that the Riemann integral will continue to exist for functions which are bounded and merely possess a.e. a limit from one side or the other: the side need *not* always be the same and may vary from point to point.

In this form, our theorem not only includes Lebesgue's, but also the little known existence theorems of Dini [1], p. 246 et seq., and Pasch [4], who weakened du Bois-Reymond's condition by employing the notion of one-sided oscillation of a function at a point (however, the side does not vary with the point). It also includes an analogous weakening of Lebesgue's condition, published in [2], where, again, limits are always taken on the same side.

Once our theorem has been established, we can, of course, conclude that the conditions that we have imposed upon f must make it continuous a.e., since that is *necessary* for integrability. The same conclusion could have been drawn, independently of any integrability considerations, from a remarkable theorem of W. H. Young [9] (that can be found also in Saks [6], (ii), p. 261) which implies that there are at most countably many points where a real function can have a limit from one side or the other, without being continuous at the point. This means that the truth of our theorem could have been inferred from Lebesgue's. It is odd that this possibility does not seem to have been noted previously.

§2. Details of the method. We begin by recalling two preliminary propositions, both of which are entirely elementary. To fix the notation, we shall denote the lower left Dini derivate by D_- and the upper right one by D^+.

The first result is known as "Zygmund's Lemma" (v. [7]).

PROPOSITION 1. *Let F be a continuous, real-valued function on the finite interval $[a, b]$, and suppose that the image under F of the set where $D_-F(x) \geqslant 0$ holds has no interior points. Then F is monotonic, non-increasing on $[a, b]$.*

A simple proof of this lemma can be found in [7], p. 10.

The second result is a classical theorem of G. C. Young [8]. As the reader will see from the short proof given in Saks [6], p. 261, this proposition is elementary and ought not be to confounded with the other, deeper results of this type that are associated, in the case of continuous (fn ctions, with the name of Denjoy.

PROPOSITION 2. *The set of points x where a real-valued function F has $D_-F(x) > D^+F(x)$ is at most countable.*

We now turn to our main result.

THEOREM. *Let f be a bounded, real-valued function on the finite interval $[a, b]$ which has, at almost every point, a limit from one side or the other. Then f is Riemann integrable on $[a, b]$.*

Proof. Take any $\lambda > 0$ and set, for each x in $[a, b]$,

$$F_\lambda(x) = \overline{\int_a^x} f(u)\,du - \underline{\int_a^x} f(u)\,du - 3\lambda(x-a).$$

Note that, because of the boundedness of f, F_λ is Lipschitzian in x.

Suppose that $f(x+0)$ exists at x. Then, for all u in some interval $x, x+h)$, $h > 0$, there holds $|f(x+0)-f(u)| < \lambda$. Hence

$$F_\lambda(x+h) - F_\lambda(x) = \overline{\int_x^{x+h}} f(u)\,du - \underline{\int_x^{x+h}} f(u)\,du - 3\lambda h$$
$$\leqslant [f(x+0)+\lambda]h - [f(x+0)-\lambda]h - 3\lambda h = -\lambda h.$$

Since this also holds for any smaller value of $h > 0$, we get that $D^+F_\lambda(x) \leqslant -\lambda < 0$. It then follows from Prop. 2 that $D_-F_\lambda(x) < 0$ at all but countably many of the points where $f(x+0)$ exists.

A similar estimate, with $h < 0$, shows directly that $D_-F_\lambda(x) < 0$ at all points where $f(x-0)$ exists.

Accordingly, the set where $D_-F_\lambda \geqslant 0$ holds is a null set, and consequently, so is its image under F_λ, since F_λ is Lipschitzian. Zygmund's Lemma thus applies to F_λ and yields that F_λ is monotonic non-increasing on $[a, b]$. In particular, $F_\lambda(b) \leqslant F_\lambda(a) = 0$, so that

$$\overline{\int_a^b} f(u)\,du - \underline{\int_a^b} f(u)\,du \leqslant 3\lambda(b-a).$$

Since $\lambda > 0$ is arbitrary, it follows that

$$\overline{\int_a^b} f(u)\,du \leqslant \underline{\int_a^b} f(u)\,du.$$

But the opposite inequality is trivial, hence f is integrable on $[a, b]$.

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Walsh equiconvergence for best l_2-approximates

by

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Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. We obtain equiconvergence results for polynomials interpolating at a subsed of the roots of unity and best approximating f in the l_2-sense on the complementary set.

1. Introduction. Let A_ϱ $(1 < \varrho < \infty)$ denote the class of functions $f(z)$ analytic in $|z| < \varrho$ but not in $|z| \leqslant \varrho$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, then let $p_{m-1}(z; f)$ denote the Lagrange polynomial which interpolates f in the m roots of unity. If $S_{m-1}(z; f) = \sum_{j=0}^{m-1} a_j z^j$, then a beautiful theorem of Walsh [5] states that

$$\lim_{m\to\infty} \{p_{m-1}(z; f) - S_{m-1}(z; f)\} = 0 \quad \text{for} \quad |z| < \varrho^2, \tag{1.1}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^2$. Moreover the result is best possible in the sense that for every z with $|z| = \varrho^2$, there is an $f \in A_\varrho$ for which (1.1) fails.

Recently extensions of this theorem have been made in various directions. We refer the reader to a survey article by R. S. Varga [4] for further references. Here we generalize a result of Rivlin [3] which extends Walsh's theorem in the l_2-sense. If $m = nq + c$ and if $p_{n,m}(z; f) \in \pi_n$ minimizes

$$\sum_{k=0}^{m-1} |P_n(\omega^k; f) - f(\omega^k)|^2, \qquad \omega^m = 1 \tag{1.2}$$

over all polynomials $P_n \in \pi_n$, then Rivlin [3] showed that

$$\lim_{n\to\infty} \{P_{n,m}(z; f) - S_n(z; f)\} = 0 \quad \text{for} \quad |z| < \varrho^{1+q}, \tag{1.3}$$

the convergence being uniform and geometric in $|z| \leqslant \tau < \varrho^{1+q}$. Moreover, the result is best possible in the same sense as described above.

In Section 2, we obtain equiconvergence results for polynomials