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Remarks on $K\{M_p\}'$-spaces

by

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To my teacher
Professor Jan Mikusiński
on his 70th birthday

Abstract. The characterization of elements of the dual of the space $K\{M_p\}$ given by I. M. Gelfand and G. E. Shilov, and also the characterization of the convergence in $K\{M_p\}'$ given by L. Kitchens and C. Swartz are simplified under an additional condition on the sequence $\{M_p\}$. In particular, a simple description of the convergence in various spaces of distributions is obtained.

1. The space $K\{M_p\}$, introduced in [2] by means of a non-decreasing sequence of extended real-valued functions M_p, embraces various spaces of test functions considered in the theory of distributions. On the other hand, the space $K\{M_p\}'$ (the dual of $K\{M_p\}$) embraces various types of spaces of distributions of finite order.

In [2] (p. 113) we find a representation of elements of $K\{M_p\}'$ under conditions (M), (N), (P), imposed on the sequence $\{M_p\}$. This representation can be written in the form of a finite sum of derivatives (in a generalized sense) of functions which become bounded after dividing by a function of the sequence $\{M_p\}$. In terms of such representations, the convergence in $K\{M_p\}'$ is characterized in [5] under the same conditions on $\{M_p\}$.

However, in all known particular cases of the space $K\{M_p\}'$, e.g., in the spaces $\mathcal{D}'_K$, $\mathcal{S}'$ (see [7]), $\mathcal{K}'_p$ (see [6]), H'_r (see [8]), $\mathcal{D}'_{\mathcal{A}_a}$ (see [4]), elements can be described in a simpler way by using one derivative of finite order. Similarly, the convergence in $\mathcal{S}'$ (see [1], p. 169), in $\mathcal{D}'_{\mathcal{A}_a}$ and in $\mathcal{K}'_2$ (see [4]) can be expressed by means of single distributional derivatives. Therefore the natural question arises when elements of $K\{M_p\}'$ and the convergence in $K\{M_p\}'$ can be characterized in that simplified way.

In this note we give an additional condition, constituting a modification of (N) (denoted by (N')), which guarantees such characterizations. Note that the system of conditions (M), (N), (N'), (P) is a little stronger than the system (M), (N), (P) and, in R^1, condition (N′) coincides with (N). We do not know if condition (N′) can be omitted in the simplified characterizations.

2. The functions considered in the note are supposed to be defined on R^q and complex-valued in general.

Some notation is adopted from [1]. In particular, the symbol P^q will denote the set of all non-negative multi-indices, i.e., $k = (\varkappa_1, \ldots, \varkappa_q) \in P^q$ if all coordinates $\varkappa_j$ are non-negative integers. The symbol $\varphi^{(k)}$ for $k \in P^q$ will denote the derivative of order $k \in P^q$ of a smooth function φ and the symbol

$$\int\limits_0^x F(t)\,dt^k$$

will stand for the iterated integral of order $k \in P^q$ of a locally integrable function F (see [1], p. 62). Moreover, let $|k| = \varkappa_1 + \ldots + \varkappa_q$ for $k = (\varkappa_1, \ldots, \varkappa_q) \in P^q$.

As in [2] (p. 86), let $\{M_p\}$ be a sequence of functions $M_p\colon R^q \to [1, \infty]$, continuous on the sets

$$Q_p = \{x \in R^q\colon\ M_p(x) < \infty\},$$

which are supposed to be equal for all $p \in N$ (let $Q = Q_p$ for $p \in N$) and such that

$$M_p(x) \leqslant M_{p+1}(x) \qquad (p \in N,\ x \in R^q). \tag{1}$$

Let us consider the following conditions on the sequence $\{M_p\}$:

(M) For each $j = 1, \ldots, q$ and $p \in N$, there exists $C_{jp} > 1$ such that if $|\xi_j'| \leqslant |\xi_j''|$ and $\xi_j' \cdot \xi_j'' \geqslant 0$, then

$$M_p(\xi_1, \ldots, \xi_j', \ldots, \xi_q) \leqslant C_{jp} \cdot M_p(\xi_1, \ldots, \xi_j'', \ldots, \xi_q); \tag{2}$$

(N) For each $p \in N$, there exists an integer $p' > p$ such that $m_{pp'} \in L^1(R^q)$ and $m_{pp'}(x) \to 0$ as $|x| \to \infty$, where $m_{pp'}(x) = M_p(x)/M_{p'}(x)$ (the convention $\infty/\infty = 0$ is adopted);

(P) For each $\varepsilon > 0$ and $p \in N$, there exist an integer $p' > p$ and a positive number T such that $m_{pp'}(x) < \varepsilon$ if $M_p(x) > T$.

The above conditions have been introduced in an equivalent form in [2], pp. 87 and 111.

Note that it follows from condition (N) that

$$\mu(\bar{x}) = \int\limits_{-\infty}^{\infty} m_{pp'}(x)\,d\xi_j < \infty$$

for each $j = 1, \ldots, q$ and almost all $\bar{x} = (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_q) \in R^{q-1}$. The function μ need not be bounded (almost everywhere) in R^{q-1} in general. We postulate this property additionally:

(N′) For each $j = 1, \ldots, q$ and $p \in N$, there exist an integer $p_j > p$ and a positive constant $B_{jp} > 0$ such that

$$\mu(\bar{x}) = \int\limits_{-\infty}^{\infty} m_{pp'}(x)\,d\xi_j \leqslant B_{jp} \tag{3}$$

for each $j = 1, \ldots, q$ and almost all $\bar{x} \in R^{q-1}$.

For $q = 1$, condition (N′) reduces to (N).

As in [2] (p. 86), for a given sequence $\{M_p\}$ let $K\{M_p\}$ denote the space of smooth functions φ such that 1° $\varphi^{(k)}(x) = 0$ for $k \in P^q$ and $x \notin Q$ and 2° $M_p\varphi^{(k)}$ is a continuous bounded function on R^q for all $p \in N$ and $k \in P^q$. The space $K\{M_p\}$ is endowed with the locally convex topology generated by the norms

$$\|\varphi\|_p = \sup\{M_p(x)\,|\varphi^{(k)}(x)|\colon\ x \in R^q,\ |k| \leqslant p\}. \tag{4}$$

It can be shown that, for every $r \geqslant 1$, the sequence of norms

$$\|\varphi\|_{p,r} = \sup_{|k| \leqslant p} \Big[\int\limits_{R^q} (M_p|\varphi^{(k)}|)^r\Big]^{1/r} \qquad (p \in N)$$

generates the same locally convex topology in $K\{M_p\}$ as the norms in (4) (cf. [2], pp. 111–112 and [5], Lemma 5).

Examples. It is easy to see that the spaces $\mathcal{D}_K$, $\mathcal{S}$ (see [7]), $\mathcal{K}_1$ (see [3] and [9]) and H_r (see [8]) are $K\{M_p\}$-spaces for particular sequences $\{M_p\}$ satisfying conditions (M), (N), (N′) and (P) (cf. [5]). The space $\mathcal{K}_r$, $r > 1$ (see [6]) is a $K\{M_p\}$-space with the sequence $M_p(x) = \exp(p|x|^r)$ for $p \in N$, $x = (\xi_1, \ldots, \xi_q) \in R^q$, where $|x| = \sqrt{\xi_1^2 + \ldots + \xi_q^2}$, and the space $\mathcal{A}_a$, $a > 0$ (see [4]) is a $K\{M_p\}$-space with $M_p(x) = \exp[a(1-1/p)|x|^2]$ for $p \in N$, $x \in R^q$. In the last two cases, the sequence $\{M_p\}$ fulfils conditions (M), (N), (N′), (P).

3. In [2] (p. 113), elements of the dual of $K\{M_p\}$ are characterized under conditions (M), (N), (P) as functionals f of the form

$$(f, \varphi) = \sum_{|i| \leqslant p} \int\limits_{R^q} M_p F_i \varphi^{(i)} \qquad (\varphi \in K\{M_p\}) \tag{5}$$

for some $p \in N$, where F_i are bounded measurable functions.

Defining the mth derivative:

$$(g^{(m)}, \varphi) = (-1)^{|m|}(g, \varphi) \qquad (\varphi \in K\{M_p\})$$

for $g \in K\{M_p\}'$ and $m \in P^q$ (cf. [2], pp. 106–108) and identifying measurable functions G (finite almost everywhere) such that $M_p^{-1}G$ is bounded almost everywhere for some $p \in N$ with functionals $\tilde{G} \in K\{M_p\}'$ of the form

$$(\tilde{G}, \varphi) = \int_{R^q} G\varphi \qquad (\varphi \in K\{M_p\}),$$

we can express (5) in the form

$$f = \sum_{|i| \leqslant p} G_i^{(i)}, \tag{6}$$

where G_i are measurable functions (finite almost everywhere) such that $M_p^{-1}G_i$ are bounded almost everywhere ($G_i = (-1)^{|i|}M_pF_i$ on Q).

In [5], the convergence $f_n \to 0$ in $K\{M_p\}'$ is characterized, under assumptions (M), (N), (P), by the condition:

(a) there exist $p \in N$ and, for each $i \in P^q$ with $|i| \leqslant p$, a sequence of measurable functions G_{in} such that

$$f_n = \sum_{|i| \leqslant p} G_{in}^{(i)} \quad \text{and} \quad M_p^{-1}G_{in} \to 0 \quad \text{in} \quad L^2(R^q)$$

for each $i \in P^q$, $|i| \leqslant p$.

Moreover, under (M), (N), (P) and an additional assumption (F), we find in [5] another characterization:

(b) there exist $p, s \in N$ and, for each $i \in P^q$ with $|i| \leqslant p$, a sequence of continuous functions G_{in} such that

$$f_n = \sum_{|i| \leqslant s} G_{in}^{(i)} \quad \text{and} \quad \sup_{x \in R^q} \big(M_p^{-1}(x)\,|G_{in}(x)|\big) \to 0$$

for each $i \in P^q$, $|i| \leqslant p$.

Under conditions (M), (N), (N′) and (P), we shall give characterizations of elements of $K\{M_p\}'$ and of the convergence in $K\{M_p\}'$ which have a simpler form than (6) and (a)–(b) (see Sections 4 and 5).

We shall need two lemmas.

LEMMA 1. *Let $\{M_p\}$ fulfil conditions* (M), (N) *and* (N′). *Suppose that F is a continuous function on R^q for which there exists a $p \in N$ such that*

$$G \in L^\alpha(R^q) \tag{7}$$

for $G = M_p^{-1}F$ and for all α with $1 \leqslant \alpha \leqslant \infty$. Then, for each $k \in P^q$, there exist a $p_k \in N$ with $p_k \geqslant p$ and a continuous function F_k such that $F_k^{(k)} = F$ on R^q and (7) *holds for $G = M_{p_k}^{-1}F_k$ and for all α with $1 \leqslant \alpha \leqslant \infty$.*

Proof. The assertion is obvious for $k = 0 = (0, \ldots, 0)$.

Suppose that it is true for some $k \in P^q$, an integer $p_k \geqslant p$ and a continuous function F_k such that (7) holds with $G = M_{p_k}^{-1}F_k$. Let j be a fixed

index from the set $\{1, \ldots, q\}$. By (N′), there exist an integer $r > p_k$ and a constant $B_k > 0$ such that

$$\mu_k(\bar{x}) = \int_{-\infty}^{\infty} m_{p_k, r}(x)\, d\xi_j \leqslant B_k$$

for almost all $\bar{x} = (\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_q) \in R^{q-1}$. According to (N), find $s > r$ such that $m_{rs} \in L^1(R^q)$. In virtue of (M), we have

$$M_r^{-1}(x)\Big|\int_0^{\xi_j} F_k(x_\tau)\,d\tau\Big| \leqslant C_{jr}^{-1}\Big|\int_0^{\xi_j} M_r^{-1}(x_\tau)F_k(x_\tau)\,d\tau\Big|$$

$$\leqslant A_k \int_{-\infty}^{\infty} m_{p_k, r}(x_\tau)\,d\tau \leqslant A_k B_k$$

for almost all $x \in R^q$, where $x_\tau = (\xi_1, \ldots, \xi_{j-1}, \tau, \xi_{j+1}, \ldots, \xi_q)$ and $A_k = \sup_{x \in R^q} |M_{p_k}^{-1}(x)F_k(x)|$. Since the function

$$F_{k+e_j}(x) = \int_0^{\xi_j} F_k(x_\tau)\,d\tau \tag{8}$$

is continuous on R^q, we get

$$\sup_{x \in R^q} |M_s^{-1}(x)F_{k+e_j}(x)| \leqslant A_k B_k. \tag{9}$$

Moreover, for every $\alpha \geqslant 1$, we have

$$\int_{R^q} |M_s^{-1}(x)F_{k+e_j}(x)|^\alpha dx \leqslant (A_k B_k)^\alpha \int_{R^q} m_{rs}(x)\,dx < \infty, \tag{10}$$

since $m_{rs}(x) \leqslant 1$. In view of (8), (9) and (10), the assertion holds for $p_{k+e_j} = s$ and so the lemma is proved by induction.

In a similar way, one can prove the following lemma:

LEMMA 2. *Let $\{M_p\}$ fulfil conditions* (M), (N) *and* (N′). *Suppose that $\{F_n\}$ is a sequence of continuous functions on R^q for which there exists a $p \in N$ such that*

$$G_n \to 0 \quad \textit{in} \quad L^\alpha(R^q) \tag{11}$$

for $G_n = M_p^{-1}F_n$ and for all α with $1 \leqslant \alpha \leqslant \infty$. Then, for each $k \in P^q$, there exist a $p_k \in N$ with $p_k \geqslant p$ and continuous functions F_{kn} such that $F_{kn}^{(k)} = F_n$ on R^q and (11) *holds for $G_n = M_{p_k}^{-1}F_{kn}$ and for all α with $1 \leqslant \alpha \leqslant \infty$.*

4. Elements of the dual of $K\{M_p\}$ can be characterized as follows:

THEOREM 1. *Let $\{M_p\}$ satisfy conditions* (M), (N), (N′) *and* (P). *The following conditions are equivalent:*

(i) $f \in K\{M_p\}'$;

(ii) *there exist* $p \in N$, $k \in P^q$ *and a measurable* (*continuous*) *function* F *such that*

$$f = F^{(k)} \tag{12}$$

and

$$M_p^{-1} F \in L^a(R^q) \tag{13}$$

for all a *with* $1 \leqslant a \leqslant \infty$;

(iii) *there exist* $p \in N$, $k \in P^q$ *and a measurable* (*continuous*) *function* F *such that* (12) *holds and* (13) *is valid for some* a *with* $1 \leqslant a \leqslant \infty$.

Proof. Suppose that (i) holds. By Theorem in [2], p. 113, the functional f is of the form (6) for measurable functions G_i ($i \in P^q$, $|i| \leqslant p$) such that $M_p^{-1} G_i$ are bounded.

Let r and s be integers such that $s > r > p$ and $m_{pr}, m_{rs} \in L^1(R^q)$. The function

$$\tilde{G}_i(x) = \int_0^x G_i(u)\,du \quad (x \in R^q) \tag{14}$$

is continuous and, by (M),

$$\sup_{x \in R^q} |M_s^{-1}(x)\tilde{G}_i(x)| \leqslant \sup_{x \in R^q} |M_r^{-1}(x)\tilde{G}_i(x)| \leqslant A_i B C, \tag{15}$$

where

$$A_i = \sup_{u \in R^q} |M_p^{-1}(u) G_i(u)|, \quad B = \int_{R^q} m_{pr}(u)\,du$$

and

$$C = \prod_{j=1}^{q} C_{jr}.$$

Moreover, for each $a \geqslant 1$, we have

$$\int_{R^q} |M_s^{-1}(x)\tilde{G}_i(x)|^a\,dx \leqslant (A_i B C)^a \int_{R^q} m_{rs}(x)\,dx < \infty, \tag{16}$$

owing to (15) and (1). Inequalities (15) and (16) imply that $M_s^{-1}\tilde{G}_i \in L^a(R^q)$ for each i ($|i| \leqslant p$) and each a ($1 \leqslant a \leqslant \infty$).

By Lemma 1, there exist continuous functions F_i and integers $s_i \geqslant s$ such that

$$F_i^{(\tilde{p}-i)} = \tilde{G}_i, \tag{17}$$

where $\tilde{p} = (p, \ldots, p) \in P^q$, and

$$M_{s_i}^{-1} F_i \in L^a(R^q), \tag{18}$$

whenever $1 \leqslant a \leqslant \infty$.

Let $t = \max\{s_i: |i| \leqslant p\}$ and put

$$F = \sum_{|i| \leqslant p} F_i. \tag{19}$$

Clearly, F is a continuous function and

$$F^{(\tilde{p}+1)} = f,$$

which results from (19), (17), (14) and (6). To obtain (ii), it remains to note that

$$M_t^{-1} F \in L^a(R^q) \quad (1 \leqslant a \leqslant \infty),$$

in view of (18).

Implication (ii) $\rightarrow$ (iii) is trivial.

Now, assume (iii) for $k \in P^q$, $p \in N$, a measurable function F and some a such that $1 \leqslant a \leqslant \infty$. Choosing an integer $r > p$ such that $m_{pr} \in L^1(R^q)$ and putting

$$\tilde{F}(x) = \int_0^x F(u)\,du \quad (x \in R^q),$$

we notice that G is continuous and $\tilde{F}^{(k+1)} = f$.

Moreover, applying (M), we get

$$\sup_x |M_r^{-1}(x)\tilde{F}(x)| \leqslant C \sup |M_p^{-1}(u) F(u)| \cdot \int_{R^q} m_{pr}(x)\,dx$$

in the case $a = \infty$, and

$$\sup_x |M_r^{-1}(x)\tilde{F}(x)| \leqslant C \Big[\int_{R^q} (m_{pr})^\beta\Big]^{1/\beta} \cdot \Big[\int_{R^q} |M_p^{-1} F|^a\Big]^{1/a}$$

in the case $a < \infty$, where $C = C_{1r} \ldots C_{qr}$ and $\beta = a/(1-a)$.

Consequently, condition (ii) holds for a continuous function F such that (13) is valid for $a = \infty$. This already yields condition (i). In fact, if (12) and (13) hold for $a = \infty$, then the functional

$$(f, \varphi) = (-1)^k \int_{R^q} F\varphi^{(k)} \quad (\varphi \in K\{M_p\})$$

is linear and continuous on $K\{M_p\}$. The proof is complete.

5. The following theorem describes the convergence in the dual of $K\{M_p\}$.

THEOREM 2. *Let* $\{M_p\}$ *satisfy conditions* (M), (N), (N′) *and* (P) *and let* $f_n \in K\{M_p\}'$ *for* $n \in N$. *The following conditions are equivalent*:

(i) $f_n \rightarrow 0$ *weakly* (*strongly*) *in* $K\{M_p\}'$;

(ii) *there exist* $p \in N$, $k \in P^q$ *and measurable* (*continuous*) *functions* F_n *such that*

$$f_n = F_n^{(k)} \tag{20}$$

and

(21) $$M_p^{-1}F_n \to 0 \quad \text{in} \quad L^a(R^q)$$

for all a *with* $1 \leqslant a \leqslant \infty$;

(iii) *there exist* $p \in N$, $k \in P^q$ *and measurable (continuous) functions* F_n *such that* (20) *and* (21) *hold for some* a *with* $1 \leqslant a \leqslant \infty$;

(iv) *there exist* $p \in N$, $k \in P^q$ *and measurable functions* F_n *such that* (20) *holds, the functions* $M_p^{-1}F_n$ *are commonly bounded (almost everywhere in* R^q*) and* $F_n \to 0$ *almost everywhere in* R^q.

Proof. Assume that (i) holds. In view of Theorem 6 in [5], condition (a) holds for some $p \in N$ and some sequences of measurable functions G_{in}, where $|i| \leqslant p$ (see Section 3).

As in the proof of Theorem 1, we choose integers r, s such that $p < r < s$ and $m_{pr}, m_{rs} \in L^1(R^q)$. The functions

(22) $$\tilde{G}_{in}(x) = \int_0^x G_{in}(u)\,du \qquad (x \in R^q)$$

are continuous and, in virtue of (M),

$$\sup_x |M_s^{-1}(x)\tilde{G}_{in}(x)| \leqslant \sup |M_r^{-1}(x)\tilde{G}_{in}(x)|$$
$$\leqslant C\int_{R^q} m_{pr}M_p^{-1}|G_{in}| \leqslant \varepsilon_n \to 0,$$

where

$$C = C_{1r}\ldots C_{qr} \quad \text{and} \quad \varepsilon_n = \Big(\int_{R^q} m_{pr}\Big)^{1/2}\Big(\int_{R^q} M_p^{-2}|G_{in}|^2\Big)^{1/2}\cdot C.$$

Hence we have also

$$\int_{R^q} |M_s^{-1}(x)\tilde{G}_{in}(x)|^a\,dx \leqslant \varepsilon_n^a \int_{R^q} m_{rs} \to 0$$

whenever $1 \leqslant a < \infty$. Now, we apply Lemma 2, finding continuous functions F_{in} and integers $s_i \geqslant s$ such that

(23) $$F_{in}^{(\tilde{p}-i)} = \tilde{G}_{in},$$

where $\tilde{p} = (p, \ldots, p) \in P^q$, and

(24) $$M_t^{-1}F_{in} \to 0 \quad \text{in} \quad L^a(R^q)$$

for each i ($|i| \leqslant p$) and a ($1 \leqslant a \leqslant \infty$), where $t = \max\{s_i\colon |i| \leqslant p\}$.

Putting

$$F_n = \sum_{|i| \leqslant p} F_{in},$$

and using (22), (23) and (b), we obtain (20) for $k = \tilde{p}+1$. Moreover, (21) holds for $p = t$ and all a such that $1 \leqslant a \leqslant \infty$. Thus the implication (i) $\to$ (ii) is proved.

The implication (ii) $\to$ (iii) is evident.

Suppose that condition (iii) is fulfilled for some $k \in P^q$, $p \in N$, $1 \leqslant a \leqslant \infty$ and measurable functions F_n. It is easy to see that for the functions

$$\tilde{F}_n(x) = \int_0^x F_n(u)\,du \qquad (x \in R^q)$$

we have $\tilde{F}_n^{(k+1)} = F_n$ and

(25) $$M_r^{-1}\tilde{F}_n \to 0 \quad \text{in} \quad L^\infty(R^q)$$

for any $r > p$ such that $m_{pr} \in L^1(R^q)$. In particular, relation (25) implies the pointwise convergence and boundedness of the sequence $\{M_r^{-1}\tilde{F}_n\}$, and so (iv) follows from (iii).

Finally, note that (iv) implies (i). In fact, if (20) holds for measurable functions F_n such that for some $p \in N$ the sequence $\{M_r^{-1}F_n\}$ is bounded, then the functionals

$$(f_n, \varphi) = (-1)^{|k|}\int_{R^q} F_n(u)\varphi^{(k)}(u)\,du \qquad (\varphi \in K\{M_p\})$$

are linear and continuous on $K\{M_p\}$ and, by the Lebesgue theoremf

$$(f_n, \varphi) \to 0$$

or every $\varphi \in K\{M_p\}$, i.e., $f_n \to 0$ weakly in $K\{M_p\}'$. Thus Theorem 2 is proved.

Remark. Theorem 2 supplies descriptions of the convergence in various spaces of distributions, for instance in the duals of the spaces mentioned in the Examples in Section 2: $\mathcal{D}'_K$, $\mathcal{S}'$ (see [1]), $\mathcal{D}'_{\mathcal{A}_a}$ (see [4]), H'_r and $\mathcal{K}'_r$. Let us write, e.g., the following characterizations of the convergence in $\mathcal{K}'_r$ for $r \geqslant 1$.

Corollary. *Let* $f_n \in \mathcal{K}'_r$ $(n \in N)$ *for some* $r \geqslant 1$. *The following conditions are equivalent:*

(i) $f_n \to 0$ *weakly (strongly) in* $\mathcal{K}'_r$;

(ii) *there exist* $k \in P^q$, $p \in N$, $1 \leqslant a \leqslant \infty$ *and measurable (continuous) functions* $F_n \in L^a(R^q)$ *such that*

(26) $$f_n(x) = [e^{p|x|^r}F_n(x)]^{(k)}$$

and

$$F_n \to 0 \quad \text{in} \quad L^a(R^q);$$

(iii) *there exist* $k \in P^q$, $p \in N$ *and continuous bounded functions* F_n *such that* (26) *holds and* $F_n \to 0$ *uniformly in* R^q;

(iv) *there exist* $k \in P^q$, $p \in N$ *and measurable functions* F_n *such that* $f_n = F_n^{(k)}$, $F_n \to 0$ *almost everywhere in* R^q *and*

$$|F_n(x)| < C\exp(p|x|^r)$$

for some $C > 0$ *and almost all* $x \in R^q$.

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Sobolev's and local derivatives

by

KRYSTYNA SKÓRNIK (Katowice)

Dedicated to my teacher
Professor Jan Mikusiński
on his 70th birthday

Abstract. The paper deals with local derivatives of functions of several variables having values in a fixed Banach space. It is shown that the local derivative and Sobolev's derivative are equivalent.

Local derivatives of functions of one real variable with values in a Banach space were considered by J. Mikusiński in [2], and earlier, in [4], local derivatives of functions of q real variables with values in a Hilbert space were introduced. In [6] the author gave a list of properties, theorems and also some comments concerning local derivatives. The functions considered in this paper are defined in a q-dimensional Euclidean space R^q; their values are elements of a Banach space $\mathcal{X}$.

By a local derivative of a function f of a real variable we mean the local limit of the expression

$$\frac{1}{h}[f(x+h)-f(x)]$$

as $h \to 0$. In other words, g is a local derivative of f if

$$\lim_{h\to 0}\int_a^b \left|\frac{1}{h}[f(x+h)-f(x)]-g(x)\right| dx = 0 \tag{1}$$

holds for every bounded interval (a, b). (We assume that the integrand in (1) is locally integrable).

Let f and g be locally integrable functions on an open set $\mathcal{O}$. If, for each real valued infinitely derivable function φ with bounded support in $\mathcal{O}$, the following equality holds:

$$\int g\varphi\,dx = -\int f\varphi'\,dx,$$

then g is called *weak derivative of* f or *Sobolev's derivative* f' of f (cf. [3], p. 172).