in this paper may be extended to functions of several variables in a straightforward manner, upon replacing the various series by corresponding series of products representing each variable one at a time. Many quantities merely require a vector interpretation of the indices of summation.

References

[1] R. R. Goldberg, *Fourier Transforms*, Cambridge University Press, Cambridge 1965, p. 21.
[2] J. Mikusiński, *The Bochner Integral*, Academic Press, New York–San Francisco 1978, p. 3.

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Some Fourier transform inversion theorem

by

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Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. It is shown that if $x^{\mu}\frac{\partial^{|\nu|}}{\partial x^{\nu}}f \in L^2(R^q)$ for $|\mu+\nu| \leqslant k$, $k > q/2$ then $(\mathscr{F}^{-1}\circ\mathscr{F})f(x) = f(x)$ for $x \in R^q$, where $\mathscr{F}f$ and $\mathscr{F}^{-1}f$ denote the Fourier integral and the inversion Fourier integral of f.

It is well known that the operation of differentiation of functions corresponds to the operation of multiplication by the argument of their Fourier transforms. This makes it possible to look for solutions of differential equations by means of the Fourier transformation. However, this method of solving differential equations is useful provided the inversion formula for the Fourier transform can be applied.

The Fourier transform of a function of the class $\mathscr{S}(R^q)$ of rapidly decreasing smooth functions is defined by means of the formula

$$\mathscr{F}f(\xi) = (2\pi)^{-q/2}\int_{R^q} e^{i\langle x,\xi\rangle}f(x)\,dx, \tag{1}$$
$$\langle x, \xi\rangle = x_1\xi_1+\ldots+x_q\xi_q; \qquad x, \xi \in R^q.$$

Then, the inversion formula is expressed in the form:

$$\mathscr{F}^{-1}f(\xi) = (2\pi)^{-q/2}\int_{R^q} e^{-i\langle x,\xi\rangle}f(x)\,dx. \tag{2}$$

For f in $\mathscr{S}(R^q)$, we have

$$(\mathscr{F}^{-1}\circ\mathscr{F})f(x) = (\mathscr{F}\circ\mathscr{F}^{-1})f(x) \quad \text{for each } x \in R^q. \tag{3}$$

We can ask for what other functions formula (3) holds, where the Fourier transform and the inversion Fourier transform are given by Fourier integrals (1) and (2), respectively. The author knows only the following

THEOREM 1. *If $f \in L^1(R^q)$ and $\mathscr{F}f \in L^1(R^q)$, then*

$$(\mathscr{F}^{-1}\circ\mathscr{F})f(x) = f(x) \text{ almost everywhere.}$$

Proof. See [2], p. 1005 and [4], p. 186.

This theorem is not useful for applications, because we have to check whether the Fourier transform $\mathscr{F}f$ of a given function f of the class $L^1(R^q)$ is also in $L^1(R^q)$.

In this note we give an example of a space larger than $\mathscr{S}(R^q)$ such that formula (3) holds for its elements.

Let us consider the differential operator

$$\mathfrak{M} = -\frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_q^2} + x_1^2 + \ldots + x_q^2 + qI,$$

where I denotes the identical operator. We shall take the space $\mathscr{S}(R^q)$ as the domain of $\mathfrak{M}$. Integration by parts shows that for each f and g belonging to $\mathscr{S}(R^q)$ the relations

$$(\mathfrak{M}f, g) = (f, \mathfrak{M}g), \tag{4}$$

$$(f, f) \leqslant (\mathfrak{M}f, f) \tag{5}$$

hold. The symbol (f, g) denotes the inner product of f and g in $L^2(R^q)$. By induction on k, we have

$$(\mathfrak{M}^k f, f) \leqslant (\mathfrak{M}^{k+1} f, f) \quad \text{for } f \text{ in } \mathscr{S}(R^q). \tag{6}$$

From (5) and (6), we infer that $(\mathfrak{M}^k\cdot, \cdot)$ is a positive-definite Hermitean form on $\mathscr{S}(R^q)$. To each Hermitean form $(\mathfrak{M}^k\cdot, \cdot)$ corresponds the norm $\|\cdot\|_k$, $\|f\|_k^2 = (\mathfrak{M}^k f, f)$. These norms satisfy the following inequalities:

$$\|f\|_{L^2} = \|f\|_0 \leqslant \|f\|_1 \leqslant \ldots \leqslant \|f\|_k \leqslant \ldots \tag{7}$$

Let us denote by $\mathscr{S}_k(R^q)$ the completion of $\mathscr{S}(R^q)$ with respect to the norm $\|\cdot\|_k$. By inequalities (7), the Hilbert spaces $\mathscr{S}_k(R^q)$ (with respect to the norms (7)) satisfy the following relations:

$$L^2(R^q) = \mathscr{S}_0(R^q) \supset \mathscr{S}_1(R^q) \supset \ldots \supset \mathscr{S}_k(R^q) \supset \ldots \tag{8}$$

The Hermitean functions

$$h_n(t) = \frac{(-1)^n}{\sqrt{n!\,2^n\sqrt{2\pi}}} e^{-t^2/2} H_{n-1}(t),$$

where $H_n(t) = (-1)^n e^{t^2} (e^{-t^2})^{(n)}$, $n = 0, 1, 2, \ldots$, are eigenfunctions of the operator $-\frac{d^2}{dt^2} + t^2 + 1$ corresponding to the eigenvalues $2n$. The functions h_n, $n = 1, 2, 3, \ldots$ form an orthogonal complete system in each space $\mathscr{S}_k(R^1)$ ([3]). The functions $h_\nu(x) = h_{\nu_1}(x_1)\ldots h_{\nu_q}(x_q)$, $\nu = (\nu_1, \ldots, \nu_q) \in N^q$ are eigenfunctions of $\mathfrak{M}$ corresponding to eigenvalues $2|\nu|$, $|\nu| = \nu_1 + \ldots + \nu_q$. By using the Fubini theorem, we can show, as in the one-dimensional case, that the system $\{h_\nu: \nu \in N^q\}$ is an orthogonal complete system in each space $\mathscr{S}_k(R^q)$. It is easy to show that

$$\mathscr{F}(h_\nu) = i^{|\nu|} h_\nu, \quad \nu \in N^q \quad ([1], \text{ p. } 147). \tag{9}$$

This implies that the Fourier transformation is an isometry of the linear space $\mathscr{S}_k^0(R^q)$ spanned by the set $\{h_\nu: \nu \in N^q\}$ with respect to the norm $\|\cdot\|_k$. Since the space $\mathscr{S}_k^0(R^q)$ is a dense set in $\mathscr{S}_k(R^q)$, the Fourier transformation of $\mathscr{S}_k^0(R^q)$ onto $\mathscr{S}_k^0(R^q)$ can be extended to an isometry of $\mathscr{S}_k(R^q)$ onto $\mathscr{S}_k(R^q)$. This isometry is called to *Fourier transformation* of $\mathscr{S}_k(R^q)$.

Obviously, if $\mathscr{S}_k(R^q)$ is contained in $L^1(R^q)$, then the Fourier transformation on $\mathscr{S}_k(R^q)$ is represented by the Fourier integral (1). Our statements indicate that identity (3) is true if $\mathscr{S}_k(R^q) \subset L^1(R^q)$.

The following theorem gives some characterization of elements belonging to $\mathscr{S}_k(R^q)$.

THEOREM 2. *The function f belonging to $L^2(R^q)$ is in $\mathscr{S}_k(R^q)$ if and only if its distributional derivatives $D^\nu f$ satisfy the following condition: $T^\mu D^\nu f \in L^2(R^q)$ for $|\mu+\nu| \leqslant k$, where $T^\mu f(x) = x_1^{\mu_1}\ldots x_q^{\mu_q} f(x)$, $x = (x_1, \ldots, x_q)$.*

Proof. See [3].

In the sequel we shall need the following

LEMMA. *If f is a measurable function such that $T^\mu f \in L^2(R^q)$ for $|\mu| \leqslant k$, $k > q/2$, then $f \in L^1(R^q)$.*

Proof. Let $T^\mu f \in L^2(R^q)$ for $|\mu| \leqslant k$, $k > q/2$; then we have

$$\int_{R^q} |f(x)|\,dx = \int_{R^q} (1+|x|^2)^{-k/2}(1+|x|^2)^{k/2}|f(x)|\,dx$$
$$\leqslant \Big[\int_{R^q} (1+|x|^2)^{-k}\,dx\Big]^{1/2}\Big[\int_{R^q} (1+|x|)^k |f(x)|^2\,dx\Big]^{1/2}.$$

The first integral on the right-hand side is finite. We put

$$C = \Big[\int_{R^q} (1+|x|^2)^{-k}\,dx\Big]^{1/2};$$

then we have

$$\int_{R^q} |f(x)|\,dx \leqslant C\Big[\int_{R^q} (1+x_1^2+\ldots+x_q^2)^k |f(x)|^2\,dx\Big]^{1/2}$$
$$\leqslant C\cdot \sum_{|\tilde{\mu}|=k} \frac{k!}{\mu_0!\mu!} \|T^\mu f\|_{L^2(R^q)},$$

where $\tilde{\mu} = (\mu_0, \mu_1, \ldots, \mu_q)$, $\mu = (\mu_1, \ldots, \mu_q)$. This finishes the proof of the lemma.

From Theorem 2 and the lemma we obtain

COROLLARY. *If $k > q/2$, then $\mathscr{S}_k(R^q) \subset L^1(R^q)$.*

Our considerations lead to the following

THEOREM 3. *If $k > q/2$, then formula (3) is true for f belonging to $\mathscr{S}_k(R^q)$.*

References

[1] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of Distributions*, Elsevier Scientific Publishing Company, Amsterdam and PWN, Warszawa 1973.

[2] R.E. Edwards, *Functional Analysis*, Mir, Moskva 1969 [Russian translation].

[3] W. Kierat, *Applications of quadratic forms generated by Sturm–Liouville operators in the theory of generalized functions*, Prace Naukowe Uniwersytetu Śląskiego 388, Katowice 1980.

[4] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, New York 1966.

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On some spaces of distributions

by

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Abstract. The spaces $\mathscr{D}'_{\mathscr{A}_a}$ $(a > 0)$ of distributions are considered as a generalization of the space $\mathscr{D}'_{\mathscr{A}}$ introduced by Z. Sadlok and Z. Tyc. Some equivalent descriptions of the class $\mathscr{D}'_{\mathscr{A}_a}$ and of the convergence in $\mathscr{D}'_{\mathscr{A}_a}$ are given in terms of distributional and tempered derivatives. In particular, the spaces $\mathscr{D}'_{\mathscr{A}_a}$ turn out to be subspaces of the space $\mathscr{K}'_2$, introduced by G. Sampson and Z. Zieleźny, such that $\mathscr{K}'_2$ is an inductive limit of $\mathscr{D}'_{\mathscr{A}_p}$ $(p = 1, 2, \ldots)$. As a consequence, a characterization of the convergence in $\mathscr{K}'_2$ is obtained.

1. In connection with the theory of Hermite expansions of distributions (see [1]), Z. Sadlok and Z. Tyc introduced the class $\mathscr{D}'_{\mathscr{A}}$ of distributions such that 1° $\mathscr{D}'_{\mathscr{A}} \supset \mathscr{S}'$, 2° the Hermite coefficients $a_n = (f, h_n)$, where h_n are the Hermite functions, are uniquely defined for all distributions $f \in \mathscr{D}'_{\mathscr{A}}$ (see [6]).

Distributions of the class $\mathscr{D}'_{\mathscr{A}}$ were introduced in [6] as tempered derivatives of some order (for the definition see [1], p. 175) of functions belonging to some class $\mathscr{A}$. It is rather strange that the number 4 appears in the definition of the class $\mathscr{A}$, given in [6]. This is connected with the choice, made in [1] and [6], of constants in the definitions of the Hermite functions and tempered derivatives. However, the choice of constants is meaningless for the theory of Hermite expansions.

In this paper, we consider the general case, introducing the classes $\mathscr{A}_a$ of functions and $\mathscr{D}'_{\mathscr{A}_a}$ of distributions for arbitrary $a > 0$. For $a = \frac{1}{4}$, these classes coincide with $\mathscr{A}$ and $\mathscr{D}'_{\mathscr{A}}$, respectively.

We give several characterizations of distributions belonging to $\mathscr{D}'_{\mathscr{A}_a}$ in terms of distributional derivatives (Section 3).

Moreover, we introduce the convergence in $\mathscr{D}'_{\mathscr{A}_a}$ in some equivalent ways, by using distributional and tempered derivatives (Section 4).

It should be noted that the space $\mathscr{D}'_{\mathscr{A}_a}$ is of type $K\{M_p\}'$ in the sense of [2] and the convergence in $\mathscr{D}'_{\mathscr{A}_a}$ coincides with the weak and strong