STUDIA MATHEMATICA, T. LXXVII. (1983)

On some basic relations in the abstract Hardy algebra theory

by

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To Professor J. Mikusiński on the 70th birthday

Abstract. The present paper is devoted to the theory of function algebras. It deals with the abstract Hardy algebra theory, that is the local part of the abstract analytic function theory in the sense of Barbey–König (Lecture Notes in Math. Vol. 593, Springer, 1977). The paper extends the basic representation theorem for the maximum functional over the representative functions from the bounded to arbitrary real-valued measurable functions. This result has numerous applications, in particular to the abstract conjugation. The paper also contains a representation theorem for certain conjugable functions in terms of limits of isotone sequences which seems to be new even in the classical unit disk situation. It permits a direct development of the abstract conjugation theory.

The present paper is devoted to the abstract Hardy algebra theory as developped in Barbey–König [1] and summarized in König [2]. We adopt the former definitions and notation. Let (H, φ) be a Hardy algebra situation on the nonzero finite positive measure space (X, Σ, m); it is defined to consist of a weak* closed complex subalgebra $H \subset L^\infty(m)$ which contains the constants, and of a nonzero weak* continuous multiplicative linear functional $\varphi\colon H \to C$. The class $M \subset \mathrm{Re}\, L^1(m)$ is defined to consist of the nonnegative functions $V \in \mathrm{Re}\, L^1(m)$ with $\varphi(u) = \int uV\,dm\ \forall u \in H$. We assume (H, φ) to be reduced, that is there exist functions $V \in M$ such that $V > 0$ on the entire space. We also recall the function classes $L^\#$ und $H^\#$ with the extended functional $\varphi\colon H^\# \to C$ and the function class H^+ which dominate the presentation in [1].

A central result in the abstract Hardy algebra theory is the equation

$$\mathrm{Sup}\{\textstyle\int PV\,dm\colon\ V \in M\} = \mathrm{Inf}\{\mathrm{Re}\,\varphi(u)\colon\ u \in H \text{ with } \mathrm{Re}\, u \geqslant P\} \quad \forall P \in \mathrm{Re}\, L^\infty(m);$$

see [1], IV. 3.10 with IV. 2.5. In the present paper we shall prove in Section 2 that the above equation extends, after appropriate redefinition of the two terms involved, to all $P \in \mathrm{Re}\, L(m)$, where $L(m)$ is defined to consist of all equivalence classes modulo m of measurable complex-valued functions. This result has numerous applications some of which will be presented in Sections 3 and 4.

In Section 3 we shall obtain the well-known modification lemma without the usual boundedness assumption; compare with [4], Lemma 2.2. Next consider the norm-like functional

$$|||\cdot|||: \ |||P||| = \mathrm{Sup}\{\textstyle\int |P|V\,dm:\ V \in M\} \leqslant \infty \quad \forall P \in L(m).$$

An example shows that the functions $P \in L(m)$ with $|||P||| < \infty$ need not be in the $|||\cdot|||$ closure of $L^\infty(m)$. However, we shall prove that this is true for the powers $|P|^\tau$ with $0 < \tau < 1$.

Section 4 is devoted to applications to the abstract conjugation. In [1], Chapter VI, the conjugation is based upon $H^\#$:

A function $P \in \mathrm{Re}L(m)$ is said to be *conjugable* iff there exists a function $Q \in \mathrm{Re}L(m)$ such that $e^{t(P+iQ)} \in H^\# \ \forall t \in \boldsymbol{R}$. The function Q is then unique up to an additive real constant, and unique under the extra condition that $\varphi(e^{t(P+iQ)}) = e^{t\alpha(P)} \ \forall t \in \boldsymbol{R}$ for some unique real $\alpha(P)$, and named the *conjugate function* to P.

The class E of conjugable functions is closed with respect to $|||\cdot|||$ by [1], VI. 2.4 with IV. 3.9. It is obvious that $\mathrm{Re}H \subset E$, and that for $P \in \mathrm{Re}H$ the conjugate function is the unique $Q \in \mathrm{Re}H$ with $P+iQ \in H$ and $\mathrm{Im}\varphi(P+iQ) = 0$. Hence E contains the $|||\cdot|||$ closure of $\mathrm{Re}H$; it is unknown whether this closure is equal to E.

It is a natural question to characterize the conjugable functions in terms of M. At present no complete answer is known. Define a function $P \in L(m)$ to be *univalent* iff $|||P||| < \infty$ and $\int PV\,dm$ is independent of $V \in M$. Of course the functions in $\mathrm{Re}H$ are univalent, and hence the functions in the $|||\cdot|||$ closure of $\mathrm{Re}H$ as well.

One could expect that E coincides with the class of univalent functions. In the sixties several authors observed that a bounded function $P \in \mathrm{Re}L^\infty(m)$ is conjugable iff it is univalent. Then in [1], VI. 3.8, it is proved that a function $P \in \mathrm{Re}L(m)$ in the $|||\cdot|||$ closure of $\mathrm{Re}L^\infty(m)$ is conjugable iff it is univalent; but it is unknown whether E is contained in that closure, and even whether $|||P||| < \infty$ for all $P \in E$. In [1] the last theorem has a complicated proof, while in Section 4 below it becomes a rapid consequence of our basic results in Section 2. Furthermore we shall prove certain representations of $\alpha(P)$ for $P \in E$ under less restrictive conditions, which permit to obtain univalence results as well.

In Section 4 we shall also prove the Kolmogorov estimation for all conjugable functions; for the special cases known so far see [1], VI. 5.1 and VI. 7.1 and [7], Satz 2.6.

The final Section 5 is related to the doctoral thesis of Loch [6]; see also [7]. Loch develops an alternative version of the abstract conjugation, in that he substitutes for $H^\#$ a certain subclass $H^* \subset H^\#$ but preserves the form of the definition.

It turns out that the class E^* of conjugable functions in the sense of Loch consists of the functions $P \in E$ which are in the $|||\cdot|||$ closure of $\mathrm{Re}L^\infty(m)$. Hence by the result from [1] discussed above E^* consists of the functions $P \in \mathrm{Re}L(m)$ in the $|||\cdot|||$ closure of $\mathrm{Re}L^\infty(m)$ which are univalent. Thus E^* can be characterized in terms of M, in contrast to E. Loch obtains two further remarkable results: E^* is the $|||\cdot|||$ closure of $\mathrm{Re}H$; and each $P \in E^*$ can be represented as the difference of two nonnegative functions in E^*, a statement which is open for E.

In his work the basic tool is the function class $H^\wedge$, defined to consist of the limits (to be taken in the appropriate sense) of the sequences of functions $h_n \in H$ $(n = 1, 2, \ldots)$ such that the $\mathrm{Re}h_n \geqslant 0$ form an increasing sequence and the $\varphi(h_n) = \mathrm{Re}\varphi(h_n)$ form a bounded sequence. It is clear that $\mathrm{Re}H^\wedge \subset E^*$, and that for $P \in \mathrm{Re}H^\wedge$ the conjugate function is the unique $Q \in \mathrm{Re}L(m)$ with $P+iQ \in H^\wedge$. Hence $\mathrm{Re}H^\wedge - \mathrm{Re}H^\wedge \subset E^*$.

Now the main theorem in Section 5 is that in fact $E^* = \mathrm{Re}H^\wedge - \mathrm{Re}H^\wedge$, in the sharpened sense that for nonnegative $P \in E^*$ the second term can be chosen arbitrarily small with respect to $|||\cdot|||$. The theorem implies at once the two results of Loch quoted above. It is a representation theorem of key nature since for the first time it opens the road to develop the abstract conjugation in a direct manner as in the classical unit disk situation, and, in particular, without the idea to invoke the exponential function. Let us mention that, in order to be as short and simple as possible, we shall develop our results without explicit occurrence of H^* and E^*; for the connection it suffices to refer to [6] or [7].

In the meantime our main theorem in Section 5 could have been transferred to a different and more abstract context: to the Daniell–Stone theory of integration. A connection with the abstract Hardy algebra theory became clear which led to a new idea how to do the integral extension procedure without the lattice condition, and the above theorem turned into a representation theorem for integrable functions. For this topic we refer to Leinert [5] and König [3].

1. Preliminaries on the functionals ϑ and θ. Let (H, φ) be a reduced Hardy algebra situation on the nonzero finite positive measure space (X, Σ, m). We define the functional ϑ: $\mathrm{Re}L(m) \to [-\infty, \infty]$ to be

$$\vartheta(P) = \mathrm{Sup}\{\textstyle\int PV\,dm:\ V \in M \text{ such that } \int PV\,dm\,\exists\} \ \forall P \in \mathrm{Re}L(m),$$

where $\exists$ means existence in the extended sense, that is $\int P^+V\,dm$ and $\int P^-V\,dm$ are not both $= \infty$. We adopt the usual conventions $\mathrm{Inf}\varnothing := \infty$ and $\mathrm{Sup}\varnothing := -\infty$, also $0(\pm\infty) := 0$ and $\infty + (-\infty) := \infty$; one verifies that $u \leqslant v \Rightarrow u + x \leqslant v + x \ \forall u, v, x \in [-\infty, \infty]$.

1.1. PROPERTIES. (i) $\mathrm{Inf}P \leqslant \vartheta(P) \leqslant \mathrm{Sup}P$.

(ii) $P \leqslant Q \Rightarrow \vartheta(P) \leqslant \vartheta(Q)$.

(iii) $\vartheta(P+Q) \leqslant \vartheta(P)+\vartheta(Q)$ *if* $\int Q^- V\,dm < \infty$ $\forall\, V \in M$.

(iv) $\vartheta(cP) = c\vartheta(P)$ *for real* $c \geqslant 0$.

Proof. (i) is obvious.

(ii): We can assume that $\int PV\,dm$ exists in the extended sense for some $V \in M$. Let $V \in M$ be such that this is true. If $\int P^- V\,dm < \infty$, then $Q^- \leqslant P^-$ implies $\int Q^- V\,dm < \infty$, so that $\int QV\,dm$ exists in the extended sense as well and $\int PV\,dm \leqslant \int QV\,dm \leqslant \vartheta(Q)$. If $\int P^- V\,dm = \infty$, then $\int PV\,dm = -\infty \leqslant \vartheta(Q)$. Hence $\vartheta(P) \leqslant \vartheta(Q)$.

(iii): We can assume that $\vartheta(Q) < \infty$ and hence $\int |Q|\,V\,dm < \infty$ $\forall\, V \in M$. For $V \in M$, $\int (P+Q)V\,dm$ exists in the extended sense iff $\int PV\,dm$ exists in the extended sense, and in this case $\int (P+Q)V\,dm = \int PV\,dm + \int QV\,dm \leqslant \int PV\,dm + \vartheta(Q)$. From this the assertion follows.

(iv) is obvious for $c > 0$ and true for $c = 0$ in view of the conventions.

1.2. Remark. *For* $P \in \mathrm{Re}L(m)$ *bounded below we have* $\vartheta(\mathrm{Min}(P,t))\uparrow \uparrow\vartheta(P)$ *for* $t\uparrow\infty$.

Proof. If $a < \vartheta(P)$, then $a < \int PV\,dm$ for some $V \in M$. Hence after the Beppo-Levi theorem, $a < \int \mathrm{Min}(P,t)V\,dm \leqslant \vartheta(\mathrm{Min}(P,t))$ for t sufficiently large. The assertion follows.

Next we define the modified functional $\theta\colon \mathrm{Re}L(m) \to [-\infty, \infty]$ to be

$$\theta(P) = \lim_{t\downarrow-\infty} \vartheta(\mathrm{Max}(P,t)) \qquad \forall\, P \in \mathrm{Re}L(m).$$

We list the properties which follow from 1.1.

1.3. Properties. (i) $\mathrm{Inf} P \leqslant \vartheta(P) \leqslant \theta(P) \leqslant \mathrm{Sup} P$. *For* $P \in \mathrm{Re}L(m)$ *bounded below we have* $\vartheta(P) = \theta(P)$.

(ii) $P \leqslant Q \Rightarrow \theta(P) \leqslant \theta(Q)$.

(iii) $\theta(P+Q) \leqslant \theta(P)+\theta(Q)$ *in all cases.*

(iv) $\theta(cP) = c\theta(P)$ *for real* $c \geqslant 0$.

The norm-like functional $|||\cdot|||\colon L(m) \to [0, \infty]$ defined in the Introduction is therefore

$$|||P||| = \vartheta(|P|) = \theta(|P|) \qquad \forall\, P \in L(m).$$

The reducedness assumption implies that $|||P||| = 0 \Rightarrow P = 0$. Note that $\int |P|\,V\,dm < \infty$ $\forall\, V \in M$ implies that $|||P||| < \infty$; for otherwise there were $V_n \in M$ with $\int |P|\,V_n\,dm > 2^n$ $(n = 1, 2, \ldots)$, which would lead to $\int |P|\,V\,dm = \infty$ for $V := \sum_{n=1}^{\infty} \frac{1}{2^n} V_n \in M$.

Moreover, we have reason to introduce the functional $\vartheta_+\colon \mathrm{Re}L(m) \to [0, \infty]$ defined to be

$$\vartheta_+(P) = \lim_{t\uparrow\infty} \vartheta((P-t)^+) \qquad \forall\, P \in \mathrm{Re}L(m).$$

Its importance will be clear from the next remark.

1.4. Remark. (i) *A function* $P \in \mathrm{Re}L(m)$ *is in the* $|||\cdot|||$ *closure of* $\{f \in \mathrm{Re}L(m)\colon f$ bounded above$\}$ *iff* $\vartheta_+(P) = 0$.

(ii) *A function* $P \in \mathrm{Re}L(m)$ *is in the* $|||\cdot|||$ *closure of* $\mathrm{Re}L^\infty(m)$ *iff* $\vartheta_+(P) = \vartheta_+(-P) = 0$.

Proof. (i): For $t \in \boldsymbol{R}$ we have $P - \mathrm{Min}(P,t) = (P-t)^+ \geqslant 0$ and hence $|||P - \mathrm{Min}(P,t)||| = \vartheta((P-t)^+)$. On the other hand, for $f \in \mathrm{Re}L(m)$ bounded above and $t \geqslant f$ we have $P - t \leqslant P - f \leqslant |P-f|$ and hence $(P-t)^+ \leqslant |P-f|$, so that $\vartheta((P-t)^+) \leqslant |||P-f|||$. The assertion follows.

(ii): For $t > 0$ one verifies that

$$|P - \mathrm{Min}(\mathrm{Max}(P, -t), t)| = \mathrm{Max}((P-t)^+, (-P-t)^+)$$

and hence

$$|||P - \mathrm{Min}(\mathrm{Max}(P, -t), t)||| \leqslant \vartheta((P-t)^+) + \vartheta((-P-t)^+).$$

On the other hand, for $f \in \mathrm{Re}L^\infty(m)$ and $t \geqslant |f|$ we obtain $\vartheta((\pm P - t)^+) \leqslant |||P-f|||$ as above. The assertion follows.

1.5. Remark. *For* $P \in \mathrm{Re}L(m)$ *the following conditions are equivalent:*

(i) $\theta(P) < \infty$.

(ii) $\vartheta(\mathrm{Max}(P,t)) < \infty$ *for some* $t \in \boldsymbol{R}$ *and hence* $\forall\, t \in \boldsymbol{R}$.

(iii) $\vartheta((P-t)^+) < \infty$ *for some* $t \in \boldsymbol{R}$ *and hence* $\forall\, t \in \boldsymbol{R}$.

(iv) $\vartheta_+(P) < \infty$.

Proof. We have $(P-t)^+ = \mathrm{Max}(P,t) - t$ for $t \in \boldsymbol{R}$. Furthermore for $s < t$ we have

$$\mathrm{Max}(P,s) \leqslant \mathrm{Max}(P,t) \leqslant \mathrm{Max}(P,s) + (t-s),$$

$$(P-s)^+ - (t-s) \leqslant (P-t)^+ \leqslant (P-s)^+.$$

From these relations and from the definitions the assertions follow.

We turn to the basic relations between ϑ and θ in terms of ϑ_+. They can be essentially sharpened under the assumption that $M \subset \mathrm{Re}L^1(m)$ is weakly compact.

1.6. Proposition. (i) $\theta(P) \leqslant \vartheta(P) + \vartheta_+(-P)$ *for all* $P \in \mathrm{Re}L(m)$.

(ii) *Assume that* M *is weakly compact. Then* $\theta(P) \leqslant \vartheta(P) + \vartheta_+(P)$ *for all* $P \in \mathrm{Re}L(m)$.

Proof. (i): For $t \in \boldsymbol{R}$ we have

$$\mathrm{Max}(P,t) = P + (t-P)^+ = P + (-P-(-t))^+$$

and hence

$$\theta(P) \leqslant \vartheta(\mathrm{Max}(P,t)) \leqslant \vartheta(P) + \vartheta((-P-(-t))^+)$$

by 1.1 (iii). The assertion follows.

(ii): We first prove that $\theta(P) \leqslant \vartheta(P)$ and hence $\theta(P) = \vartheta(P)$ if P is bounded above. We have

$$P_t := \operatorname{Max}(P, t) \in \operatorname{Re} L^\infty(m) \qquad \forall t \in \boldsymbol{R}.$$

Fix a real $a > \vartheta(P)$. For each $V \in M$, $\int PV\,dm \leqslant \vartheta(P) < a$, so that by Beppo Levi there exists $t(V) \in \boldsymbol{R}$ with $\int P_{t(V)} V\,dm < a$. Then

$$M(V) := \{W \in M : \int P_{t(V)} W\,dm < a\}$$

is an open subset of M in the weak topology $\sigma(\operatorname{Re} L^1(m), \operatorname{Re} L^\infty(m))|M$ which contains V. Hence there exist $V_1, \ldots, V_r \in M$ such that $M = M(V_1) \cup \ldots \cup M(V_r)$. Put $t := \operatorname{Min}(t(V_1), \ldots, t(V_r))$. Then $\int P_t V\,dm < a$ for all $V \in M$ and hence $\theta(P) \leqslant \vartheta(P_t) \leqslant a$. It follows that $\theta(P) \leqslant \vartheta(P)$. Now for an arbitrary $P \in \operatorname{Re} L(m)$ and $t \in \boldsymbol{R}$ we have $P = \operatorname{Min}(P, t) + (P-t)^+$ and hence

$$\theta(P) \leqslant \theta(\operatorname{Min}(P, t)) + \theta((P-t)^+) = \vartheta(\operatorname{Min}(P, t)) + \vartheta((P-t)^+) \leqslant \vartheta(P) + \vartheta((P-t)^+).$$

The assertion follows.

1.7. Proposition. (i) *$\vartheta_+(P) \leqslant \vartheta(P) + \theta(-P)$ for all $P \in \operatorname{Re} L(m)$.*

(ii) *Assume that M is weakly compact. Then $\vartheta_+(P) \leqslant \theta(P) + \vartheta(-P)$ for all $P \in \operatorname{Re} L(m)$.*

Proof. (i): For $t \in \boldsymbol{R}$ we have $(P-t)^+ = P + \operatorname{Max}(-P, -t)$ and hence $\vartheta((P-t)^+) \leqslant \vartheta(P) + \vartheta(\operatorname{Max}(-P, -t))$ after 1.1 (iii). The assertion follows.

(ii): Assume first that P is bounded below. Then $-P$ is bounded above, so that 1.6 (ii) implies that $\theta(-P) = \vartheta(-P)$ and hence (i) that $\vartheta_+(P) \leqslant \vartheta(P) + \vartheta(-P)$. Now for an arbitrary $P \in \operatorname{Re} L(m)$ and $t \in \boldsymbol{R}$ we have

$$\vartheta_+(P) \leqslant \vartheta_+(\operatorname{Max}(P, t)) \leqslant \vartheta(\operatorname{Max}(P, t)) + \vartheta(-\operatorname{Max}(P, t)) \leqslant \vartheta(\operatorname{Max}(P, t)) + \vartheta(-P).$$

For $t \downarrow -\infty$ the assertion follows.

Next we consider the univalent functions defined in the Introduction.

1.8. Proposition. (i) *For $P \in \operatorname{Re} L(m)$ we have $\vartheta(P) + \vartheta(-P) < 0$ iff $\int P^+ V\,dm = \int P^- V\,dm = \infty \; \forall V \in M$.*

(ii) *For univalent $P \in \operatorname{Re} L(m)$ we have $\vartheta(P) + \vartheta(-P) = 0$.*

(iii) *Assume that $P \in \operatorname{Re} L(m)$ is such that $\vartheta(P) + \vartheta(-P) \leqslant 0$ and that $\int PV\,dm$ exists in the extended sense $\forall V \in M$. Then P is univalent.*

Proof. (i): If $\int P^+ V\,dm = \int P^- V\,dm = \infty \; \forall V \in M$, then $\vartheta(P) = \vartheta(-P) = -\infty$ and hence $\vartheta(P) + \vartheta(-P) = -\infty$. Assume now that $\int P^+ V\,dm < \infty$ for some $V \in M$. In case $\int P^- V\,dm = \infty$ then $\int (-P) V\,dm = \infty$ and hence

$$\infty = \int PV\,dm + \int (-P) V\,dm \leqslant \vartheta(P) + \vartheta(-P),$$

and in case $\int P^- V\,dm < \infty$ we have

$$0 = \int PV\,dm + \int (-P) V\,dm \leqslant \vartheta(P) + \vartheta(-P).$$

Thus $\vartheta(P) + \vartheta(-P) \geqslant 0$. The conclusion is the same if one assumes that $\int P^- V\,dm < \infty$ for some $V \in M$.

(ii) is obvious.

(iii): For each $V \in M$ we have

$$\infty > \vartheta(P) \geqslant \int PV\,dm = \int P^+ V\,dm - \int P^- V\,dm,$$

$$\infty > \vartheta(-P) \geqslant \int (-P) V\,dm = \int P^- V\,dm - \int P^+ V\,dm;$$

therefore $\int P^+ V\,dm, \int P^- V\,dm < \infty$ or $\int |P| V\,dm < \infty$. It follows that $|||P||| < \infty$. Now for any $V, W \in M$ we have

$$\int PV\,dm + \int (-P) W\,dm \leqslant \vartheta(P) + \vartheta(-P) \leqslant 0 \quad \text{or} \quad \int PV\,dm \leqslant \int PW\,dm.$$

It follows that P is univalent.

1.9. Proposition. (i) *For $P \in \operatorname{Re} L(m)$ univalent we have $\theta(P) = \vartheta(P) + \vartheta_+(-P)$.*

(ii) *For $P \in \operatorname{Re} L(m)$ we have $\theta(P) + \theta(-P) \geqslant 0$; and $\theta(P) + \theta(-P) = 0$ iff P is univalent and $\vartheta_+(P) = \vartheta_+(-P) = 0$.*

Proof. (i): From 1.7 (i) we have $\vartheta_+(-P) \leqslant \vartheta(-P) + \theta(P)$, which for univalent P means that $\vartheta(P) + \vartheta_+(-P) \leqslant \theta(P)$. Combine this with 1.6 (i) to obtain the assertion.

(ii): From 1.7 (i) we see that $\theta(P) + \theta(-P)$ is $\geqslant \vartheta_+(P)$ and $\geqslant \vartheta_+(-P)$, and hence $\geqslant 0$. In case $\theta(P) + \theta(-P) = 0$ we have $\vartheta_+(P) = \vartheta_+(-P) = 0$, so that $\vartheta(P^+), \vartheta(P^-) < \infty$; furthermore $\vartheta(P) + \vartheta(-P) \leqslant 0$, so that P is univalent by 1.8 (iii). And if P is univalent and $\vartheta_+(P) = \vartheta_+(-P) = 0$, then $\theta(P) + \theta(-P) = 0$ by (i) and 1.8 (ii).

1.10. Proposition. *Assume that M is weakly compact. For univalent $P \in \operatorname{Re} L(m)$, $\vartheta_+(P) = \vartheta_+(-P)$. In particular, $\vartheta_+(P) = 0$ if P is bounded below.*

Proof. Combine 1.7 (ii) with 1.6 (i).

Let us define $L^\wedge$ to consist of the functions $P \in \operatorname{Re} L(m)$ with $\vartheta_+(P) = 0$. The connection between $L^\wedge$ and $L^\#$ will be discussed in 2.9 and 3.5(9).

2. The functional σ^+ and the main theorem. We define the functional $\sigma^+ \colon \operatorname{Re} L(m) \to [-\infty, \infty]$ to be

$$\sigma^+(P) = \operatorname{Inf}\{\operatorname{Re}\varphi(u) - c \colon u \in H^+ \text{ and } c \geqslant 0 \text{ with } \operatorname{Re} u - c \geqslant P\} \quad \forall\, P \in \operatorname{Re} L(m).$$

From the definition of H^+ we see that, in particular,

$$\sigma^+(P) = \operatorname{Inf}\{\operatorname{Re}\varphi(u) \colon u \in H^+ \text{ with } \operatorname{Re} u \geqslant P\} \quad \text{for} \quad P \geqslant 0.$$

Note that $\sigma^+(P) < \infty$ iff there exists some $u \in H^+$ with $\operatorname{Re} u \geqslant P$.

2.1. PROPERTIES. (i) $\theta(P) \leqslant \sigma^+(P) \leqslant \operatorname{Sup} P$.

(ii) $P \leqslant Q \Rightarrow \sigma^+(P) \leqslant \sigma^+(Q)$.

(iii) $\sigma^+(P+Q) \leqslant \sigma^+(P) + \sigma^+(Q)$.

(iv) $\sigma^+(cP) = c\sigma^+(P)$ *for real* $c \geqslant 0$.

Proof. (i): We have to prove the first inequality and can assume that $\sigma^+(P) < \infty$. Let $u \in H^+$ and $c \geqslant 0$ with $\operatorname{Re} u - c \geqslant P$. Then $\operatorname{Re} u - c \geqslant \operatorname{Max}(P, -c)$. Thus the basic result [1], V.4.1.3, shows that

$$\int \operatorname{Max}(P, -c)\, V\, dm \leqslant \int (\operatorname{Re} u)\, V\, dm - c \leqslant \operatorname{Re}\varphi(u) - c \qquad \forall V \in M,$$

and hence

$$\theta(P) \leqslant \vartheta\big(\operatorname{Max}(P, -c)\big) \leqslant \operatorname{Re}\varphi(u) - c.$$

The assertion follows. (ii)–(iv) are obvious.

2.2. Remark. *For $P \in \operatorname{Re} L(m)$ bounded above we have*

$$\sigma^+(P) = \operatorname{Inf}\{\operatorname{Re}\varphi(u) \colon u \in H \text{ with } \operatorname{Re} u \geqslant P\} =: \sigma(P).$$

Proof. (1) Each function $u \in H$ can be written $u = v - c$ with $c \geqslant 0$ and $v \in H$ and $\operatorname{Re} v \geqslant 0$, so that $v \in H^+$. Thus $\sigma^+(P) \leqslant \sigma(P)$ is clear. We have to prove $\sigma(P) \leqslant \sigma^+(P)$ and can assume that $\sigma^+(P) < \infty$.

(2) Let $u \in H^+$ and $c \geqslant 0$ with $\operatorname{Re} u - c \geqslant P$. Fix $R > 0$ with $P + c \leqslant R$ and $\lambda > 1$; then $t > 0$ with $2tR + 1 < \lambda$. From [1], V.4.1, we know that

$$u_t := \frac{u}{1+tu} \in H \quad \text{with} \quad \operatorname{Re} u_t = \frac{\operatorname{Re} u + t|u|^2}{|1+tu|^2} \geqslant 0.$$

We have

$$\begin{aligned}(P+c)|1+tu|^2 &\leqslant \operatorname{Min}(\operatorname{Re} u, R)|1+tu|^2 \\ &= \operatorname{Min}(\operatorname{Re} u, R)(1 + 2t\operatorname{Re} u + t^2|u|^2) \\ &\leqslant \operatorname{Re} u + 2tR\operatorname{Re} u + t^2 R|u|^2 \\ &\leqslant \lambda \operatorname{Re} u + tR(t|u|^2) \leqslant \lambda(\operatorname{Re} u + t|u|^2),\end{aligned}$$

so that $P + c \leqslant \lambda \operatorname{Re} u_t$. Thus $\lambda u_t - c \in H$ with $\operatorname{Re}(\lambda u_t - c) \geqslant P$, so that the definition of $\sigma(P)$ implies that

$$\sigma(P) \leqslant \operatorname{Re}\varphi(\lambda u_t - c) = \lambda \operatorname{Re}\varphi(u_t) - c = \lambda \operatorname{Re}\frac{\varphi(u)}{1 + t\varphi(u)} - c.$$

For $t \downarrow 0$ we obtain $\sigma(P) \leqslant \lambda \operatorname{Re}\varphi(u) - c$, for $\lambda \downarrow 1$ then $\sigma(P) \leqslant \operatorname{Re}\varphi(u) - c$. Hence $\sigma(P) \leqslant \sigma^+(P)$.

2.3. Remark. *For any $P \in \operatorname{Re} L(m)$ we have $\sigma^+\big(\operatorname{Max}(P, t)\big) \downarrow \sigma^+(P)$ for $t \downarrow -\infty$.*

Proof. We can assume that $\sigma^+(P) < \infty$. Let $u \in H^+$ and $c \geqslant 0$ with $\operatorname{Re} u - c \geqslant P$. Then $\operatorname{Re} u - c \geqslant \operatorname{Max}(P, -c)$, and hence $\operatorname{Re}\varphi(u) - c \geqslant \sigma^+\big(\operatorname{Max}(P, -c)\big)$. The result follows.

The next lemma is the key result for the future development.

2.4. LEMMA. *Let $P \in \operatorname{Re} L(m)$ be nonnegative with $\sigma^+\big(\operatorname{Min}(P, t)\big) \uparrow \lambda < \infty$ for $t \uparrow \infty$. Then there exists $u \in H^+$ with $\operatorname{Re} u \geqslant P$ and $\operatorname{Re}\varphi(u) \leqslant \lambda$.*

Proof. (1) For $t \geqslant 0$ we fix real $\lambda(t) > \lambda$ such that $\lambda(t) \downarrow \lambda$ for $t \uparrow \infty$. Then in view of $\lambda(t) > \lambda \geqslant \sigma^+\big(\operatorname{Min}(P, t)\big)$ and $\operatorname{Min}(P, t) \geqslant 0$ there exists $u \in H^+$ with $\operatorname{Re} u_t \geqslant \operatorname{Min}(P, t)$ and $\varphi(u_t) = \operatorname{Re}\varphi(u_t) < \lambda(t)$.

(2) We invoke [1], V. 4.2, to obtain the function

$$h_t := \frac{1 - u_t}{1 + u_t} \in \operatorname{Ball}(H),$$

which is not the constant -1. We have

$$u_t = \frac{1 - h_t}{1 + h_t} \quad \text{and hence} \quad \operatorname{Re} u_t = \frac{1 - |h_t|^2}{|1 + h_t|^2},$$

$$\varphi(u_t) = \frac{1 - \varphi(h_t)}{1 + \varphi(h_t)} \quad \text{and hence} \quad \operatorname{Re}\varphi(u_t) = \frac{1 - |\varphi(h_t)|^2}{|1 + \varphi(h_t)|^2}.$$

It follows that

$$|h_t|^2 + \operatorname{Min}(P, t)|1 + h_t|^2 \leqslant 1,$$
$$|\varphi(h_t)|^2 + \lambda(t)|1 + \varphi(h_t)|^2 > 1.$$

(3) For $\tau \geqslant 0$ let $M(\tau) \subset L^\infty(m)$ be the weak* closure of the set $\{h_t \colon t \geqslant \tau\} \subset \operatorname{Ball}(H)$. Thus $M(\tau) \subset \operatorname{Ball}(H)$ is weak* compact and $\neq \varnothing$, and $\tau \leqslant \tau'$ implies that $M(\tau') \subset M(\tau)$. Thus there exists some $h \in \operatorname{Ball}(H)$ such that $h \in M(\tau)$ for all $\tau \geqslant 0$. For some fixed $V \in M$ there exists a sequence of numbers $0 \leqslant t(n) \uparrow \infty$ such that $\int (h_{t(n)} - h)\, V\, dm \to 0$. Therefore $\varphi(h_{t(n)}) \to \varphi(h)$. It follows that

$$(+) \qquad |\varphi(h)|^2 + \lambda|1 + \varphi(h)|^2 \geqslant 1.$$

Also h cannot be the constant -1 since otherwise $\varphi(h_{t(n)})\to\varphi(h) = -1$ or $\varphi(u_{t(n)})\to\infty$ which contradicts the last formula in (1).

(4) For $\tau \geqslant 0$ define

$$B(\tau):=\{h\in H:\ |h|^2+\mathrm{Min}(P,\tau)|1+h|^2\leqslant 1\}\subset \mathrm{Ball}(H).$$

It has the following properties.

(i) $B(\tau)$ is convex. In fact, for $f, g \in L(m)$ and real $\alpha, \beta \geqslant 0$ with $\alpha+\beta = 1$ we have

$$|\alpha f+\beta g|^2\leqslant(\alpha|f|+\beta|g|)^2 = \alpha^2|f|^2+\beta^2|g|^2+2\alpha\beta|f|\,|g|$$
$$\leqslant \alpha^2|f|^2+\beta^2|g|^2+\alpha\beta(|f|^2+|g|^2) = \alpha|f|^2+\beta|g|^2.$$

From this relation the assertion is clear.

(ii) $B(\tau)$ is weak* closed. In fact, a simple argument involving $L^2(m)$ shows that a bounded convex subset of $L^\infty(m)$ is weak* closed if it is closed with respect to pointwise convergent sequences; see the proof of the Krein–Smulian result [1], IV. 3.14.

(iii) We have $h_t\in B(\tau)\,\forall t\geqslant\tau$ by (2) and hence $M(\tau)\subset B(\tau)$ in view of (ii). It follows that the function $h\in\mathrm{Ball}(H)$ obtained in (3) satisfies $h\in B(\tau)$ for all $\tau\geqslant 0$. Therefore

$$(-)\qquad |h|^2+P|1+h|^2\leqslant 1.$$

(5) We have thus obtained a function $h\in\mathrm{Ball}(H)$ which is not the constant -1 and satisfies $(+)$ and $(-)$. For the corresponding function $u:=\dfrac{1-h}{1+h}\in H^+$ we have therefore

$$\mathrm{Re}\,u = \frac{1-|h|^2}{|1+h|^2}\geqslant P,$$

$$\mathrm{Re}\,\varphi(u) = \frac{1-|\varphi(h)|^2}{|1+\varphi(h)|^2}\leqslant\lambda.$$

The proof is complete.

2.5. Proposition. *For $P\in\mathrm{Re}L(m)$ bounded below we have*

$$\sigma^+\big(\mathrm{Min}(P,t)\big)\uparrow\sigma^+(P)\quad \textit{for}\quad t\uparrow\infty.$$

Proof. From 2.1 we have $\sigma^+(P+c) = \sigma^+(P)+c\ \forall c\in\boldsymbol{R}$, so that we can assume that $P\geqslant 0$. Now $\sigma^+\big(\mathrm{Min}(P,t)\big)\uparrow\lambda\leqslant\sigma^+(P)$ for $t\uparrow\infty$. We have to show that $\sigma^+(P)\leqslant\lambda$ and can assume that $\lambda<\infty$. But then $\sigma^+(P)\leqslant\lambda$ is immediate by 2.4.

2.6. Theorem. *Let $P\in\mathrm{Re}L(m)$ be nonnegative with $\sigma^+(P)<\infty$. Then there exists $u\in H^+$ with $\mathrm{Re}\,u\geqslant P$ and $\mathrm{Re}\,\varphi(u) = \sigma^+(P)$.*

Proof. Immediate by 2.4 and 2.5.

We turn to the functional

$$a:\ \{F\in\mathrm{Re}L(m):\ F\geqslant 0\}\to[0,\infty]$$

which is a basic tool in [1]. The definition is

$$a(F) = \mathrm{Sup}\{|\varphi(u)|:\ u\in H \text{ with } |u|\leqslant F\}\quad \forall F\geqslant 0.$$

It follows that $a(F)a(G)\leqslant a(FG)\,\forall F, G\geqslant 0$.

2.7. Properties. (i) $a(e^P)\geqslant e^{-\sigma^+(-P)}\,\forall P\in\mathrm{Re}L(m)$.

(ii) $\big(a(e^{tP})\big)^{1/t}\to e^{-\sigma(-P)}$ *for* $t\downarrow 0$ $\forall P\in\mathrm{Re}L^\infty(m)$.

(iii) $a(e^P)\geqslant e^{-\vartheta(-P)}\,\forall P\in\mathrm{Re}L(m)$ *with* $e^P\in L^{\#}$.

Here (i) is an immediate consequence of the definitions involved. (ii) appears as [1], IV. 2.5, with a simple direct proof. However, (iii) is the central result of [1], Chapter IV. It appears to be somewhat weaker than [1], IV. 3.8, but combines at once with the elementary proposition [1], IV. 3.11, to the full assertion of [1], IV. 3.8.

An immediate consequence of 2.7 (ii) and (iii) is that for $P\in\mathrm{Re}L^\infty(m)$ we have $\vartheta(P)\geqslant\sigma(P)$ and hence $\vartheta(P) = \theta(P) = \sigma^+(P) = \sigma(P)$; see [1], IV. 3.10. This is a central theorem in the abstract Hardy algebra theory as well. Our former results permit to extend it to the entire $\mathrm{Re}L(m)$.

2.8. Theorem. $\theta(P) = \sigma^+(P)$ *for all* $P\in\mathrm{Re}L(m)$.

Proof. Assume first that P is bounded below. For $t\in\boldsymbol{R}$, $\mathrm{Min}(P,t)\in\mathrm{Re}L^\infty(m)$ and hence $\vartheta\big(\mathrm{Min}(P,t)\big) = \sigma\big(\mathrm{Min}(P,t)\big) = \sigma^+\big(\mathrm{Min}(P,t)\big)$. For $t\uparrow\infty$ we obtain $\vartheta(P) = \sigma^+(P)$ in view of 1.2 and 2.5. Consider now an arbitrary $P\in\mathrm{Re}L(m)$. For $t\in\boldsymbol{R}$, $\mathrm{Max}(P,t)$ is bounded below and hence $\vartheta\big(\mathrm{Max}(P,t)\big) = \sigma^+\big(\mathrm{Max}(P,t)\big)$. For $t\downarrow-\infty$ we obtain $\theta(P) = \sigma^+(P)$ in view of 2.3.

2.9. Corollary. *We have* $P\in L^{\hat{}}\Rightarrow e^P\in L^{\#}$ (an example in Section 3 will show that the converse need not be true).

Proof. The assumption is $\sigma^+\big((P-t)^+\big) = \vartheta\big((P-t)^+\big)\to 0$ for $t\uparrow\infty$. Hence there are functions $u_t\in H^+$ with $\mathrm{Re}\,u_t\geqslant(P-t)^+$ such that $\varphi(u_t) = \mathrm{Re}\,\varphi(u_t)\to 0$ for $t\uparrow\infty$. In view of the Kolmogorov estimation [1], V. 5.6, and the Beppo Levi theorem we can find a sequence $t(n)\uparrow\infty$ such that $u_{t(n)}\to 0$ pointwise for $n\to\infty$. Thus the functions $v_n:=\exp(-u_{t(n)})\in H$ fulfil $|v_n|\leqslant 1$ and $v_n\to 0$ and $|v_n|e^P = \exp(P-\mathrm{Re}\,u_n)\leqslant\exp\big(t(n)\big)$. It follows that $e^P\in L^{\#}$ by the definition.

Another consequence of 2.8 is the following extension of 2.7 (ii):

2.10. Proposition. $\big(a(e^{tP})\big)^{1/t}\to e^{-\sigma^+(-P)}$ *for* $t\downarrow 0$ $\forall P\in L^{\hat{}}$.

Proof. (1) Assume that P is bounded above. For $s\in\boldsymbol{R}$, $\mathrm{Max}(P,s)\in\mathrm{Re}L^\infty(m)$ and hence by 2.7 (ii)

$$\limsup_{t\downarrow 0}\big(a(e^{tP})\big)^{1/t}\leqslant\lim_{t\downarrow 0}\big(a(e^{t\,\mathrm{Max}(P,s)})\big)^{1/t}$$
$$= e^{-\sigma^+(-\mathrm{Max}(P,s))} = e^{-\sigma^+(\mathrm{Min}(-P,-s))}.$$

From 2.5 we have $\sigma^+(\mathrm{Min}(-P, -s)) \to \sigma^+(-P)$ for $s \downarrow -\infty$, so that

$$\limsup_{t\downarrow 0} \big(a(e^{tP})\big)^{1/t} \leqslant e^{-\sigma^+(-P)}.$$

(2) For $P \in \mathrm{Re}\,L(m)$ we have

$$\big(a(e^{tP})\big)^{1/t} \leqslant \big(a(e^{t\,\mathrm{Min}(P,s)})\big)^{1/t} e^{\sigma^+((P-s)^+)} \quad \forall s \in \boldsymbol{R} \text{ and } t > 0.$$

In fact, from $P = \mathrm{Min}(P, s) + (P-s)^+$ and 2.7 (i) we obtain

$$a(e^{t\,\mathrm{Min}(P,s)}) = a(e^{tP}e^{-t(P-s)^+}) \geqslant a(e^{tP})\,a(e^{-t(P-s)^+}) \geqslant a(e^{tP})\,e^{-t\sigma^+((P-s)^+)}.$$

From (1) it follows that

$$\limsup_{t\downarrow 0} \big(a(e^{tP})\big)^{1/t} \leqslant e^{-\sigma^+(-P)} e^{\sigma^+((P-s)^+)} \quad \forall s \in \boldsymbol{R},$$

so that for $P \in L^{\wedge}$ we conclude from 2.8 that

$$\limsup_{t\downarrow 0} \big(a(e^{tP})\big)^{1/t} \leqslant e^{-\sigma^+(-P)}.$$

In view of 2.7 (i) the assertion follows.

3. Applications of the main theorem.

3.1. Proposition. *Let $P \in \mathrm{Re}\,L(m)$ be nonnegative and $\vartheta(P) \leqslant \alpha\beta$ with real $\alpha, \beta > 0$. Then there exists a function $u \in H^+$ such that $|e^{-u}|P \leqslant \beta$ and*

$$\varphi(u) = \mathrm{Re}\,\varphi(u) \leqslant \alpha \quad \textit{and} \quad |||1 - e^{-u}||| \leqslant \sqrt{\alpha}.$$

Proof. From 2.8 and 2.6 we obtain $v \in H^+$ with $\mathrm{Re}\,v \geqslant P$ and $\varphi(v) = \mathrm{Re}\,\varphi(v) = \vartheta(P) \leqslant \alpha\beta$. Put $u := v/e\beta \in H^+$. Then $\mathrm{Re}\,u \geqslant P/e\beta$ and hence

$$|e^{-u}|P = e^{-\mathrm{Re}\,u}P \leqslant e\beta \exp\left(-\frac{P}{e\beta}\right)\frac{P}{e\beta} \leqslant \beta$$

since $xe^{-x} \leqslant 1/e \ \forall x \geqslant 0$. Furthermore $\varphi(u) = \mathrm{Re}\,\varphi(u) \leqslant \alpha/e < \alpha$ and hence

$$\varphi(e^{-u}) = e^{-\varphi(u)} \geqslant 1 - \varphi(u) \geqslant 1 - \alpha/e.$$

It follows for $V \in M$ that

$$\int |1 - e^{-u}|^2 V\,dm = \int (1 + |e^{-u}|^2 - 2\,\mathrm{Re}\,e^{-u})V\,dm \leqslant 2 - 2\,\mathrm{Re}\,\varphi(e^{-u}) \leqslant \frac{2}{e}\alpha < \alpha,$$

$$\int |1 - e^{-u}|\,V\,dm \leqslant \Big(\int |1 - e^{-u}|^2 V\,dm\Big)^{1/2} < \sqrt{\alpha},$$

so that $|||1 - e^{-u}||| \leqslant \sqrt{\alpha}$ as claimed.

3.2. Corollary (Extended Modification Theorem). *Let $P_n \in \mathrm{Re}\,L(m)$ be nonnegative $(n = 1, 2, \ldots)$ with $\vartheta(P_n) \to 0$. Then there exist functions $u_n \in H^+$ $(n = 1, 2, \ldots)$ such that*

$$v_n := \exp(-u_n) \in \mathrm{Ball}(H)$$

fulfil

$$|||1 - v_n||| \to 0 \quad \textit{and} \quad \|v_nP_n\| := \mathrm{Sup}\,|v_nP_n| \to 0.$$

The next application requires a lemma which is of interest in itself. For $u \in H^+$ define

$$u_t := \frac{tu}{t+u} \quad \text{for} \quad t > 0.$$

Then $u_t \in H$ with $|u_t| \leqslant t$ and $\mathrm{Re}\,u_t \geqslant 0$, and $u_t \to u$ for $t \uparrow \infty$.

3.3. Lemma. *Let $u = P + iQ \in H^+$ with $\mathrm{Im}\,\varphi(u) = 0$ be such that $|Q| \leqslant cP$ for some real $c > 0$. Then $|||u - u_t||| \to 0$ for $t \uparrow \infty$. If $|Q| \leqslant P$, then in addition $\mathrm{Re}\,u_t \uparrow P$ for $t \uparrow \infty$.*

Proof. (0) For a complex number $z = x + iy$ with $x \geqslant 0$ the function

$$F\colon F(t) = \frac{tz}{t+z} \quad \text{for} \quad t > 0$$

has the derivative

$$F'(t) = \left(\frac{z}{t+z}\right)^2.$$

Hence

$$(\mathrm{Re}\,F)'(t) = \mathrm{Re}\,F'(t) = \frac{t^2(x^2-y^2) + 2tx|z|^2 + |z|^4}{|t+z|^4}.$$

Thus we have the last assertion.

(1) For $t > 0$ we have

$$u - u_t = \int_t^\infty \left(\frac{u}{s+u}\right)^2 ds, \quad |u - u_t| \leqslant \int_t^\infty \left|\frac{u}{s+u}\right|^2 ds,$$

$$\int |u - u_t|\,V\,dm \leqslant \int_t^\infty \left(\int \left|\frac{u}{s+u}\right|^2 V\,dm\right) ds \quad \text{for} \quad V \in M.$$

From the Fubini result [1], X. 2.1, we obtain

$$\int \left|\frac{u}{s+u}\right|^2 V\,dm = \int_{]0,\infty[} (Vm)\left(\left[\left|\frac{u}{s+u}\right|^2 \geqslant x\right]\right) dx = \int_{]0,1[} (Vm)\left(\left[\left|\frac{u}{s+u}\right|^2 \geqslant x\right]\right) dx \leqslant \int_{]0,1[} (Vm)([|u|^2 \geqslant s^2x])\,dx.$$

(2) We can assume $|Q| \leqslant cP$ for some $c > 1$. Thus $c = \tan(\tau\pi/2)$ with $1/2 < \tau < 1$ or $1/2 < 1/2\tau < 1$. Then the assumption implies that the principal value $u^{1/\tau} = : v$ satisfies $\mathrm{Re}\, v \geqslant 0$. Now $u^{1/2\tau} \in H^+$ by [1], V. 4.9, and hence $v = u^{1/2\tau}u^{1/2\tau} \in H^+$ by [1], V. 4.12. Furthermore $\varphi(v) =$ the principal value $(\varphi(u))^{1/\tau}$ and hence real $\geqslant 0$.

(3) We combine $u = v^\tau$ with (1) and [1], V. 5.4, to obtain

$$\int \left|\frac{u}{s+u}\right|^2 V\,dm \leqslant \int_{]0,1[} (Vm)([|v| \geqslant s^{1/\tau}x^{1/2\tau}])\,dx$$

$$\leqslant 2\varphi(v)s^{-1/\tau}\int_0^1 x^{-1/2\tau}dx = \frac{4\tau}{2\tau-1}\varphi(v)s^{-1/\tau}.$$

It follows from (1) that for $t > 0$ we have

$$\int |u-u_t|V\,dm \leqslant \frac{4\tau}{2\tau-1}\varphi(v)\int_t^\infty s^{-1/\tau}ds$$

$$= \frac{4\tau^2}{(2\tau-1)(1-\tau)}\varphi(v)t^{-(1-\tau)/\tau} \quad \text{for} \quad V \in M,$$

$$|||u-u_t||| \leqslant \frac{4\tau^2}{(2\tau-1)(1-\tau)}\varphi(v)t^{-(1-\tau)/\tau}.$$

The proof is complete.

3.4. THEOREM. *Let $P \in \mathrm{Re}\,L(m)$ be nonnegative with $\vartheta(P) < \infty$. For $0 < \tau < 1$, $P^\tau \in L^\wedge$, that is P^τ is in the $|||\cdot|||$ closure of $\mathrm{Re}\,L^\infty(m)$.*

Proof. From 2.8 and 2.6 we obtain $v \in H^+$ with $\mathrm{Re}\,v \geqslant P$ and $\varphi(v) = \mathrm{Re}\,\varphi(v) = \vartheta(P)$. We have

$$u := v^\tau \in H^+ \quad \text{with} \quad \varphi(u) = (\varphi(v))^\tau = (\vartheta(P))^\tau$$

by [1], V. 4.9. From 3.3 we obtain $|||u-u_t||| \to 0$ and hence $|||\,|u|-|u_t|\,||| \to 0$ for $t \uparrow \infty$. Hence $\vartheta_+(|u|) = 0$. Now $P^\tau \leqslant (\mathrm{Re}\,h)^\tau \leqslant |h|^\tau = |h^\tau| = |u|$, so that $\vartheta_+(P^\tau) = 0$.

The remainder of the section is devoted to an example. It is similar to [1], VI. 4.10, but more complicated.

3.5. EXAMPLE. Let D denote the open unit disk and S the unit circle in C with $\lambda :=$ arc length on S normalized to $\lambda(S) = 1$. We start with the classical situations $(H^\infty(D), \varphi_z)$ on $(S, \text{Baire}, \lambda)$, where

$$\varphi_z:\ \varphi_z(u) = \int uP(z,\cdot)\,d\lambda =: \langle u\lambda\rangle(z) \qquad \forall u \in H^\infty(D),$$

with $P(z,\cdot) :=$ the Poisson kernel, is the evaluation at $z \in D$.

We fix a sequence of points $0 < a_1 < \ldots < a_n < \ldots < 1$ with $a_n \uparrow 1$ and put $X := S \cup \{a_n : n \in N\}$. We further fix a sequence of numbers

$\alpha_n > 0$ $(n = 1, 2, \ldots)$ with $\sum_{n=1}^\infty \alpha_n < \infty$ and form the finite positive Baire measure m on X such that $m|S = \lambda$ and $m(\{a_n\}) = \alpha_n$ $\forall n \in N$. For $f \in L(m)$ we write $f^0 := f|S \in L(\lambda)$. On (X, Baire, m) we consider the Hardy algebra situation (H, φ) defined to be

$$H := \{u \in L^\infty(m):\ u^0 \in H^\infty(D) \text{ and } u(a_n) = \langle u^0\lambda\rangle(a_n)\ \forall n \in N\},$$

$$\varphi:\ \varphi(u) = \varphi_0(u^0) = \int u^0 d\lambda = \langle u^0\lambda\rangle(0) \qquad \forall u \in H.$$

To prove the weak* closedness of H it is convenient to use the Krein–Smulian result [1], IV. 3.14.

(1) We first determine the function class M. For a nonnegative function $V \in \mathrm{Re}\,L^1(m)$ we have

$$\int\Big(\sum_{n=1}^\infty P(a_n,\cdot)V(a_n)\alpha_n\Big)d\lambda = \sum_{n=1}^\infty V(a_n)\alpha_n = \int_{X\setminus S} V\,dm < \infty,$$

so that $\sum_{n=1}^\infty P(a_n,\cdot)V(a_n)\alpha_n$ is finite λ-a.e. on S. For $u \in H$ we obtain

$$\int uV\,dm = \int u^0V^0 d\lambda + \sum_{n=1}^\infty \langle u^0\lambda\rangle(a_n)V(a_n)\alpha_n$$

$$= \int u^0\Big(V^0 + \sum_{n=1}^\infty P(a_n,\cdot)V(a_n)\alpha_n\Big)d\lambda.$$

Thus $V \in M$ iff $V^0 + \sum_{n=1}^\infty P(a_n,\cdot)V(a_n)\alpha_n = 1$. This requires that

$$\sum_{n=1}^\infty P(a_n, s)V(a_n)\alpha_n \leqslant 1$$

for λ-almost all $s \in S$; but since

$$\Big\{s \in S:\ \sum_{n=1}^\infty P(a_n, s)V(a_n)\alpha_n \leqslant 1\Big\}$$

is closed, this means that

$$\sum_{n=1}^\infty P(a_n, s)V(a_n)\alpha_n \leqslant 1 \quad \text{for all } s \in S.$$

Now

$$\mathrm{Max}\,P(a_n,\cdot) = P(a_n, 1) = \frac{1+a_n}{1-a_n}.$$

Thus $V \in M$ iff

$$\sum_{n=1}^{\infty} \frac{1+a_n}{1-a_n} V(a_n) a_n \leqslant 1 \quad \text{and} \quad V^0 = 1 - \sum_{n=1}^{\infty} P(a_n, \cdot) V(a_n) a_n.$$

(2) Consider a function $P \in \operatorname{Re} L(m)$ with $\int |P^0| d\lambda < \infty$. For $V \in M$,

$$\sum_{n=1}^{\infty} |\langle P^0 \lambda\rangle(a_n)| V(a_n) a_n \leqslant \sum_{n=1}^{\infty} \Big(\int |P^0| P(a_n, \cdot) d\lambda\Big) V(a_n) a_n$$
$$= \int |P^0| \Big(\sum_{n=1}^{\infty} P(a_n, \cdot) V(a_n) a_n\Big) d\lambda \leqslant \int |P^0| d\lambda < \infty.$$

If $\int PV\, dm$ exists in the extended sense, that is if the series $\sum_{n=1}^{\infty} P(a_n) V(a_n) a_n$ converges in the extended sense, then

$$\int PV\, dm = \int P^0 V^0 dm + \sum_{n=1}^{\infty} P(a_n) V(a_n) a_n$$
$$= \int P^0 \Big(1 - \sum_{n=1}^{\infty} P(a_n, \cdot) V(a_n) a_n\Big) d\lambda + \sum_{n=1}^{\infty} P(a_n) V(a_n) a_n$$
$$= \int P^0 d\lambda + \sum_{n=1}^{\infty} \big(P(a_n) - \langle P^0 \lambda\rangle(a_n)\big) V(a_n) a_n.$$

(3) From (1) and (2) we conclude that a function $P \in \operatorname{Re} L(m)$ is univalent iff $\int |P^0| d\lambda < \infty$ and $P(a_n) = \langle P^0 \lambda\rangle(a_n)$ $\forall n \in N$.

(4) For $P \in \operatorname{Re} L(m)$ with $\int |P^0| d\lambda < \infty$ we claim that

$$\vartheta(P) = \int P^0 d\lambda + \operatorname*{Sup}_{n \in N} \frac{1-a_n}{1+a_n} \operatorname{Max}\big(0, P(a_n) - \langle P^0 \lambda\rangle(a_n)\big).$$

It then follows that it is equal to $\theta(P)$ as well. For the proof let A denote the supremum in question, so that $0 \leqslant A \leqslant \infty$.

(i) In order to prove $\vartheta(P) \leqslant \int P^0 d\lambda + A$ we can assume that $A < \infty$. Then we have

$$P(a_n) \leqslant \langle P^0 \lambda\rangle(a_n) + \frac{1+a_n}{1-a_n} A,$$

so that by (1) and (2) for each $V \in M$ the series $\sum_{n=1}^{\infty} P(a_n) V(a_n) a_n$ converges in the extended sense, and we have

$$\int PV\, dm \leqslant \int P^0 d\lambda + \sum_{n=1}^{\infty} \frac{1+a_n}{1-a_n} V(a_n) a_n A \leqslant \int P^0 d\lambda + A.$$

It follows that $\vartheta(P) \leqslant \int P^0 d\lambda + A$.

(ii) In order to prove $\vartheta(P) \geqslant \int P^0 d\lambda + A$ we can assume that $A > 0$.

For $\alpha > 0$ with $\alpha < A$,

$$\frac{1-a_p}{1+a_p} \big(P(a_p) - \langle P^0 \lambda\rangle(a_p)\big) > \alpha$$

for some $p \in N$. We take the $V \in M$ with

$$\frac{1+a_p}{1-a_p} V(a_p) a_p = 1 \quad \text{and} \quad V(a_n) = 0 \quad \forall n \neq p.$$

From (2) we obtain

$$\int PV\, dm = \int P^0 d\lambda + \big(P(a_p) - \langle P^0 \lambda\rangle(a_p)\big) V(a_p) a_p > \int P^0 d\lambda + \alpha.$$

It follows that $\vartheta(P) \geqslant \int P^0 d\lambda + A$.

(5) We shall need the lemma: For $F \in L^1(\lambda)$ we have

$$\frac{1-R}{1+R} \operatorname*{Max}_{|z|=R} |\langle F\lambda\rangle(z)| \to 0 \quad \text{for} \quad R \uparrow 1.$$

We have to prove that for any sequence of points $z_l \in D$ $(l = 1, 2, \ldots)$ with $z_l \to c \in S$ we have

$$\frac{1-|z_l|}{1+|z_l|} \langle F\lambda\rangle(z_l) = \int F(s) \frac{1-|z_l|}{1+|z_l|} P(z_l, s)\, d\lambda(s) \to 0.$$

But $\dfrac{1-|z_l|}{1+|z_l|} P(z_l, s) \leqslant 1$ and $\to 0$ for $l \to \infty$ for all $s \in S$ except for $s = c$, so that the assertion follows from the dominated convergence theorem.

(6) For $P \in \operatorname{Re} L(m)$ we claim that

$$\vartheta_+(P) = \begin{cases} \infty & \text{if} \quad \int (P^0)^+ d\lambda = \infty, \\ \limsup\limits_{n\to\infty} \dfrac{1-a_n}{1+a_n} \big(P(a_n)\big)^+ & \text{if} \quad \int (P^0)^+ d\lambda < \infty. \end{cases}$$

In fact, in the case $\int (P^0)^+ d\lambda = \infty$ we have

$$\vartheta\big((P-t)^+\big) \geqslant \int (P^0 - t)^+ d\lambda = \infty \qquad \forall t \in \boldsymbol{R}$$

and hence $\vartheta_+(P) = \infty$. So assume that $\int (P^0)^+ d\lambda < \infty$. For $t \in \boldsymbol{R}$, $\int (P^0 - t)^+ d\lambda < \infty$, so that from (2) we have

$$\vartheta\big((P-t)^+\big)$$
$$= \int (P^0 - t)^+ d\lambda + \operatorname*{Sup}_{n \in N} \frac{1-a_n}{1+a_n} \operatorname{Max}\big(0, (P-t)^+(a_n) - \langle (P^0-t)^+ \lambda\rangle(a_n)\big).$$

(i) For $n \in N$ and $t \in R$ we obtain

$$\frac{1-a_n}{1+a_n}P(a_n) \leqslant \frac{1-a_n}{1+a_n}t + \frac{1-a_n}{1+a_n}\langle (P^0-t)^+\lambda\rangle(a_n) + \vartheta\big((P-t)^+\big);$$

hence for $t \geqslant 0$ the same with $\big(P(a_n)\big)^+$ in place of $P(a_n)$. In view of (5) it follows that

$$\limsup_{n\to\infty} \frac{1-a_n}{1+a_n}\big(P(a_n)\big)^+ \leqslant \vartheta\big((P-t)^+\big) \quad \forall t \geqslant 0,$$

so that the limsup is $\leqslant \vartheta_+(P)$.

(ii) For fixed $r \in N$ and $t \geqslant \operatorname{Max}\big(0, P(a_1), \ldots, P(a_r)\big)$ we have

$$\vartheta\big((P-t)^+\big) \leqslant \int (P^0-t)^+ d\lambda + \operatorname*{Sup}_{n>r} \frac{1-a_n}{1+a_n}\big(P(a_n)\big)^+.$$

For $t\uparrow\infty$ it follows that

$$\vartheta_+(P) \leqslant \operatorname*{Sup}_{n>r} \frac{1-a_n}{1+a_n}\big(P(a_n)\big)^+ \quad \forall r \in N,$$

$$\vartheta_+(P) \leqslant \limsup_{n\to\infty} \frac{1-a_n}{1+a_n}\big(P(a_n)\big)^+.$$

The proof is complete.

(7) For $P \in \operatorname{Re}L(m)$ univalent we have $\vartheta_+(P) = \vartheta_+(-P) = 0$. This is clear from (3) combined with (5) and (6).

(8) The function $P \in \operatorname{Re}L(m)$ defined to be

$$P^0 := 0 \quad \text{and} \quad P(a_n) = \frac{1+a_n}{1-a_n} \quad \forall n \in N$$

is $\geqslant 0$ and satisfies $\vartheta(P) = \vartheta_+(P) = 1$ by (4) and (6). It follows that $\vartheta(P) < \infty$ does not enforce that $P \in L^\wedge$. This counterexample has been announced in the Introduction. Note that $P = \operatorname{Re}\dfrac{1+Z}{1-Z}$ with $Z: Z(z) = z$ $\forall z \in X$, so that $Z \in \operatorname{Ball}(H)$ and hence $\dfrac{1+Z}{1-Z} \in H^+$ by [1], V. 4.2.

(9) Let us assume that $\sum_{n=1}^{\infty}(1-a_n) < \infty$. Then for $P \in \operatorname{Re}L(m)$ we assert that $e^P \in L^\#$ iff $e^P|S$ is in the classical $L^\#(D)$, that is iff $\int (P^0)^+ d\lambda < \infty$. In particular, $e^P \in L^\#$ for all $P \in \operatorname{Re}L(m)$ with $P^0 = 0$. Thus in view of (6) the implication $e^P \in L^\# \Rightarrow \vartheta(P) = \theta(P) < \infty$ for $P \geqslant 0$ is false. A fortiori, the implication $e^P \in L^\# \Rightarrow P \in L^\wedge$ is false. This counterexample has been announced in 2.9.

For the proof consider the Blaschke products

$$B_n: B_n(z) = \prod_{l=n}^{\infty} \frac{a_l - z}{1-a_l z} \quad \forall z \in D \ (n = 1, 2, \ldots).$$

We have $|B_n| \leqslant 1$ and $B_n \to 1$ on D. Furthermore $B_n = \langle b_n\lambda\rangle$ for the limit functions $b_n \in H^\infty(D)$ with $|b_n| = 1$. In view of

$$\int |1-b_n|^2 d\lambda = \int (1+|b_n|^2 - 2\operatorname{Re}b_n)\,d\lambda = 2\big(1-\operatorname{Re}B_n(0)\big) \to 0$$

there exists a subsequence $1 \leqslant n(1) < \ldots < n(l) < \ldots$ such that $b_{n(l)} \to 1$. Now assume that $P \in \operatorname{Re}L(m)$ fulfils $e^P|S \in L^\#(D)$. Then there are functions $u_l \in H^\infty(D)$ $(l = 1, 2, \ldots)$ with $|u_l| \leqslant 1$ and $u_l \to 1$ such that $u_l(e^P|S) \in L^\infty(\lambda)$. The functions $v_l \in H$ defined to be $v_l|S := b_{n(l)}u_l$ and $v_l|X\setminus S := B_{n(l)}\langle u_l\lambda\rangle$ thus fulfil $|v_l| \leqslant 1$ and $v_l \to 1$. Furthermore the function $v_l e^P$ belongs to $L^\infty(m)$ since it is bounded on $X\setminus S$ for the trivial reason that it vanishes outside of a finite subset. It follows that $e^P \in L^\#$. The converse implication is obvious.

4. Applications to the abstract conjugation. We start with the Kolmogorov estimation. In view of 2.8 the proof of [1], VI. 5.1, carries over.

4.1. THEOREM. *For $0 < \tau < 1$ we have*

$$\cos\frac{\tau\pi}{2} ||| \, |P+iQ|^\tau \, ||| \leqslant 2^{1-\tau} |||P|||^\tau \quad \forall P \in E,$$

with $Q \in \operatorname{Re}L(m)$ the conjugate function to P.

Proof. We put $h := P+iQ$.

(1) We can assume that $|||P||| = \sigma^+(|P|) < \infty$. Fix $u \in H^+$ with $\operatorname{Re}u \geqslant |P|$ and $\operatorname{Im}\varphi(u) = 0$. Then for $u^\pm := u \pm h$ we have $\operatorname{Re}u^\pm \geqslant 0$ and $u^\pm \in H^+$ since for $t \geqslant 0$ we have

$$e^{-tu^\pm} = e^{-tu}e^{\mp th} \in H^\# \cap L^\infty(m) = H.$$

Furthermore

$$e^{-t\varphi(u^\pm)} = \varphi(e^{-tu^\pm}) = e^{-t\varphi(u)}e^{\mp t\alpha(P)},$$

so that $\varphi(u^\pm) = \varphi(u) \pm \alpha(P)$ which is real $\geqslant 0$. From the Kolmogorov estimation for H^+ [1], V. 5.6, we obtain

$$\cos\frac{\tau\pi}{2} ||| \, |u^\pm|^\tau ||| \leqslant \big(\varphi(u) \pm \alpha(P)\big)^\tau.$$

(2) We shall need the simple inequalities $(a+b)^\tau \leqslant a^\tau + b^\tau$ and $a^\tau + b^\tau \leqslant 2^{1-\tau}(a+b)^\tau$ for $a, b \geqslant 0$.

(3) We have $h = \frac{1}{2}(u^+ - u^-)$ and hence

$$|h|^\tau \leqslant 2^{-\tau}(|u^+|+|u^-|)^\tau \leqslant 2^{-\tau}(|u^+|^\tau+|u^-|^\tau).$$

It follows that

$$\cos\frac{\tau\pi}{2}\,|||\,|h|^\tau\,||| \leqslant 2^{-\tau}\cos\frac{\tau\pi}{2}\,\big(|||\,|u^+|^\tau\,|||+|||\,|u^-|^\tau\,|||\big)$$
$$\leqslant 2^{-\tau}\Big(\big(\varphi(u)+\alpha(P)\big)^\tau+\big(\varphi(u)-\alpha(P)\big)^\tau\Big)\leqslant 2^{1-\tau}\big(\varphi(u)\big)^\tau.$$

Hence by the definition of $\sigma^+(|P|)$ the first member is $\leqslant 2^{1-\tau}\big(\sigma^+(|P|)\big) = 2^{1-\tau}|||P|||^\tau$.

We turn to the new proof of [1], VI. 3.8, announced in the Introduction.

4.2. THEOREM. *Assume that $P\in \mathrm{Re}L(m)$ satisfies $\vartheta_+(P) = \vartheta_+(-P) = 0$. Then $P\in E$ iff P is univalent. In this case we have $\alpha(P) = \int PV\,dm$ $\forall V\in M$.*

Proof. (1) For $t\in \mathbf{R}$ it follows from 2.7 (i) and 2.8 that $a(e^{tP})\geqslant e^{-\theta(-tP)}$ and hence

$$a(e^{tP})\,a(e^{-tP})\geqslant e^{-\theta(-tP)-\theta(tP)}$$

which by 1.9 (ii) is $=1$ if P is univalent. Hence $P\in E$.

(2) Assume that $P\in E$ with conjugate function $Q\in\mathrm{Re}L(m)$ and $h:=P+iQ$. For $\varepsilon=\pm1$,

$$e^{\varepsilon\alpha(P)} = (e^{t\varepsilon\alpha(P)})^{1/t} = \big(\varphi(e^{t\varepsilon h})\big)^{1/t}\leqslant \big(a(e^{t\varepsilon P})\big)^{1/t}\quad \forall t>0,$$

so that from 2.10 and 2.8 we obtain $\varepsilon\alpha(P)\leqslant -\theta(-\varepsilon P)$ or $\alpha(P)\leqslant -\theta(-P)\leqslant \theta(P)\leqslant \alpha(P)$. Hence P is univalent by 1.9 (ii) and $\alpha(P) = \theta(P) = \vartheta(P)$ by 1.6 (i).

We collect some more results which are rather simple consequences of the theorems in Section 2 as well.

4.3. PROPOSITION. *Let $P\in E$. Assume that $G\in\mathrm{Re}L(m)$ satisfies $G\geqslant 0$, $G\geqslant P$ and $\vartheta(G)<\infty$. Then $\vartheta(G-P)<\infty$ and hence $|||P|||<\infty$. Furthermore*

$$\vartheta(G) = \alpha(P)+\vartheta(G-P).$$

Proof. Let $Q\in\mathrm{Re}L(m)$ be the conjugate function to P and $h:=P+iQ$.

(1) In view of 2.6 and 2.8 there exists a function $u\in H^+$ with $\mathrm{Re}\,u\geqslant G\geqslant P$ and $\mathrm{Re}\,\varphi(u) = \vartheta(G)$. We have $\mathrm{Re}(u-h)\geqslant 0$ and $u-h\in H^+$ since for $t\geqslant 0$ we have

$$e^{-t(u-h)} = e^{-tu}e^{th}\in H^\#\cap L^\infty(m) = H.$$

Furthermore

$$\varphi(e^{-t(u-h)}) = \varphi(e^{-tu})\varphi(e^{th}),$$

so that $\varphi(u-h) = \varphi(u)-\alpha(P)$. From [1], V. 4.1.3), we conclude that

$$\vartheta(G-P)\leqslant\vartheta(\mathrm{Re}\,u-P) = \vartheta\big(\mathrm{Re}(u-h)\big)\leqslant \mathrm{Re}\,\varphi(u-h)$$
$$= \mathrm{Re}\,\varphi(u)-\alpha(P) = \vartheta(G)-\alpha(P).$$

In particular, $\vartheta(G-P)<\infty$. Hence from $P = G-(G-P)$, $|P|\leqslant G+(G-P)$ we conclude that $|||P|||<\infty$.

(2) In view of 2.6 and 2.8 there exists a function $v\in H^+$ with $\mathrm{Re}\,v\geqslant G-P$ and $\mathrm{Re}\,\varphi(v) = \vartheta(G-P)$. We have $\mathrm{Re}(v+h) = \mathrm{Re}\,v+P\geqslant G\geqslant 0$. As above we conclude that $v+h\in H^+$ and $\varphi(v+h) = \varphi(v)+\alpha(P)$. Then from [1], V. 4.1.3) it follows that

$$\vartheta(G)\leqslant\vartheta\big(\mathrm{Re}(v+h)\big)\leqslant\mathrm{Re}\,\varphi(v+h) = \mathrm{Re}\,\varphi(v)+\alpha(P) = \vartheta(G-P)+\alpha(P).$$

The proof is complete.

4.4. COROLLARY. *Let $P\in E$. Assume that $G\in\mathrm{Re}L(m)$ satisfies $G\geqslant P$ and $\theta(G)<\infty$. Then $\vartheta(G-P)<\infty$ and $\theta(-G)<\infty$, and hence $|||G|||<\infty$ and $|||P|||<\infty$. Furthermore*

$$\theta(G) = \alpha(P)+\lim_{t\uparrow\infty}\vartheta\big((G-P)+(-G-t)^+\big).$$

Proof. For $t\in\mathbf{R}$ we have

$$(G+t)^+-(P+t) = (G-P)+(-G-t)^+.$$

Furthermore $(G+t)^+\geqslant 0$ and $(G+t)^+\geqslant G+t\geqslant P+t$, and $\vartheta\big((G+t)^+\big)<\infty$ in view of 1.5 since $\theta(G+t)\leqslant\theta(G)+t<\infty$. Thus from 4.3 applied to $P+t\in E$ and $(G+t)^+$ we obtain $\vartheta(G-P)<\infty$ and $\vartheta\big((-G-t)^+\big)<\infty$, or $\theta(-G)<\infty$ after 1.5. Once more by 1.5 we have $\vartheta(G^+)$, $\vartheta(G^-)<\infty$ and hence $|||G|||<\infty$, and thus $|||P|||<\infty$ since $|P|\leqslant|G|+(G-P)$. Furthermore 4.3 implies that

$$\vartheta\big((G+t)^+\big) = \alpha(P+t)+\vartheta\big((G-P)+(-G-t)^+\big),$$

which in view of $(G+t)^+ = t+\mathrm{Max}(G,-t)$ means that

$$\vartheta\big(\mathrm{Max}(G,-t)\big) = \alpha(P)+\vartheta\big((G-P)+(-G-t)^+\big).$$

For $t\uparrow\infty$ we obtain the last assertion.

The special cases $G = P$ and $G = P^+$ lead to remarkable representation formules for $\alpha(P)$.

4.5. COROLLARY. *Let $P\in E$ with $\theta(P)<\infty$. Then $|||P|||<\infty$ and*

$$\alpha(P) = \theta(P)-\vartheta_+(-P) = \vartheta(P^+)-\vartheta(P^-).$$

We conclude the section with a univalence result which follows from 4.3.

4.6. PROPOSITION. *Let $P \in E$. Assume that there exists a univalent function $G \in \operatorname{Re} L(m)$ such that $|P| \leqslant G$. Then P is univalent and*

$$a(P) = \int PV\,dm \quad \forall V \in M.$$

Proof. We have $G \geqslant 0$, so that 4.3 can be applied to G and εP with $\varepsilon = \pm 1$. For $V \in M$ we obtain

$$\int GV\,dm = \vartheta(G) = a(\varepsilon P) + \vartheta(G - \varepsilon P) \geqslant a(\varepsilon P) + \int (G - \varepsilon P) V\,dm,$$

and hence $\varepsilon a(P) = a(\varepsilon P) \leqslant \varepsilon \int PV\,dm$. It follows that $a(P) = \int PV\,dm$.

In 5.3 we shall prove that for $P \in \operatorname{Re} L(m)$ with $\vartheta_+(P) = 0$ there exist nonnegative univalent functions $G \in \operatorname{Re} L(m)$ such that $G \geqslant P$. Therefore 4.6 contains the relevant direction in 4.2. On the other hand, it is not clear whether a nonnegative univalent function $G \in \operatorname{Re} L(m)$ satisfies $\vartheta_+(G) = 0$. However, in view of 1.10, this is true if M is weakly compact.

5. The representation theorems in terms of $H^\wedge$.

5.1. PROPOSITION. *Let $u_n \in H^+$ $(n = 1, 2, \ldots)$ with $\operatorname{Im} \varphi(u_n) = 0$ and $\sum_{n=1}^{\infty} \operatorname{Re} \varphi(u_n) < \infty$. Then there exists a unique function $h \in H^+$ such that*

$$(*) \qquad \Big|\Big|\Big| \, \Big| h - \sum_{l=1}^{n} u_l \Big|^\tau \Big|\Big|\Big| \to 0 \quad \text{for} \quad n \to \infty \quad \forall 0 < \tau < 1.$$

Furthermore we have

(1) *For each sequence $1 \leqslant p(1) < \ldots < p(n) < \ldots$ with*

$$\sum_{l=p(n)+1}^{\infty} \varphi(u_l) \leqslant \frac{1}{2^n}$$

we have $\sum_{l=1}^{p(n)} u_l \to h$ pointwise for $n \to \infty$.

(1′) $\sum_{l=1}^{\infty} \operatorname{Re} u_l = \operatorname{Re} h$ *pointwise.*

(2) *For $0 < \tau < 1$ we have*

$$\cos \frac{\tau\pi}{2} \Big|\Big|\Big| \, \Big| h - \sum_{l=1}^{n} u_l \Big|^\tau \Big|\Big|\Big| \leqslant \Big(\sum_{l=n+1}^{\infty} \varphi(u_l) \Big)^\tau \quad \forall n \in \mathbf{N}.$$

$$(2') \qquad \Big|\Big|\Big| \operatorname{Re} h - \sum_{l=1}^{n} \operatorname{Re} u_l \Big|\Big|\Big| \leqslant \sum_{l=n+1}^{\infty} \varphi(u_l) \quad \forall n \in \mathbf{N}.$$

$$(3) \qquad \sum_{l=1}^{\infty} \varphi(u_l) = \varphi(h).$$

The function $h \in H^+$ thus obtained will be denoted by $\bigoplus_{n=1}^{\infty} u_n$.

Proof. (i) There is at most one function $h \in H^+$ with $(*)$. This is clear in view of $(a+b)^\tau \leqslant a^\tau + b^\tau$ for $a, b \geqslant 0$.

(ii) We put $h_n := \sum_{l=1}^{n} u_l \in H^+$ for $n \in \mathbf{N}$. From the Kolmogorov estimation on H^+ [1], V. 5.6, we obtain for $0 < \tau < 1$ and $V \in M$

$$(**) \qquad \cos \frac{\tau\pi}{2} \int |h_q - h_p|^\tau V\,dm \leqslant \Big(\sum_{l=p+1}^{q} \varphi(u_l) \Big)^\tau \quad \forall 1 \leqslant p < q.$$

We fix a sequence $1 \leqslant p(1) < \ldots < p(n) < \ldots$ with

$$\sum_{l=p(n)+1}^{\infty} \varphi(u_l) \leqslant \frac{1}{2^n} \quad \forall n \in \mathbf{N}.$$

Then

$$\cos \frac{\tau\pi}{2} \int |h_{p(n+1)} - h_{p(n)}|^\tau V\,dm \leqslant \frac{1}{2^{\tau n}} \quad \forall n \in \mathbf{N}.$$

For some $V \in M$ which is > 0 on the entire space it follows from the Beppo Levi theorem that the series $\sum_{n=1}^{\infty} |h_{p(n+1)} - h_{p(n)}|^\tau$ is pointwise convergent. Hence $\sum_{n=1}^{\infty} |h_{p(n+1)} - h_{p(n)}|$ is pointwise convergent as well, so that the $h_{p(n)}$ tend to some $h \in L(m)$ for $n \to \infty$. In view of [1], V. 4.5, we have $h \in H^+$ and $\varphi(h_{p(n)}) \to \varphi(h)$, so that (3) follows. Furthermore from $(**)$ and the Fatou theorem we obtain

$$\cos \frac{\tau\pi}{2} \int |h - h_p|^\tau V\,dm \leqslant \Big(\sum_{l=p+1}^{\infty} \varphi(u_l) \Big)^\tau \quad \forall p \in \mathbf{N},$$

so that (2) follows. Thus our function $h \in H^+$ satisfies $(*)$, and also (1) by its definition.

(iii) From [1], V. 3.1 (3), we have $\int (\operatorname{Re} u_l) V\,dm \leqslant \varphi(u_l)$ for $l \in \mathbf{N}$ and $V \in M$. Thus by the Beppo Levi theorem the series $\sum_{l=1}^{\infty} \operatorname{Re} u_l$ is pointwise convergent. Since a subsequence of the sequence of the partial sums tends to $\operatorname{Re} h$, we obtain (1′). Furthermore

$$\int (\operatorname{Re} h) V\,dm = \sum_{l=1}^{\infty} \int (\operatorname{Re} u_l) V\,dm \quad \forall V \in M.$$

For $n \in \boldsymbol{N}$ it follows that

$$\int \Big(\mathrm{Re}\,h - \sum_{l=1}^{n} \mathrm{Re}\,u_l\Big) V\,dm = \sum_{l=n+1}^{\infty} \int (\mathrm{Re}\,u_l) V\,dm \leqslant \sum_{l=n+1}^{\infty} \varphi(u_l),$$

and hence (2′). The proof is complete.

5.2. Remark. *Let* $u_{pq} \in H^+$ $(p, q = 1, 2, \ldots)$ *with* $\mathrm{Im}\,\varphi(u_{pq}) = 0$ *and* $\sum_{p,q=1}^{\infty} \mathrm{Re}\,\varphi(u_{pq}) < \infty$. *Then the functions*

$$h_p := \bigoplus_{q=1}^{\infty} u_{pq} \in H^+ \quad \forall p \in \boldsymbol{N} \text{ and } h := \bigoplus_{p=1}^{\infty} h_p \in H^+$$

are well-defined. And for each bijection $\boldsymbol{N} \to \boldsymbol{N} \times \boldsymbol{N}$: $l \mapsto (p(l), q(l))$ *we have*

$$h = \bigoplus_{l=1}^{\infty} u_{p(l),q(l)}.$$

This follows from 5.1 via a chain of standard estimations.

As in [6] and [7] we define $H^\wedge$ to consist of the functions

$$h = \bigoplus_{l=1}^{\infty} u_l \quad \text{with } u_l \in H \ (l = 1, 2, \ldots) \text{ such that } \mathrm{Re}\,u_l \geqslant 0 \text{ and}$$

$$\mathrm{Im}\,\varphi(u_l) = 0 \text{ and } \sum_{l=1}^{\infty} \mathrm{Re}\,\varphi(u_l) < \infty.$$

Thus $\{h \in H: \mathrm{Re}\,h \geqslant 0 \text{ and } \mathrm{Im}\,\varphi(h) = 0\} \subset H^\wedge \subset H^+$. And from 5.2 we conclude that for any sequence of functions $h_n \in H^\wedge$ $(n = 1, 2, \ldots)$ with $\sum_{n=1}^{\infty} \varphi(h_n) < \infty$ we have $\bigoplus_{n=1}^{\infty} h_n \in H^\wedge$.

For $h \in H^\wedge$ we see from 5.1 (2′) that the nonnegative function $\mathrm{Re}\,h \in \mathrm{Re}\,L(m)$ belongs to $L^\wedge$ and is univalent with $\varphi(h) = \mathrm{Re}\,\varphi(h) = \vartheta(\mathrm{Re}\,h)$. All this is far from true for the functions $h \in H^+$. For in 3.5 we have found in (8) an example of a function $h \in H^+$ such that $\mathrm{Re}\,h \notin L^\wedge$, so that in view of (7) the function $\mathrm{Re}\,h$ is not univalent as well, and moreover there is no univalent $G \in \mathrm{Re}\,L(m)$ with $G \geqslant \mathrm{Re}\,h$.

As in [6] and [7] we define the functional $\sigma^\wedge$: $\mathrm{Re}\,L(m) \to [-\infty, \infty]$ to be

$$\sigma^\wedge(P) = \mathrm{Inf}\{\varphi(u) - c:\ u \in H^\wedge \text{ and } c \geqslant 0 \text{ with } \mathrm{Re}\,u - c \geqslant P\} \quad \forall P \in \mathrm{Re}\,L(m).$$

Note that $\sigma^\wedge(P) < \infty$ iff there exists some $u \in H^\wedge$ with $\mathrm{Re}\,u \geqslant P$. The functional $\sigma^\wedge$ shares the properties of σ^+ listed in 2.1. Furthermore

$$\sigma^+(P) \leqslant \sigma^\wedge(P) \quad \forall P \in \mathrm{Re}\,L(m).$$

The above example shows that for a function $h \in H^+$ it can happen that $\sigma^\wedge(\mathrm{Re}\,h) = \infty$, whereas of course $\sigma^+(\mathrm{Re}\,h) < \infty$. Thus σ^+ and $\sigma^\wedge$ need not be equal. However, in [6] the following theorem is proved which we include for the sake of completeness.

5.3. Theorem. *For* $P \in \mathrm{Re}\,L(m)$ *we have* $\sigma^\wedge(P) < \infty$ *iff* $P \in L^\wedge$. *In this case* $\sigma^\wedge(P) = \sigma^+(P) = \theta(P)$.

Proof. (i) The implication $\sigma^\wedge(P) < \infty \Rightarrow P \in L^\wedge$ is obvious.

(ii) For each sequence of numbers $0 = t_0 < t_1 < \ldots < t_n < \ldots$ with $t_n \uparrow \infty$ one verifies the formula

$$x = \sum_{n=0}^{\infty} \big(\mathrm{Min}(x, t_{n+1}) - t_n\big)^+ \quad \text{for all real } x \geqslant 0.$$

(iii) Let $P \in L^\wedge$. Then $P^+ \in L^\wedge$ as well. It suffices to prove $\sigma^\wedge(P^+) < \infty$ so that we can assume $P \geqslant 0$. We have $\vartheta\big((P-t)^+\big) \to 0$ for $t \uparrow \infty$, so that we can choose $0 = t_0 < t_1 < \ldots < t_n < \ldots$ with $t_n \uparrow \infty$ such that $\vartheta\big((P-t_n)^+\big) < 1/2^n$ for $n \geqslant 1$. Thus $\vartheta\Big(\big(\mathrm{Min}(P, t_{n+1}) - t_n\big)^+\Big) < 1/2^n$ for $n \geqslant 1$. In view of 2.2 and 2.8 there exist functions $u_n \in H$ $(n = 1, 2, \ldots)$ with $\mathrm{Re}\,u_n \geqslant \big(\mathrm{Min}(P, t_{n+1}) - t_n\big)^+ \geqslant 0$ and $\varphi(u_n) = \mathrm{Re}\,\varphi(u_n) < 1/2^n$. In addition let $u_0 := t_1$. Then $h := \bigoplus_{n=0}^{\infty} u_n \in H^\wedge$ in view of (ii) satisfies

$$\mathrm{Re}\,h = \sum_{n=0}^{\infty} \mathrm{Re}\,u_n \geqslant \sum_{n=0}^{\infty} \big(\mathrm{Min}(P, t_{n+1}) - t_n\big)^+ = P.$$

It follows that $\sigma^\wedge(P) < \infty$.

(iv) It remains to prove that $\sigma^\wedge(P) \leqslant \sigma^+(P)$ for $P \in L^\wedge$. This is clear for $P \in \mathrm{Re}\,L(m)$ bounded above since then $\sigma^\wedge(P) \leqslant \sigma(P) = \sigma^+(P)$ in view of the definitions and 2.2. Let now $P \in L^\wedge$, and let $h = \bigoplus_{l=1}^{\infty} u_l \in H^\wedge$ as in the definition of $H^\wedge$ with $\mathrm{Re}\,h \geqslant P$. For $n \in \boldsymbol{N}$ and real $t \geqslant \sum_{l=1}^{n} \mathrm{Re}\,u_l$, $(P-t)^+ \leqslant \sum_{l=n+1}^{\infty} \mathrm{Re}\,u_l$ and hence $\sigma^\wedge\big((P-t)^+\big) \leqslant \sum_{l=n+1}^{\infty} \varphi(u_l)$ by the definition of $\sigma^\wedge$. Now $P = \mathrm{Min}(P, t) + (P-t)^+$ and hence

$$\sigma^\wedge(P) \leqslant \sigma^\wedge\big(\mathrm{Min}(P, t)\big) + \sigma^\wedge\big((P-t)^+\big) \leqslant \sigma^+\big(\mathrm{Min}(P, t)\big) + \sigma^\wedge\big((P-t)^+\big)$$
$$\leqslant \sigma^+(P) + \sum_{l=n+1}^{\infty} \varphi(u_l).$$

For $n \to \infty$ the assertion follows.

In contrast to the case σ^+ the next remark is nontrivial.

5.4. Remark. *For nonnegative* $P \in \mathrm{Re}\,L(m)$ *we have*

$$\sigma^\wedge(P) = \mathrm{Inf}\{\varphi(u):\ u \in H^\wedge \text{ with } \mathrm{Re}\,u \geqslant P\}.$$

Proof. For $0 \leqslant P \in \mathrm{Re}\,L(m)$ let $\tau(P)$ denote the infimum in question.

It is clear that $\sigma^\wedge(P) \leqslant \tau(P)$, and that $\tau(P) \leqslant \sigma(P) = \sigma^+(P) = \sigma^\wedge(P)$ if P is bounded. Furthermore τ is subadditive. We have to prove that $\tau(P) \leqslant \sigma^\wedge(P)$ and can assume that $\sigma^\wedge(P) < \infty$. So let $h = \bigoplus_{l=1}^{\infty} u_l \in H^\wedge$ as in the definition of $H^\wedge$ with $\mathrm{Re}\,h \geqslant P$. We have to repeat the last lines of the last proof. For $n \in N$ and $t \geqslant \sum_{l=1}^{n} \mathrm{Re}\,u_l$ we have $(P-t)^+ \leqslant \sum_{l=n+1}^{\infty} \mathrm{Re}\,u_l$ and hence $\tau\big((P-t)^+\big) \leqslant \sum_{l=n+1}^{\infty} \varphi(u_l)$. Furthermore

$$\tau(P) \leqslant \tau\big(\mathrm{Min}(P,t)\big) + \tau\big((P-t)^+\big) \leqslant \sigma^\wedge\big(\mathrm{Min}(P,t)\big) + \tau\big((P-t)^+\big)$$
$$\leqslant \sigma^\wedge(P) + \sum_{l=n+1}^{\infty} \varphi(u_l),$$

so that for $n \to \infty$ the assertion follows.

We turn to the connection with the abstract conjugation. For $h \in H^\wedge$ we have $0 \leqslant \mathrm{Re}\,h \in L^\wedge$. Hence $e^{t\mathrm{Re}h} \in L^\# \ \forall t \geqslant 0$ by 2.9. We conclude from 5.1 that

$$e^{th} \in H^\# \quad \text{and} \quad \varphi(e^{th}) = e^{t\varphi(h)} \quad \forall t \in \boldsymbol{R}.$$

Thus $\mathrm{Re}\,h \in E$; this can also be deduced from 4.2 since $\mathrm{Re}\,h$ is known to be univalent. And since $\varphi(h)$ is real, it follows that the conjugate function to $\mathrm{Re}\,h$ is $\mathrm{Im}\,h$, and that $a(\mathrm{Re}\,h) = \varphi(h)$.

Thus for $h = P + iQ \in H^\wedge - H^\wedge$ we have $P \in E$ with conjugate function Q, and $a(P) = \varphi(h)$. Furthermore it is clear that $\vartheta_+(P) = \vartheta_+(-P) = 0$. The main results of the present section are in the opposite direction.

5.5. Theorem. *Assume that $P \in E$ is such that $0 \leqslant P \in L^\wedge$. Then for each $\varepsilon > 0$ there is a representation $P = G - F$ with $F, G \in \mathrm{Re}\,H^\wedge$ and $\vartheta(F) < \varepsilon$.*

Proof. (1) Let $\varepsilon = \sum_{l=1}^{\infty} \varepsilon_l$ with $\varepsilon_l > 0$ $(l = 1, 2, \ldots)$. We put $P_0 := P$ and choose $P_l \in E$ with $0 \leqslant P_l \in L^\wedge$ and $h_l \in H^\wedge$ $(l = 1, 2, \ldots)$ via induction as follows.

$l = 1$: By 5.3 and 5.4 there exists $h_1 \in H^\wedge$ such that $\mathrm{Re}\,h_1 \geqslant P = P_0$ and

$$\varphi(h_1) = \mathrm{Re}\,\varphi(h_1) = \vartheta(\mathrm{Re}\,h_1) < \sigma^\wedge(P) + \varepsilon_1 = \vartheta(P) + \varepsilon_1.$$

Then put

$$P_1 := \mathrm{Re}\,h_1 - P = \mathrm{Re}\,h_1 - P_0,$$

so that $P_1 \in E$ with $0 \leqslant P_1 \in L^\wedge$. Furthermore $\vartheta(P_1) < \varepsilon_1$ since the functional ϑ is of course linear on the subspace of the univalent functions $\subset \mathrm{Re}\,L(m)$.

$1 \leqslant l-1 \Rightarrow l$: By 5.3 and 5.4 there exists $h_l \in H^\wedge$ such that $\mathrm{Re}\,h_l \geqslant P_{l-1}$ and

$$\varphi(h_l) = \mathrm{Re}\,\varphi(h_l) = \vartheta(\mathrm{Re}\,h_l) < \sigma^\wedge(P_{l-1}) + \varepsilon_l = \vartheta(P_{l-1}) + \varepsilon_l.$$

Then put

$$P_l := \mathrm{Re}\,h_l - P_{l-1},$$

so that $P_l \in E$ with $0 \leqslant P_l \in L^\wedge$ and $\vartheta(P_l) < \varepsilon_l$.

(2) In view of $\vartheta(P_l) < \varepsilon_l$ for $l \geqslant 1$ and $\sum_{l=1}^{\infty} \varepsilon_l < \infty$ the Beppo Levi theorem implies that $P_l \to 0$ pointwise for $l \to \infty$.

(3) We have $\mathrm{Re}\,h_l = P_{l-1} + P_l$ for $l \geqslant 1$ and hence $\varphi(h_l) = \vartheta(\mathrm{Re}\,h_l) < \varepsilon_{l-1} + \varepsilon_l$ for $l \geqslant 2$. Therefore we can form

$$f := \bigoplus_{l=1}^{\infty} h_{2l} \in H \quad \text{and} \quad g := \bigoplus_{l=0}^{\infty} h_{2l+1} \in H^\wedge.$$

We have

$$F := \mathrm{Re}\,f = \sum_{l=1}^{\infty} \mathrm{Re}\,h_{2l} \quad \text{and} \quad G := \mathrm{Re}\,g = \sum_{l=0}^{\infty} \mathrm{Re}\,h_{2l+1},$$

$$\vartheta(F) = \varphi(f) = \sum_{l=1}^{\infty} \varphi(h_{2l}) < \sum_{l=1}^{\infty} (\varepsilon_{2l-1} + \varepsilon_{2l}) = \varepsilon.$$

(4) Now we have

$$(-1)^{l-1}\mathrm{Re}\,h_l = (-1)^{l-1}P_{l-1} - (-1)^l P_l \quad \text{for} \quad l \geqslant 1,$$
$$P = \sum_{l=1}^{n} (-1)^{l-1}\mathrm{Re}\,h_l + (-1)^n P_n \quad \text{for} \quad n \geqslant 1,$$
$$P = \sum_{l=0}^{n-1} \mathrm{Re}\,h_{2l+1} - \sum_{l=1}^{n} \mathrm{Re}\,h_{2l} + P_{2n} \quad \text{for} \quad n \geqslant 1.$$

For $n \to \infty$ we obtain $P = G - F$ and hence the assertion.

5.6. Corollary. *A function $P \in \mathrm{Re}\,L(m)$ satisfies $P \in E$ and $0 \leqslant P \in L^\wedge$ iff there exists a sequence of functions $P_n \in \mathrm{Re}\,H^\wedge$ $(n = 1, 2, \ldots)$ with $P_n \downarrow P$ for $n \to \infty$.*

Proof. (1) Assume that $P_n \in \mathrm{Re}\,H^\wedge$ $(n = 1, 2, \ldots)$ with $P_n \downarrow P$ for $n \to \infty$. Then $0 \leqslant P \in L^\wedge$. And P is univalent and hence belongs to E in view of 4.2.

(2) Assume that $P \in E$ with $0 \leqslant P \in L^\wedge$. Choose $\varepsilon_l > 0$ $(l = 1, 2, \ldots)$ with $\sum_{l=1}^{\infty} \varepsilon_l < \infty$. By 5.5 we have $P = G_l - F_l$ with $F_l, G_l \in \mathrm{Re}\,H^\wedge$ and

$\vartheta(F_l) < \varepsilon_l$. For the functions $f_l \in H^\wedge$ with $\mathrm{Re} f_l = F_l$ we have

$$\sum_{l=1}^{\infty} \varphi(f_l) = \sum_{l=1}^{\infty} \vartheta(F_l) < \sum_{l=1}^{\infty} \varepsilon_l < \infty.$$

Therefore we can form the functions $h_n := \bigoplus_{l=n}^{\infty} f_l \in H^\wedge$ for $n \geqslant 1$. We have $\mathrm{Re} h_n = \sum_{l=n}^{\infty} F_l$ pointwise and hence $\mathrm{Re} h_n \downarrow 0$ pointwise for $n \to \infty$. For $n \geqslant 1$ now $P = (G_n + \mathrm{Re} h_{n+1}) - \mathrm{Re} h_n = P_n - \mathrm{Re} h_n$ with $P_n := G_n + \mathrm{Re} h_{n+1} \in \mathrm{Re} H^\wedge$. It follows that $P_n = P + \mathrm{Re} h_n \downarrow P$ for $n \to \infty$ and hence the assertion.

5.7. THEOREM. *For $P \in \mathrm{Re} L(m)$ the following are equivalent:*

(i) *$P \in E$ and $\vartheta_+(P) = \vartheta_+(-P) = 0$.*

(ii) *P is univalent and $\vartheta_+(P) = \vartheta_+(-P) = 0$.*

(iii) *P is in the $|||\cdot|||$ closure of $\mathrm{Re} H \subset \mathrm{Re} L^\infty(m)$.*

(iv) *$P \in \mathrm{Re} H^\wedge - \mathrm{Re} H^\wedge$.*

Proof. (iv) $\Rightarrow$ (iii) is clear from 5.1, (iii) $\Rightarrow$ (ii) is obvious, and (ii) $\Leftrightarrow$ (i) is 4.2. Thus it remains to prove (i) $\Rightarrow$ (iv). $\vartheta_+(P) = 0$ means that $P \in L^\wedge$ and hence $\sigma^\wedge(P) < \infty$ by 5.3. Thus there exists a function $h \in H^\wedge$ with $\mathrm{Re} h \geqslant P$. For $Q := \mathrm{Re} h - P \geqslant 0$ we have $Q \in E$, and $Q \in L^\wedge$ in view of $\vartheta_+(-P) = 0$ or $-P \in L^\wedge$. From 5.5 we obtain a representation $Q = G - F$ with $F, G \in \mathrm{Re} H^\wedge$. It follows that $P = \mathrm{Re} h - Q = (\mathrm{Re} h + F) - G \in \mathrm{Re} H^\wedge - \mathrm{Re} H^\wedge$. The proof is complete.

We conclude with a remark on the relation between $H^\wedge$ and H^+ which follows from the above results combined with our previous Lemma 3.3.

5.8. PROPOSITION. *Let $h = P + iQ \in H^+$ with $\mathrm{Im}\varphi(h) = 0$ be such that $|Q| \leqslant cP$ for some real $c > 0$. Then $h \in H^\wedge - H^\wedge$ and for each $\varepsilon > 0$ there is a representation $h = g - f$ with $f, g \in H^\wedge$ and $\varphi(f) = \vartheta(\mathrm{Re} f) < \varepsilon$. If, in particular, $|Q| \leqslant P$, then $h \in H^\wedge$.*

Proof. (1) The functions $h_t \in H$ for $t > 0$ dealt with in 3.3 have the properties $\mathrm{Re} h_t \geqslant 0$ and $\mathrm{Im}\varphi(h_t) = 0$ and $h_t \to h$ pointwise and $|||h - h_t||| \to 0$ for $t \uparrow \infty$. In particular, $|||P - \mathrm{Re} h_t||| \to 0$, so that P is in the $|||\cdot|||$ closure of $\mathrm{Re} H$. Hence $P \in E$ and $0 \leqslant P \in L^\wedge$ by 5.7. From 5.5 we obtain functions $f, g \in H^\wedge$ such that $P = \mathrm{Re} g - \mathrm{Re} f$ and $\varphi(f) = \vartheta(\mathrm{Re} f) < \varepsilon$. Now we have in addition $|h| \in L^\wedge$ since $|||\,|h| - |h_t|\,||| \to 0$ for $t \uparrow \infty$ and $|h_t| \leqslant |h|$. For $s \in \boldsymbol{R}$ it follows that

$$|\exp(sh_t)| = \exp(s\,\mathrm{Re} h_t) \leqslant \exp(|s|\,|h_t|) \leqslant \exp(|s|\,|h|),$$

which belongs to $L^\#$ in view of 2.9. Hence

$$e^{sh} \in H^\# \quad \text{and} \quad \varphi(e^{sh}) = e^{s\varphi(h)} \quad \forall s \in \boldsymbol{R}.$$

Since $\varphi(h)$ is real, it follows that the conjugate function to P is $= Q$. Thus $P = \mathrm{Re} g - \mathrm{Re} f$ implies that $h = g - f$.

(2) Assume now that $|Q| \leqslant P$. Then in addition $\mathrm{Re} h_t \uparrow P$ for $t \uparrow \infty$. Thus $h_n = \sum_{l=1}^{n} u_l$ for $n \in \boldsymbol{N}$, where the functions $u_l \in H$ $(l = 1, 2, \ldots)$ have the properties $\mathrm{Re} u_l \geqslant 0$ and $\mathrm{Im}\varphi(u_l) = 0$ and $\sum_{l=1}^{\infty} \mathrm{Re}\varphi(u_l) < \infty$. From $h = \sum_{l=1}^{\infty} u_l$ combined with 5.1 we obtain $h = \bigoplus_{l=1}^{\infty} u_l \in H^\wedge$. The proof is complete.

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Received February 25, 1982 (1739)