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Completeness of sequential convergence groups

by

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Dedicated to Professor Jan Mikusiński in honour of his 70th birthday

Abstract. The completion theory developed by J. Novák for sequential convergence groups with maximal convergences is extended to the case of general convergence groups. The properties of the minimal completion are studied and, in particular, necessary and sufficient conditions are given for the minimal completion of a Fréchet convergence group to be Fréchet.

1. Convergence groups. This paper is concerned with sequential commutative convergence groups. As neither filter convergence nor the non-commutative case will be considered, the words "sequential commutative" will be omitted in the sequel.

Recall ([6], [2]) that a *convergence* $\mathfrak{L}$ on a set L is a set of pairs $(\langle x_n\rangle, x)$ satisfying the axioms

$(\mathscr{L}_0)$ $(\langle x_n\rangle, x) \in \mathfrak{L}$ and $(\langle x_n\rangle, y) \in \mathfrak{L}$ imply $x = y$,

$(\mathscr{L}_1)$ $(\langle x\rangle, x) \in \mathfrak{L}$ for each $x \in L$ and

$(\mathscr{L}_2)$ $(\langle x_n\rangle, x) \in \mathfrak{L}$ implies $(\langle x_{n_i}\rangle, x) \in \mathfrak{L}$ for each subsequence $\langle n_i\rangle$ of $\langle n\rangle$.

If $(\langle x_n\rangle, x) \in \mathfrak{L}$, then we say that the sequence $\langle x_n\rangle$ $\mathfrak{L}$*-converges* to x, in symbols $\mathfrak{L}\text{-}\lim x_n = x$. A convergence is said to be *maximal* and denoted by $\mathfrak{L}^*$ if it also satisfies the axiom

$(\mathscr{L}_3)$ Let $(\langle x_n\rangle, x) \in L^N \times L$. If for each $\langle n_i\rangle$ there exists $\langle n_{i_j}\rangle$ such that $(\langle x_{n_{i_j}}\rangle, x) \in \mathfrak{L}$, then $(\langle x_n\rangle, x) \in \mathfrak{L}$.

A *convergence space* $(L, \mathfrak{L}, \lambda)$ is a closure space where the closure operator λ is induced by the convergence $\mathfrak{L}$, i.e., $\lambda A = \{x \mid \exists \langle x_n\rangle, \bigcup (x_n) \subset A, (\langle x_n\rangle, x) \in \mathfrak{L}\}$. The *topological modification* of the closure operator λ, i.e., the finest topological closure operator coarser than λ, will be denoted by λ^{ω_1}.

The notion of a convergence group was introduced by J. Novák in [5] and a general theory of such groups was given in [7]. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space and let $(L, +)$ be a group. Then $(L, \mathfrak{L}, \lambda, +)$ is a *convergence group* if the mapping $f\colon (L, \mathfrak{L}, \lambda) \times (L, \mathfrak{L}, \lambda) \to (L, \mathfrak{L}, \lambda)$ defined by $f(x, y) = x - y$ is sequentially continuous.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space and let $(L, +)$ be a group. Then ([7]) $(L, \mathfrak{L}, \lambda, +)$ is a convergence group iff the following condition is satisfied:

$(\mathscr{S}\mathscr{G})$ If $(\langle x_n\rangle, x) \in \mathfrak{L}$ and $(\langle y_n\rangle, y) \in \mathfrak{L}$, then there is a subsequence $\langle n_i\rangle$ of $\langle n\rangle$ such that $(\langle x_{n_i} - y_{n_i}\rangle, x-y) \in \mathfrak{L}$.

If the convergence $\mathfrak{L}$ is maximal, i.e., $\mathfrak{L} = \mathfrak{L}^*$, then $(L, \mathfrak{L}, \lambda, +)$ is a convergence group iff the following condition is satisfied:

$(\mathscr{S}\mathscr{G}^*)$ If $(\langle x_n\rangle, x) \in \mathfrak{L}$ and $(\langle y_n\rangle, y) \in \mathfrak{L}$, then $(\langle x_n - y_n\rangle, x-y) \in \mathfrak{L}$.

J. Novák has shown in [7] that there are convergence groups satisfying condition $(\mathscr{S}\mathscr{G}^*)$ which are not maximal, i.e., for which $\mathfrak{L} \neq \mathfrak{L}^*$. Hence we can distinguish three classes of convergence groups.

Definition 1. A convergence group will be called a *C-group*, a convergence group satisfying condition $(\mathscr{S}\mathscr{G}^*)$ will be called a *C^+-group* and a convergence group for which $\mathfrak{L} = \mathfrak{L}^*$ will be called a *C^*-group.*

If we denote by $\boldsymbol{C}$, $\boldsymbol{C}^+$ and $\boldsymbol{C}^*$ the corresponding classes of convergence groups, then we have $\boldsymbol{C}^* \subsetneqq \boldsymbol{C}^+ \subsetneqq \boldsymbol{C}$.

Example 1. Let Q be the group of rational numbers and let $\mathfrak{L}$ be the usual convergence on Q. Then $(Q, \mathfrak{L}, \lambda, +)$ is a C^*-group.

Example 2 ([1]). Let $X = \{x = (\xi_i) \mid \xi_i \in R\ \forall i \in N, \sum_{i=1}^{\infty} |\xi_i| < \infty\}$ and let $\mathfrak{L} = \{(\langle x_n\rangle, x) \mid \xi_i^{(n)} \to \xi_i\ \forall\, i \in N, \exists\, y = (\eta_i) \in X,\ \ |\xi_i^{(n)}| \leqslant \eta_i\ \forall\, n \in N\}$. Then $(X, \mathfrak{L}, \lambda, +)$ is a C^+-group which is not a C^*-group.

Example 3. Let $X = \{[f] \mid f\colon [0,1] \to R, f\ B\text{-measurable}\}$ and let $\mathfrak{L} = \{(\langle [f_n]\rangle, [f]) \mid f_n \to f \text{ a.e.}\}$. Then $(X, \mathfrak{L}, \lambda, +)$ is a C^+-group which is not a C^*-group.

Example 4. Let $(Q, \mathfrak{L}, \lambda, +)$ be the C^*-group of rational numbers from Example 1 and let $(\langle x_n\rangle, x) \in \mathfrak{M}$ if $(\langle x_n\rangle, x) \in \mathfrak{L}$ and $\langle x_n\rangle$ is a monotone sequence. Then $(Q, \mathfrak{M}, \mu, +)$ is a C-group which is not a C^+-group.

2. Cauchy sequences in convergence groups. J. Novák has constructed a completion for C^*-groups in [8]. We are now going to consider the general case of C-groups. The first step is to decide what should be the Cauchy sequences.

We shall use the following notation ([2]). $\xi, \eta, \ldots$ will denote sequences of points, $\xi(n)$ will denote the nth term of the sequence ξ, i.e., $\xi = \langle \xi(n)\rangle$, $s, t, u, \ldots$ will denote monotone mappings of N into N, $\xi \circ s$ will denote a subsequence of the sequence ξ and $\xi \wedge \eta$ will denote the sequence whose $(2n-1)$th term is $\xi(n)$ and whose $2n$th term is $\eta(n)$.

The following natural definitions of Cauchy sequences suggest themselves. Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group. Denote

$$C = \{\xi \in L^N \mid \forall s\, \forall t\, \exists u\colon (\langle \xi\circ s\circ u(n) - \xi\circ t\circ u(n)\rangle, 0) \in \mathfrak{L}\},$$

$$C' = \{\xi \in L^N \mid \forall s\, \forall t\, (\langle \xi\circ s(n) - \xi\circ t(n)\rangle, 0) \in \mathfrak{L}\},$$

$$C'' = \{\xi \in L^N \mid \forall s\, (\langle \xi(n) - \xi\circ s(n)\rangle, 0) \in \mathfrak{L}\}.$$

Proposition 1. (i) $C' \subset C'' \subset C$.

(ii) $\mathfrak{L} = \mathfrak{L}^*$ *implies* $C' = C'' = C$.

Proof. (i): $C' \subset C''$ is obvious. Now let $\xi \in C''$ and let s and t be given. We have $(\langle \xi(n) - \xi\circ s(n)\rangle, 0) \in \mathfrak{L}$ and $(\langle \xi(n) - \xi\circ t(n)\rangle, 0) \in \mathfrak{L}$. It follows by $(\mathscr{S}\mathscr{G})$ that for some u $(\langle \xi\circ s\circ u(n) - \xi\circ t\circ u(n)\rangle, 0) \in \mathfrak{L}$ and therefore $\xi \in C$.

(ii): Suppose $\mathfrak{L} = \mathfrak{L}^*$ and $\xi \in C$ and let s and t be given. For each v there exists u such that $(\langle \xi\circ s\circ v\circ u(n) - \xi\circ t\circ v\circ u(n)\rangle, 0) \in \mathfrak{L}$ and by $(\mathscr{L}_3)$ we have $(\langle \xi\circ s(n) - \xi\circ t(n)\rangle, 0) \in \mathfrak{L}^* = \mathfrak{L}$. Hence $\xi \in C'$.

Example 5. Let $(Q, \mathfrak{L}, \lambda, +)$ be the C-group of rational numbers from Example 1. Define a convergence $\mathfrak{N}$:

$(\xi, x) \in \mathfrak{N}$ if $(\xi, x) \in \mathfrak{L}$ and $x \neq 0$,

$(\xi, 0) \in \mathfrak{N}$ if $(\xi, 0) \in \mathfrak{L}$ and ξ is a monotone sequence.

$(Q, \mathfrak{N}, \nu, +)$ is a C-group which is not a C^+-group.

(i) Put

$$x_n = 1 + 2^{-(n+2)},$$

$$y_n = 1 + 2^{-(n+2)} + (-4)^{-(n+2)},$$

$$\langle z_n\rangle = \langle x_n\rangle \wedge \langle y_n\rangle.$$

We have $(\langle z_n\rangle, 1) \in \mathfrak{N}$.

Put $z_{p_n} = z_{n+1}$. Then $(\langle z_n - z_{p_n}\rangle, 0) \notin \mathfrak{N}$ and hence $\langle z_n\rangle \notin C''$ and by Proposition 1 we have $\langle z_n\rangle \notin C'$. Notice that, in fact, $\xi \in C'$ only if there exists $x \in Q$ such that $\xi(n) = x$ for all but finitely many n. Indeed, if ξ is not of this kind, then it contains either two distinct constant subsequences or a one-to-one sequence which contains a monotone subsequence $\xi\circ s$. In the first case clearly $\xi \notin C'$. In the second case put $t(n) = 4n$, $u(2n-1) = 4(2n-1)-1$, $u(2n) = 4(2n)+1$. Then $(\langle \xi\circ s\circ t(n) - \xi\circ s\circ u(n)\rangle, 0) \notin \mathfrak{N}$ and hence $\xi \notin C'$. On the other hand, obviously $\langle z_n\rangle \in C$. Hence $C'' \neq C$.

(ii) Put $v_n = 2^{-n}$. We have $(\langle v_n\rangle, 0) \in \mathfrak{N}$ and by the argument given in (i) we have $\langle v_n\rangle \notin C'$. On the other hand, for each $\langle q_n\rangle$ we have $(\langle v_n - v_{q_n}\rangle, 0) \in \mathfrak{N}$ and therefore $\langle v_n\rangle \in C''$. Hence $C' \neq C''$.

Example 5 shows that there exist C-groups for which $C' \neq C'' \neq C$. Further it shows that in general neither C' nor C'' contain all convergent sequences. Since it is natural to expect a convergent sequence to be a Cauchy sequence, it would not be appropriate to use either C' or C'' as a set of Cauchy sequences. (The situation is different in the case of C^+-groups which will be considered in Section 4.) We shall see that C is better behaved in this respect. Therefore while $C' = C'' = C$ for C^*-groups and C' was used by J. Novák ([8]) as a set of Cauchy sequences in a C^*-group, in the general case of C-groups only C can serve as a set of Cauchy sequences. We shall now examine the set C in some detail.

PROPOSITION 2. *Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group. If $(\xi, x) \in \mathfrak{L}$, then $\xi \in C$.*

Proof. Let s and t be given. We have $(\xi \circ s, x) \in \mathfrak{L}$ and $(\xi \circ t, x) \in \mathfrak{L}$. By $(\mathscr{SG})$ there is u such that $(\langle \xi \circ s \circ u(n) - \xi \circ t \circ u(n)\rangle, 0) \in \mathfrak{L}$ and hence $\xi \in C$.

J. M. Irwin and D. C. Kent ([3] and [4]) developed a general theory of Cauchy sequences from the axiomatic point of view by introducing the notion of a sequential Cauchy space. Following [2] sequential Cauchy spaces will be called *$\mathscr{CL}$-spaces.*

Let L be a set and let $B \subset L^N$. We say that (L, B) is a *$\mathscr{CL}$-space* if the following conditions are satisfied:

$(\mathscr{CL}_1)$ $\langle x\rangle \in B$ for each $x \in L$,

$(\mathscr{CL}_2)$ $\xi \in B$ implies $\xi \circ s \in B$ for each subsequence $\xi \circ s$ of ξ,

$(\mathscr{CL}_3)$ $\xi, \eta \in B$ and $\xi \circ s = \eta \circ t$ for some s and t imply $\xi \wedge \eta \in B$.

A $\mathscr{CL}$-space is said to be *separated* if it satisfies the condition

$(\mathscr{CL}_0)$ $\langle x\rangle \wedge \langle y\rangle \in B$ implies $x = y$.

A $\mathscr{CL}$-space is said to be *maximal* if it satisfies the condition

$(\mathscr{CL}_4)$ Let $\xi \in L^N$. If the conditions

(a) For each s there exists t such that $\xi \circ s \circ t \in B$ and

(b) $\xi \circ s \in B$ and $\xi \circ t \in B$ imply $\xi \circ s \wedge \xi \circ t \in B$

are satisfied, then $\xi \in B$. If $\xi \in B$, then ξ is said to be *B-Cauchy.*

PROPOSITION 3. *Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group. Then (L, C) is a maximal separated $\mathscr{CL}$-space.*

Proof. $(\mathscr{CL}_1)$: For each $x \in L$ obviously $\langle x\rangle \in C$.

$(\mathscr{CL}_2)$: Let $\xi \in C$ and let $\xi \circ s$ be a subsequence of ξ. Again, obviously, $\xi \circ s \in C$.

$(\mathscr{CL}_3)$: Let $\xi, \eta \in C$, let there be v and w such that $\xi \circ v = \eta \circ w$ and let s and t be given. Without any loss of generality we may assume that there exists u_1 such that

$$(\xi \wedge \eta) \circ s \circ u_1 = \xi \circ p \quad \text{for some } p$$

and that there exists u_2 such that

$$(\xi \wedge \eta) \circ t \circ u_1 \circ u_2 = \eta \circ q \quad \text{for some } q.$$

We have

$$(\xi \wedge \eta) \circ s \circ u_1 \circ u_2 = \xi \circ p \circ u_2.$$

There exists u_3 such that

$$(\langle (\xi \wedge \eta) \circ s \circ u_1 \circ u_2 \circ u_3(n) - \xi \circ v \circ u_3(n)\rangle, 0) \in \mathfrak{L}.$$

There exists u_4 such that

$$(\langle \eta \circ w \circ u_3 \circ u_4(n) - (\xi \wedge \eta) \circ t \circ u_1 \circ u_2 \circ u_3 \circ u_4(n)\rangle, 0) \in \mathfrak{L}.$$

Finally, there exists u_5 such that for $u' = u_1 \circ u_2 \circ u_3 \circ u_4 \circ u_5$

$$(\langle (\xi \wedge \eta) \circ s \circ u'(n) - (\xi \wedge \eta) \circ t \circ u'(n)\rangle, 0) \in \mathfrak{L}$$

and hence $\xi \wedge \eta \in C$.

$(\mathscr{CL}_0)$: $\langle x\rangle \wedge \langle y\rangle \in C$ clearly implies $x = y$.

$(\mathscr{CL}_4)$: Let $\xi \in L^N$, let for each p there exist v such that $\xi \circ p \circ v \in C$ and let $\xi \circ q \in C$ and $\xi \circ r \in C$ imply $\xi \circ q \wedge \xi \circ r \in C$. Let s and t be given. There exists v_1 such that $\xi \circ s \circ v_1 \in C$. Further there exists v_2 such that $\xi \circ t \circ v_1 \circ v_2 \in C$. For each v_3 and v_4 there exists v_5 such that

$$(\langle (\xi \circ s \circ v_1 \circ v_2 \wedge \xi \circ t \circ v_1 \circ v_2) \circ v_3 \circ v_5(n) - \\ -(\xi \circ s \circ v_1 \circ v_2 \wedge \xi \circ t \circ v_1 \circ v_2) \circ v_4 \circ v_5(n)\rangle, 0) \in \mathfrak{L}.$$

For $v_3(n) = 2n-1$, $v_4(n) = 2n$ we have

$$(\langle \xi \circ s \circ v_1 \circ v_2 \circ v_5(n) - \xi \circ t \circ v_1 \circ v_2 \circ v_5(n)\rangle, 0) \in \mathfrak{L}$$

and hence $\xi \in C$.

3. Completion of convergence groups.

DEFINITION 2. Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group. A sequence $\xi \in L^N$ is said to be a *Cauchy sequence* in $(L, \mathfrak{L}, \lambda, +)$ if $\xi \in C$, i.e., if for each s and t there is a u such that

$$(\langle \xi \circ s \circ u(n) - \xi \circ t \circ u(n)\rangle, 0) \in \mathfrak{L}.$$

DEFINITION 3. Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group and let ξ and η be Cauchy sequences in $(L, \mathfrak{L}, \lambda, +)$. The sequences ξ and η are said to be *equivalent*, in symbols $\xi \sim \eta$, if for each s and t there is a u such that $(\langle \xi \circ s \circ u(n) - \xi \circ t \circ u(n)\rangle, 0) \in \mathfrak{L}$.

For the case of C^*-groups we obtain the notions of a Cauchy sequence and equivalence of Cauchy sequences introduced in [8].

The following statement is easy to prove.

PROPOSITION 4. *Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group. Then*

(i) *A subsequence of a Cauchy sequence is a Cauchy sequence.*

(ii) *If $(\xi, x) \in \mathfrak{L}^*$, then ξ is a Cauchy sequence.*

(iii) *The relation $\sim$ is an equivalence.*

DEFINITION 4 (cf. [8]). A C-group $(L, \mathfrak{L}, \lambda, +)$ is said to be *complete* if for each Cauchy sequence ξ there exists a point $x \in L$ such that $(\xi, x) \in \mathfrak{L}$.

PROPOSITION 5. *A complete C-group is a C^*-group.*

Proof. The assertion follows by Proposition 4.

DEFINITION 5 (cf. [8]). A C-group $(M, \mathfrak{M}, \mu, +)$ is said to be a *completion* of a C-group $(L, \mathfrak{L}, \lambda, +)$ if it is complete and contains $(L, \mathfrak{L}, \lambda, +)$ as a μ^{ω_1}-dense subspace and a subgroup.

J. Novák constructed in [8] a completion for each C^*-group in the following way. Let $(L, \mathfrak{L}, \lambda, +)$ be a C^*-group. Denote by L_1 the set of all equivalence classes $[\langle x_n\rangle]$ of Cauchy sequences $\langle x_n\rangle$. Identify points x of L with classes $[\langle x\rangle]$. Let $[\langle x_n\rangle]+[\langle y_n\rangle] = [\langle x_n+y_n\rangle]$ and let $(\langle z_n\rangle, z) \in \mathfrak{L}_1$ if there is a Cauchy sequence x_n such that $z-z_n = [\langle x_n\rangle]-x_n$. The C^*-group $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ is a completion of the C^*-group $(L, \mathfrak{L}, \lambda, +)$.

PROPOSITION 6. *Every C-group $(L, \mathfrak{L}, \lambda, +)$ has a completion $(M, \mathfrak{M}, \mu, +)$.*

Proof. By Theorem 1 in [8] the C^*-group $(L, \mathfrak{L}^*, \lambda, +)$ has the completion $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$; put $(M, \mathfrak{M}, \mu, +) = (L_1, \mathfrak{L}_1^*, \lambda_1, +)$.

DEFINITION 6. The completion $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ is said to be the *minimal completion* of the C-group $(L, \mathfrak{L}, \lambda, +)$.

PROPOSITION 7. *Let $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ be the minimal completion of a C-group $(L, \mathfrak{L}, \lambda, +)$. Then*

(i) *$(L, \mathfrak{L}, \lambda, +)$ is λ_1-dense in $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$.*

(ii) *If $(M, \mathfrak{M}, \mu, +)$ is a completion of the C-group $(L, \mathfrak{L}, \lambda, +)$, then there is a continuous homomorphism of $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ into $(M, \mathfrak{M}, \mu, +)$ leaving L pointwise fixed.*

Proof. (i) follows directly from the construction of L_1.

(ii): Define a mapping $f\colon (L_1, \mathfrak{L}_1^*, \lambda_1, +) \to (M, \mathfrak{M}, \mu, +)$ by $f(x) = \mathfrak{M}\text{-}\lim x_n$ for $x = [\langle x_n\rangle]$. Clearly $f|L = \mathrm{id}$. The mapping f is well-defined. Namely, if $[\langle x_n\rangle] = [\langle y_n\rangle]$, then $\mathfrak{L}^*\text{-}\lim(x_n-y_n) = 0$. Hence $\mathfrak{M}\text{-}\lim(x_n-y_n) = 0$ and $\mathfrak{M}\text{-}\lim y_n = \mathfrak{M}\text{-}\lim x_n = f(x)$. The mapping f is a homomorphism since $f([\langle x_n\rangle]+[\langle y_n\rangle]) = f([\langle x_n\rangle+\langle y_n\rangle]) = \mathfrak{M}\text{-}\lim(x_n+y_n) = f([\langle x_n\rangle])+f([\langle y_n\rangle])$. Finally, if $\mathfrak{L}_1^*\text{-}\lim z_n = z$, then for each subsequence $\langle n_i\rangle$ there is a subsequence $\langle n_{i_j}\rangle$ such that $\mathfrak{L}_1\text{-}\lim z_{n_{i_j}} = z$. Hence there is a Cauchy sequence $\langle x_n\rangle$ such that $z-z_{n_{i_j}} = [\langle x_n\rangle]-x_j$. We have $\mathfrak{M}\text{-}\lim f(z_{n_{i_j}}) = f(z)-f([\langle x_n\rangle])+\mathfrak{M}\text{-}\lim x_j = f(z)$. It follows by $(\mathscr{L}_3)$ that $\mathfrak{M}\text{-}\lim f(z_n) = f(z)$.

EXAMPLE 6. Let $(Q, \mathfrak{L}, \lambda, +)$ be the C^*-group of rational numbers with the usual convergence from Example 1. The minimal completion of the C^*-group $(Q, \mathfrak{L}, \lambda, +)$ is the minimal completion $(R, \mathfrak{L}_1^*, \lambda_1, +)$ constructed in [8], i.e., R is the set of real numbers and $(\langle z_n\rangle, z) \in \mathfrak{L}_1$ if there exists a real number x and a sequence $\langle x_n\rangle$ of rational numbers

converging to x in the usual way such that $z-z_n = x-x_n$. This C-group differs substantially from the topological group of real numbers. It fails to have many properties of the topological C^*-group $(Q, \mathfrak{L}, \lambda, +)$. In particular,

(i) (R, λ_1) fails to be regular,

(ii) $(R, \lambda_1^{\omega_1})$ fails to be regular,

(iii) $(R, \mathfrak{L}_1^*, \lambda_1)$ fails to be sequentially regular, i.e., $\lim f(x_n) = f(x)$ for each $f \in C\big((R, \mathfrak{L}_1^*, \lambda_1)\big)$ fails to imply $\mathfrak{L}_1^*\text{-}\lim x_n = x$ ([6], [2]).

Proof. Let $P = \bigcup_{n=1}^{\infty} (p_n^{-1/2})$ where $\langle p_n\rangle$ is the sequence of primes. Suppose there is a sequence $\langle x_n\rangle$ and a point $x \in R$ such that $(\langle x_n\rangle, x) \in \mathfrak{L}_1^*$ and $\bigcup_{n=1}^{\infty} (x_n) \subset P$. Then there is a sequence $\langle m_n\rangle$ such that $(\langle p_{m_n}^{-1/2}\rangle, x) \in \mathfrak{L}_1$. By the definition of the convergence $\mathfrak{L}_1$ there is a sequence $\langle y_n\rangle$ of rational numbers converging to a real number y such that $x-p_{m_n}^{-1/2} = y-y_n$. It follows that $p_{m_1}^{-1/2}-p_{m_2}^{-1/2} = y_1-y_2 \in Q$ which is a contradiction. Hence $\lambda_1 P = P$ and therefore also $\lambda_1^{\omega_1} P = P$. Since P is λ_1-closed, $R-P$ is a λ_1-neighborhood of 0. Let $U \subset R-P$ be a λ_1-neighborhood of 0. Then $\lambda_1 U \cap P \neq \emptyset$ and (i) holds. Since P is also $\lambda_1^{\omega_1}$-closed, $R-P$ is a $\lambda_1^{\omega_1}$-neighborhood of 0 as well. Let $V \subset R-P$ be a $\lambda_1^{\omega_1}$-neighborhood of 0. Since $\lambda_1 < \lambda_1^{\omega_1}$, it follows that V is a λ_1-neighborhood of 0 as well. Hence $\lambda_1 V \cap P \neq \emptyset$. Therefore $\lambda_1^{\omega_1} V \cap P \neq \emptyset$ and (ii) holds.

Denote $a_m = p_m^{-1/2}$. We have $0 \in R-\lambda_1 \bigcup_{m=1}^{\infty} (a_m)$. For each $m \in N$ let $\langle a_{mn}\rangle$ be a sequence of points of Q such that $0 < a_{mn} < a_m$ and $(\langle a_{mn}\rangle, a_m) \in \mathfrak{L}_1$. Let $f \in C\big((R, \mathfrak{L}_1, \lambda_1)\big)$. For each $\varepsilon > 0$ and for each $m \in N$ there is n_m such that $|f(a_m)-f(a_{mn})| < \varepsilon/2$ for each $n \geqslant n_m$. Since $(\langle a_{mn_m}\rangle, 0) \in \mathfrak{L}_1$ there is m_0 such that $|f(a_{mn_m})-f(0)| < \varepsilon/2$ for each $m \geqslant m_0$. For $m \geqslant m_0$ we have $|f(a_m)-f(0)| \leqslant |f(a_m)-f(a_{mn_m})|+|f(a_{mn_m})-f(0)| < \varepsilon/2+\varepsilon/2 = \varepsilon$. Since $(\langle a_m\rangle, 0) \notin \mathfrak{L}_1^*$, it follows that (iii) holds.

By (i) and (ii) both $(R, \lambda_1, +)$ and $(R, \lambda_1^{\omega_1}, +)$ fail to be topological groups.

EXAMPLE 7. Let $(X, \mathfrak{L}, \lambda, +)$ be the C^+-group of sequences of real numbers from Example 2. Using (v) in [1] we obtain that the minimal completion $(X, \mathfrak{L}_1^*, \lambda_1, +)$ of the C^+-group $(X, \mathfrak{L}, \lambda, +)$ is the complete metric space $l_1 = (X, \varrho)$ where $\varrho(x, y) = \sum_{i=1}^{\infty} |\xi_i-\eta_i|$, i.e., $\mathfrak{L}_1^*$ is the convergence induced by the metric ϱ.

EXAMPLE 8. Let $(X, \mathfrak{L}, \lambda, +)$ be the C^+-group of equivalence clases of B-measurable functions with convergence almost everywhere from Example 3. The minimal completion $(X, \mathfrak{L}_1^*, \lambda_1, +)$ of the C^+-group $(X, \mathfrak{L}, \lambda, +)$ is the metric space of equivalence classes of B-measurable functions with convergence in measure, i.e., $\mathfrak{L}_1^*$ is convergence in measure.

EXAMPLE 9. Let $(Q, \mathfrak{M}, \mu, +)$ be the C-group of rational numbers with the monotone convergence from Example 4. The minimal completion of the C-group $(Q, \mathfrak{M}, \mu, +)$ is the minimal completion of the topological group of rational numbers.

Example 6 shows that the minimal completion of even a topological group can fail to have "nice" properties. Let us now consider whether the property of being a Fréchet space is preserved.

DEFINITION 7 (cf. [8]). Let $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ be the minimal completion of a C-group $(L, \mathfrak{L}, \lambda, +)$. Two points x and y of L_1 are said to be *equivalent* if $x-y \in L$. The class of all points of L_1 which are equivalent to x will be denoted by $[x]$.

PROPOSITION 8. *Let $(L, \mathfrak{L}, \lambda, +)$ be a Fréchet convergence group. Then the minimal completion $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ of $(L, \mathfrak{L}, \lambda, +)$ is a Fréchet convergence group iff there is only a finite number of distinct equivalence classes of points of L_1.*

Proof. I. Suppose that the condition is fulfilled. We have to prove that λ_1 is a topological closure for L_1. Let $A \subset L_1$ be given and let $x \in \lambda_1^2 A$. Then there is a sequence $\langle x_m \rangle$ such that $\bigcup_{m=1}^{\infty} (x_m) \subset \lambda_1 A$ and $(\langle x_m \rangle, x) \in \mathfrak{L}_1$. Hence there is a Cauchy sequence $\langle a_n \rangle$ in $(L, \mathfrak{L}, \lambda, +)$ such that $x-x_m = [\langle a_n \rangle]-a_m$. It follows that $x_m = a_m+x-[\langle a_n \rangle] = x_1+a_m-a_1 = x_1+b_m$ where $(\langle b_m \rangle, x-x_1) \in \mathfrak{L}_1$. Since $x_m \in \lambda_1 A$ for each $m \in N$, there are sequences $\langle x_{mn} \rangle$ such that $\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} (x_{mn}) \subset A$ and $(\langle x_{mn} \rangle, x_m) \in \mathfrak{L}_1$. We have $x_{mn} \in [x_{m1}]$ for each m and n. Since the number of distinct equivalence classes is finite, we can assume without loss of generality that $x_{mn} \in [x_{11}]$ for each $m, n \in N$. It follows that $x_{mn} = x_{1n}+a_{mn}$ where $\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} (a_{mn}) \subset L$ and $\mathfrak{L}_1^*$-$\lim a_{mn} = x_m-x_1 = b_m \in L$ for each $m \in N$. Put $c_{mn} = a_{mn}-b_m$. Then $\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} (c_{mn}) \subset L$ and $(\langle c_{mn} \rangle, 0) \in \mathfrak{L}^*$ for each $m \in N$. Since $(L, \mathfrak{L}, \lambda, +)$ is Fréchet, 0 is not a ϱ-point of L ([7]) and hence there exists a sequence $\langle c_{m_i n_i} \rangle$ such that $(\langle c_{m_i n_i} \rangle, 0) \in \mathfrak{L}^*$. It follows that $\mathfrak{L}_1^*$-$\lim x_{m_i n_i} = \mathfrak{L}_1^*$-$\lim (x_{1n_i}+c_{m_i n_i}+b_{m_i}) = x$. Hence $x \in \lambda_1 A$ and therefore $\lambda_1^2 = \lambda_1$.

II. Suppose, on the contrary, that there is an infinite number of distinct equivalence classes. Hence there is a sequence $\langle a_n \rangle$ such that $a_i \notin [a_j]$ for $i \neq j$. Let $a_i = [\langle a_{in} \rangle]$. Put $x_m = a_1-a_{1m}$ and $x_{mn} = a_1-a_{1m}+a_{m+1}-a_{m+1\,n}$. Then $(\langle x_{mn} \rangle, x_m) \in \mathfrak{L}_1^*$ and $(\langle x_m \rangle, 0) \in \mathfrak{L}_1^*$. Hence $0 \in \lambda^2 \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} (x_{mn})$. Let $z_i = x_{m_i n_i}$. Then $z_i-z_j \notin L$ for $i \neq j$. Hence by Lemma 11 in [8] $0 \notin \lambda \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} (x_{mn})$ and therefore $\lambda_1 \neq \lambda_1^2$.

COROLLARY 1. *The minimal completion of the topological group of rational numbers is not a Fréchet convergence group.*

Proof. Clearly $p_i^{-1/2} \notin [p_j^{-1/2}]$ for $i \neq j$ where $\langle p_n \rangle$ is the sequence of primes.

Let (L, B) be a $\mathscr{CL}$-space. A sequence $\langle x_n \rangle$ of points of L is said to *converge to a point x of L* if $\langle x_n \rangle \wedge \langle x \rangle \in B$. The $\mathscr{CL}$-space (L, B) is called *complete* if for each $\langle y_n \rangle \in B$ there is $y \in L$ such that $\langle y_n \rangle$ converges to y ([3], [4]).

PROPOSITION 9. *Let $(L, \mathfrak{L}, \lambda, +)$ be a C-group and let (L, C) be the $\mathscr{CL}^*$-space of Cauchy sequences in $(L, \mathfrak{L}, \lambda, +)$. Then a sequence $\langle x_n \rangle$ converges to x in (L, C) iff $(\langle x_n \rangle, x) \in \mathfrak{L}^*$.*

Proof. I. Let $\langle x_n \rangle \wedge \langle x \rangle \in C$. Then for each $\langle n_i \rangle$ there exists $\langle n_{i_j} \rangle$ such that $(\langle x_{n_{i_j}}-x \rangle, 0) \in \mathfrak{L}$. Hence $(\langle x_n \rangle, x) \in \mathfrak{L}^*$.

II. Let $(\langle x_n \rangle, x) \in \mathfrak{L}^*$. By Proposition 4 we have $\langle x_n \rangle \in C$. Let s and t be given and consider the sequence $\xi = \langle (\langle x_n \rangle \wedge \langle x \rangle) \circ s(n) - (\langle x_n \rangle \wedge \langle x \rangle) \circ t(n) \rangle$. By a suitable choice of u we can obtain a subsequence $\xi \circ u$ of ξ which will be of one of the following forms: $\langle x_{i_n}-x_{j_n} \rangle$, $\langle x-x_{i_n} \rangle$, $\langle x_{i_n}-x \rangle$ or $\langle 0 \rangle$ and therefore $\langle x_n \rangle \wedge \langle x \rangle \in C$.

COROLLARY 2. *A C-group $(L, \mathfrak{L}, \lambda, +)$ is complete iff the $\mathscr{CL}^*$-space of Cauchy sequences in $(L, \mathfrak{L}, \lambda, +)$ is complete.*

4. Cauchy sequences in C^+-groups. In [2] the notion of an $\mathscr{L}$-group was introduced. For these groups we require that the group operation be $\mathscr{L}$-continuous instead of sequentially continuous. Let $(L, \mathfrak{L})$ be an $\mathscr{L}$-space and $(L, +)$ a group. $(L, \mathfrak{L}, +)$ is said to be an *$\mathscr{L}$-group* if the following condition is satisfied

$(\mathscr{LG})$ $(\langle x_n \rangle, x) \in \mathfrak{L}$ and $(\langle y_n \rangle, y) \in \mathfrak{L}$ imply $(\langle x_n-y_n \rangle, x-y) \in \mathfrak{L}$

R. Frič has shown that for $\mathscr{L}$-groups the following statement holds

PROPOSITION 10 (R. Frič). *Let $(L, \mathfrak{L}, +)$ be an $\mathscr{L}$-group. Then*

(i) $C'' \subset C'$.

(ii) *$(\xi, x) \in \mathfrak{L}$ implies $\xi \in C''$.*

(iii) *If $\xi \in C''$, then $\xi \circ s \in C''$ for each s.*

(iv) *If $\xi \in C''$, then ξ is either convergent or totally divergent.*

We see immediately that the following statement holds.

PROPOSITION 11. *There is a one-to-one correspondence between the class of C^+-groups and the class of $\mathscr{L}$-groups.*

Propositions 1, 10 and 11 yield the following corollary.

COROLLARY 3. *Let $(L, \mathfrak{L}, \lambda, +)$ be a C^+-group. Then*

(i) $C' = C''$.

(ii) *$(\xi, x) \in \mathfrak{L}$ implies $\xi \in C''$.*

(iii) *If* $\xi \in C''$, *then* $\xi \circ s \in C''$ *for each* s.

(iv) *If* $\xi \in C''$, *then* ξ *is either convergent or totally divergent.*

The Example 3 of the C^{+}-group $(X, \mathfrak{L}, \lambda, +)$ of equivalence classes of B-measurable functions with convergence almost everywhere shows that in general for C^{+}-groups we have $C' = C'' \subsetneqq C$. Indeed, if $f_n \to f$ in measure but not almost everywhere, then $\langle [f_n] \rangle \in C - C''$ in view of (iv) of Corollary 3. In fact, it is easy to see that $(X, \mathfrak{L}, \lambda, +)$ is "C''-complete" in the sense that each $\langle [f_n] \rangle \in C''$ converges in $(X, \mathfrak{L}, \lambda, +)$.

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On the structure of L_φ-solution sets of integral equations in Banach spaces

by

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Abstract. In this paper we consider the integral equation

$$(1)\qquad x(t) = p(t) + \int_0^t f(t, s, x(s))\,ds$$

in a Banach space X. We prove that under suitable assumptions the set of all solutions of (1), belonging to a certain Orlicz space $L_\varphi(J, X)$, is a compact R_δ.

Let X be a separable Banach space. For any compact interval J and for any N-function φ (cf. [4], [6]) we shall denote by $L_\varphi(J, X)$ the Orlicz space of all strongly measurable functions $u: J \to X$ for which the number

$$\|u\|_\varphi = \inf\Big\{r > 0:\ \int_J \varphi(\|u(s)\|/r)\,ds \leqslant 1\Big\}$$

is finite. It is well known that $\langle L_\varphi(J, X), \|\cdot\|_\varphi \rangle$ is a Banach space. Moreover, we shall denote by $E_\varphi(J, X)$ the closure in $L_\varphi(J, X)$ of the set of all bounded functions. For properties of the spaces $L_\varphi(J, X)$ and $E_\varphi(J, X)$ see [4], pp. 76–106.

In [7] we gave some conditions which guarantee that the integral equation

$$(1)\qquad x(t) = p(t) + \int_0^t f\big(t, s, x(s)\big)\,ds$$

has at least one solution x belonging to a certain space $L_\varphi(J, X)$. In this paper we shall show that under the same assumptions as in [7] the set S of all solutions $x \in L_\varphi(J, X)$ of (1) is a compact R_δ in the sense of Aronszajn, i.e., S is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts.

Let $L^1(J, X)$ denote the Lebesgue space of Bochner integrable functions $u: J \to X$ provided with the norm $\|u\|_1 = \int_J \|u(s)\|\,ds$, and let β and β_1 be the ball measures of noncompactness in X and $L^1(J, X)$, respectively.