(3) ⇒ (1): Suppose (3) holds. $\mathcal{D}(R^+_{n+1})$ and $\mathcal{D}(R^-_{n+1})$ are dense in $B^+_{p,\mu}$ and $B^-_{p,\mu}$, respectively. The map l: $\varphi_1+\varphi_2\to\varphi_1$ of $\mathcal{D}(R^+_{n+1})+\mathcal{D}(R^-_{n+1})$ into $\mathcal{D}(R^+_{n+1})$ is continuous, that is, $\|\varphi_1\|_{p,\mu}\leqslant C\|\varphi_1+\varphi_2\|_{p,\mu}$ with a positive constant C. Since $\mathcal{D}(R^+_{n+1})+\mathcal{D}(R^-_{n+1})$ is dense in $B_{p,\mu}(R_{n+1})$, it suffices to $l(\varphi)=\varphi_+$ for any $\varphi\in\mathcal{D}(R^+_{n+1})+\mathcal{D}(R^-_{n+1})$, which is an immediate consequence of the definition of the map l.

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Received October 11, 1982 (1815)

STUDIA MATHEMATICA, T. LXXVII. (1984)

Factorization in some Fréchet algebras of differentiable functions

by

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Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. It is shown that for each compact set $B\subset\mathcal{S}(\boldsymbol{R}^n)$ there exist $u\in\mathcal{S}(\boldsymbol{R}^n)$ and a compact set $B'\subset\mathcal{S}(\boldsymbol{R}^n)$ such that $uB'=B$ holds (essential part of the "compact strong factorization property"). The same property is shown for s (rapidly decreasing sequences) and $\mathring{\mathcal{B}}(\Omega)$.

Introduction. The starting point of this paper was the question of Kamiński whether $\operatorname{lin}(\mathcal{S}*\mathcal{S})=\mathcal{S}$, or equivalently, $\operatorname{lin}(\mathcal{S}\cdot\mathcal{S})=\mathcal{S}$ holds ([11], Problems 4 and 5, p. 282). We give an affirmative answer by showing that $\mathcal{S}\cdot\mathcal{S}=\mathcal{S}$ holds. More precisely, we show that $\mathcal{S}$ has the compact strong factorization property, i.e., roughly speaking, that a compact $B\subset\mathcal{S}$ can be written as uB', with $u\in\mathcal{S}$ and compact $B'\subset\mathcal{S}$. (Let us mention that $\operatorname{lin}(\mathcal{D}*\mathcal{D})=\mathcal{D}$ is known from [15], [7].)

Factorization properties are known for Fréchet algebras having a uniformly bounded approximate unit. From the known factorization theorems for Fréchet and Banach algebras we have extracted a rather strong concept of factorization property (cf. Definition 1.1), which is satisfied in Fréchet algebras having a uniformly bounded left approximate unit. We show that this factorization property is also satisfied in a certain class of Fréchet algebras of differentiable functions, which do not have a bounded approximate unit. $\mathcal{S}(\boldsymbol{R}^n)$ and $\mathring{\mathcal{B}}(\Omega)$ belong to this class, for quasi-bounded $\Omega=\mathring{\Omega}\subset\boldsymbol{R}^n$.

In Section 1 we define the concepts of Fréchet algebra, strong factorization property, (uniformly bounded) left approximate unit. We state the factorization theorem, and we note that reflexive Fréchet algebras having no unit cannot have a bounded approximate unit. We introduce $\mathring{\mathcal{B}}(\Omega)$ and show that $\mathring{\mathcal{B}}(\boldsymbol{R}^n)$ has a uniformly bounded approximate unit.

In Section 2 we define a class of Fréchet algebras $\mathcal{B}^m_\gamma(\Omega)$ of m-times differentiable functions on $\Omega=\mathring{\Omega}\subset\boldsymbol{R}^n$ $(0\leqslant m\leqslant\infty)$. If the weight function $\gamma\in C(\Omega)$ is such that there exists a certain kind of partition of unity on Ω then $\mathscr{B}^m_\gamma(\Omega)$ is shown to have the bounded strong factorization property. As a means of proof we show that the Fréchet algebra s of rapidly decreasing sequences has the compact strong factorization property, obeying some additional requirements.

In Section 3 we show that under an additional assumption on γ the partition of unity required in Section 2 exists. We apply the results of Section 2 to $\mathscr{S}(\boldsymbol{R}^n)$, $\dot{\mathscr{B}}(\Omega)$, and $\dot{\mathscr{B}}(\Omega)$.

Acknowledgement. The author is grateful to P. Dierolf for several stimulating discussions.

1. Fréchet algebras, factorization properties, approximate unit. A *Fréchet algebra* A is defined to be a Fréchet space (i.e., a complete metrizable locally convex space) over $\boldsymbol{K}(=\boldsymbol{R}$ or $\boldsymbol{C})$, which is also an algebra, and which possesses a sequence $(p_k; k\in\boldsymbol{N})$ of seminorms defining the topology of A such that

$$p_k(xy)\leqslant p_k(x)p_k(y) \tag{1.1}$$

$(x, y\in A)$ holds for all $k\in\boldsymbol{N}$ (cf. [12], Def. 4.1, p. 13). Without loss of generality we may (and shall) assume that the seminorms are chosen increasingly: $p_k\leqslant p_{k+1}$ $(k\in\boldsymbol{N})$.

If E is a Hausdorff locally convex space over $\boldsymbol{K}$ and $B=\operatorname{aco}B\subset E$ is bounded, then $E_B=(E_B, p_B)$ denotes the normed space $E_B=\operatorname{lin}B$, where p_B is the gauge of B. The embedding $E_B\hookrightarrow E$ is continuous. If B is complete, then E_B is a Banach space (cf. [16], Ch. III, § 7, p. 97).

1.1. Definition. Let A be a Fréchet algebra. Let $\mathscr{B}$ be a collection of bounded subsets of A. We say that A has the *$\mathscr{B}$-strong left factorization property* ($\mathscr{B}$-SLFP) if the following holds: For each set $B\in\mathscr{B}$ and each neighbourhood V of zero in A there exist $z\in A$ and a continuous linear mapping $T\colon A_{\overline{\operatorname{aco}B}}\to A$ with the properties:

(i) $z(Tx)=x$ for all $x\in B$;
(ii) Tx belongs to the closed left ideal generated by x, for all $x\in B$;
(iii) $Tx-x\in V$ for all $x\in B$;
(iv) $T|B\colon B\to A$ is continuous.

In particular, if $\mathscr{B}$ is the collection of compact (bounded) subsets of A, then this property is called the *compact* (*bounded*, respectively) SLFP. If A is commutative we shall omit "left", and accordingly write "SFP" instead of "SLFP".

If $\mathscr{B}$ is such that $\bigcup(B;\ B\in\mathscr{B})=A$, and A has the $\mathscr{B}$-SLFP, then obviously $A=AA$. We defer the motivation for the above definition to Remark 1.4.

1.2. Remarks. (a) Properties (i) and (iv) of Definition 1.1 imply that $T|B\colon B\to T(B)$ is a homeomorphism, with inverse $T(B)\ni y\mapsto zy\in B$.

(b) If A has the compact SLFP, and $(x_j;\ j\in\boldsymbol{N})$ is a sequence in A, $x_j\to 0$, then $\{x_j;\ j\in\boldsymbol{N}\}\cup\{0\}$ is compact; therefore there exist $z\in A$ and T as in Definition 1.1. From (iv) we obtain $y_j:=Tx_j\to 0$, and (i) implies $zy_j=x_j$ $(j\in\boldsymbol{N})$.

A left approximate unit (LAU) in the Fréchet algebra A is a net $(e_\iota;\ \iota\in J)$ in A such that $e_\iota x\to x$ for all $x\in A$. The LAU is *bounded* (BLAU) if the set $\{e_\iota;\ \iota\in J\}$ is bounded. It is called *uniformly bounded* (UBLAU), if $\sup\{p_k(e_\iota);\ \iota\in J, k\in\boldsymbol{N}\}<\infty$. "Left" and "L" will be omitted if A is commutative.

1.3. Theorem (factorization theorem). *Let A be a Fréchet algebra with a* UBLAU. *Then A has the compact* SLFP.

We shall not give a proof of this theorem. We refer to the following remarks for the sources of this result (cf. "Added in proof" 1).

1.4. Remarks. (a) A weaker form of Theorem 1.3 is due to Craw [4], where it is shown that sequences tending to zero can be factorized simultaneously (cf. Remark 1.2 (b)). The formulation of Theorem 1.3 is essentially due to Ovaert [13], Thme 1. In particular, this is the only reference (we know of) where the existence of the map T of Definition 1.1 is mentioned.

Unfortunately, we cannot refer to a precise place in the literature for a proof of Theorem 1.3. The ideas of a possible proof are sketched in [13]. (It appears, however, that a complete proof of the results announced in [13] has never been published.)

(b) For a Banach algebra A, Theorem 1.3 is a generalization of the Cohen factorization theorem [2]. We refer to [9], Thm. (32.23), p. 268; [1], Ch. I, § 11, Cor. 12, p. 62; [19], Thm. 6.4, p. 23, where the conclusion is always weaker than in Theorem 1.3. We refer to [9], Notes to § 32, p. 290, for additional references.

(c) Obviously, Definition 1.1 is motivated (and justified) by the validity of Theorem 1.3. We refer further to [3], [17], where some of the aspects were introduced which are taken into account in Definition 1.1.

The following proposition serves mainly to exclude the existence of a BLAU for a certain class of Fréchet algebras.

1.5. Proposition. *Let A be a Fréchet algebra which is reflexive (as a locally convex space), and assume that A has a* BLAU. *Then A has a left unit e $(ex=x$ for all $x\in A)$.*

Proof. Since A is reflexive, the bounded set $\{e_\iota;\ \iota\in J\}$ is relatively $\sigma(A, A')$-compact (cf. [10], ch. 3, § 8, Prop. 1, p. 227), and therefore the net $(e_\iota;\ \iota\in J)$ has a $\sigma(A, A')$-cluster point $e\in A$. For $x\in A$ the mapping $A\ni y\mapsto yx\in A$ is continuous, therefore is $\sigma(A, A')$-continuous, and so ex is a $\sigma(A, A')$-cluster point of the net $(e_\iota x;\ \iota\in J)$. Since also $e_\iota x\to x$, we obtain $ex=x$. ∎

Examples of Fréchet algebras which have a UBLAU may be found in [13], [4] (function algebras with convolution as algebra multiplication). We want to treat another class of examples. We introduce the Fréchet algebra $\mathring{\mathscr{B}}^m(\Omega)$. For $\Omega = \mathring{\Omega} \subset \boldsymbol{R}^n$, $m \in \boldsymbol{N}_0 \cup \{\infty\}$ we define

$$\mathring{\mathscr{B}}^m(\Omega) := \{f \in \mathscr{E}^m(\Omega);\ \partial^\alpha f \in C_0(\Omega) \text{ for all } \alpha \in \boldsymbol{N}_0^n,\ |\alpha| \leqslant m\}$$

(cf. [6], Ex. (4.5), p. 75, and Prop. (4.10), p. 77). We define the family of norms $(\mathring{p}_k;\ k \in \boldsymbol{N}_0,\ k \leqslant m)$ by

$$\mathring{p}_k(f) := \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \|\partial^\alpha f\|_\infty.$$

Equipped with this family of norms, $\mathring{\mathscr{B}}^m(\Omega)$ is a Fréchet space (a Banach space for $m \neq \infty$). If we define multiplication in $\mathring{\mathscr{B}}^m(\Omega)$ by pointwise multiplication $(fg(\xi) = f(\xi)g(\xi)\ (\xi \in \Omega))$ then the Leibniz rule implies that the norms satisfy (1.1). Therefore $\mathring{\mathscr{B}}^m(\Omega)$ is a Fréchet algebra.

1.6. Proposition. *In $\mathring{\mathscr{B}}^m(\boldsymbol{R}^n)$, a* UBAU *is given by the sequence* $(e_k;\ k \in \boldsymbol{N})$

$$e_k(\xi) := \exp(-\xi^2/k^2) \quad (\xi \in \Omega).$$

$\mathring{\mathscr{B}}^m(\boldsymbol{R}^n)$ *has the compact* SFP.

Proof. In view of Theorem 1.3 it is sufficient to show the first statement.

We shall restrict ourselves to proving the uniform boundedness of $(e_k;\ k \in \boldsymbol{N})$, the other properties being easy to establish. We define $h \in \mathring{\mathscr{B}}^m(\boldsymbol{R})$ by $h(s) := \exp(-s^2)$. From $e_k(\xi) = \prod_{j=1}^n h(\xi_j/k)$ and the form of the norms it follows that it is sufficient to show

$$\sum_{j=0}^\infty \frac{1}{j!} \|h^{(j)}\|_\infty < \infty. \tag{1.2}$$

Since $\boldsymbol{C} \ni r \mapsto \exp(-r^2)$ is holomorphic, and

$$\sup\{|\exp(-(s+it)^2)|;\ s, t \in \boldsymbol{R},\ |t| \leqslant 2\} = e^4,$$

the Cauchy integral formulas imply $\|h^{(j)}\|_\infty \leqslant j!\, e^4\, 2^{-j}$, and therefore (1.2) holds. ■

2. Factorization in a class of Fréchet function algebras. Let $\emptyset \neq \Omega = \mathring{\Omega} \subset \boldsymbol{R}^n$, and let $\gamma \in C(\Omega)$ satisfy

$$\gamma(\xi) \geqslant 1 \quad (\xi \in \Omega). \tag{I}$$

For this section we choose $m \in \boldsymbol{N}_0 \cup \{\infty\}$. We define

$$\mathscr{B}_\gamma^m(\Omega) := \{f \in \mathscr{E}^m(\Omega);\ \gamma(\cdot)^k \partial^\alpha f \in L_\infty(\Omega)$$
$$\text{for all } k \in \boldsymbol{N}_0,\ \alpha \in \boldsymbol{N}_0^n \text{ with } |\alpha| \leqslant m\}.$$

For $m = \infty$, the superscript "m" will be omitted; thus, $\mathscr{B}_\gamma(\Omega) = \mathscr{B}_\gamma^\infty(\Omega)$. With the sequence of norms $(p_k;\ k \in \boldsymbol{N}_0)$,

$$p_k(f) := \sum_{|\alpha| \leqslant \min(k,m)} \frac{1}{\alpha!} \|\gamma(\cdot)^k \partial^\alpha f\|_\infty,$$

$\mathscr{B}_\gamma^m(\Omega)$ becomes a Fréchet space. If we define multiplication in $\mathscr{B}_\gamma^m(\Omega)$ by pointwise multiplication of functions then the Leibniz rule implies that the norms satisfy (1.1). Therefore $\mathscr{B}_\gamma^m(\Omega)$ is a Fréchet algebra. It is easy to see that the subspace

$$\mathring{\mathscr{B}}_\gamma^m(\Omega) := \overline{\mathscr{D}(\Omega)} \quad (= \overline{\mathscr{D}^m(\Omega)})$$

(closure taken in $\mathscr{B}_\gamma^m(\Omega)$) is a closed ideal of $\mathscr{B}_\gamma^m(\Omega)$).

2.1. Lemma. *If γ is unbounded then there is no bounded net $(f_\iota;\ \iota \in J)$ in $\mathscr{B}_\gamma^m(\Omega)$ such that $f_\iota g \to g$ for all $g \in \mathring{\mathscr{B}}_\gamma^m(\Omega)$. In particular, $\mathscr{B}_\gamma^m(\Omega)$ and $\mathring{\mathscr{B}}_\gamma^m(\Omega)$ do not have a* BAU.

Proof. Only the first statement has to be proved. Assume that a net with the mentioned properties exists. There exists a sequence $(\xi_j)_{j \in \boldsymbol{N}}$ in Ω, $\gamma(\xi_j) \to \infty$. For $j \in \boldsymbol{N}$ there exists $g_j \in \mathscr{D}(\Omega) \subset \mathring{\mathscr{B}}_\gamma^m(\Omega)$, $g_j(\xi_j) = 1$. From $f_\iota g_j \to g_j$ we obtain $f_\iota(\xi_j) \underset{\iota}{\to} 1$. This implies $\sup\{p_1(f_\iota);\ \iota \in J\} \geqslant \gamma(\xi_j)$ for all j. ■

If γ is bounded then $1 \in \mathscr{B}_\gamma^m(\Omega)$, and therefore the bounded SFP for $\mathscr{B}_\gamma^m(\Omega)$ is trivial. We want to show that $\mathscr{B}_\gamma^m(\Omega)$ has the bounded SFP even for unbounded γ if the existence of a certain partition of unity is assumed. For $i \in \boldsymbol{N}$ we define

$$\Omega_i := \{\xi \in \Omega;\ j-1 < \gamma(\xi) < j+1\}.$$

Then $(\Omega_i;\ i \in \boldsymbol{N})$ is a covering of Ω by open sets, $\Omega_i \cap \Omega_j = \emptyset$ for $|i-j| > 1$. We assume that there exists a partition of unity $(\varphi_i;\ i \in \boldsymbol{N})$ on Ω, with the following properties:

(II) $\varphi_i \in \mathscr{E}^m(\Omega)$, $\varphi_i \geqslant 0$, $\operatorname{supp}\varphi_i \subset \Omega_i$ $(i \in \boldsymbol{N})$, $\sum_{i \in \boldsymbol{N}} \varphi_i(\xi) = 1$ $(\xi \in \Omega)$;

(III) for all $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, there exist $k(\alpha) \in \boldsymbol{N}_0$, $K_\alpha \geqslant 0$ such that

$$\sum_{i \in \boldsymbol{N}} |\partial^\alpha \varphi_i(\xi)| \leqslant K_\alpha \gamma(\xi)^{k(\alpha)} \quad (\xi \in \Omega). \tag{2.1}$$

In Section 3 we shall indicate a condition on γ which implies the existence of a partition of unity with these properties.

2.2. Remarks. (a) Let (I), (II), (III) be satisfied. Let $\mathscr{F} := \{F \subset \boldsymbol{N};\ F \text{ finite}\}$ be directed by inclusion. For $F \in \mathscr{F}$ we define $\varphi_F := \sum_{i \in F} \varphi_i$. Then $(\varphi_F;\ F \in \mathscr{F})$ is an AU for $\mathscr{B}_\gamma^m(\Omega)$.

(b) If (I), (II), (III) are satisfied, then we have

$$\mathring{\mathscr{B}}_\gamma^m(\Omega) = \{f \in \mathscr{E}^m(\Omega);\ \gamma(\cdot)^k \partial^\alpha f \in C_0(\Omega) \text{ for } k \in \boldsymbol{N}_0,\ |\alpha| \leqslant m\}.$$

For the nontrivial inclusion "$\supset$" it is sufficient to show that, for $f \in \{\ldots\}$, we have $\varphi_F f \in \mathring{\mathscr{B}}_\gamma^m(\Omega)$ for all $F \in \mathscr{F}$. This latter property is a consequence of [6], Prop. (4.10a), p. 77.

If additionally $1/\gamma \in C_0(\Omega)$ then we conclude $\mathring{\mathscr{B}}_\gamma^m(\Omega) = \mathscr{B}_\gamma^m(\Omega)$.

The following is the main result of this section.

2.3. Theorem. *Assume* (I), (II), (III). *Then* $\mathscr{B}_\gamma^m(\Omega)$ *has the bounded* SFP.

For the proof of this result we shall need some properties of the space s of rapidly decreasing sequences,

$$s := \{x = (x_i)_{i \in \boldsymbol{N}} \in \boldsymbol{K}^{\boldsymbol{N}};\ q_k(x) := \sup_{i \in \boldsymbol{N}} i^k |x_i| < \infty\ (k \in \boldsymbol{N}_0)\}.$$

Equipped with the sequence of norms $(q_k;\ k \in \boldsymbol{N}_0)$, s becomes a Fréchet space. Moreover, with multiplication defined by coordinatewise multiplication, the norms q_k satisfy (1.1), and therefore s a Fréchet algebra. s is a nuclear space (cf. [14], 6.1.6, p. 89), and therefore is reflexive (cf. [16], Ch. III, 5.5, p. 144). Since s has no unit, Proposition 1.5 implies that s does not have a BAU.

2.4. Lemma. *Let* $x \in s$, $d > 1$. *Then there exists* $y \in s$ *such that*:

(i) $|x_i| \leqslant y_i$ $(i \in \boldsymbol{N})$;

(ii) $d^{-1} y_i \leqslant y_{i+1} \leqslant y_i$ $(i \in \boldsymbol{N})$.

Proof. We define $\hat{y}_i := \max\{d^{-(i-j)}|x_j|;\ j = 1, \ldots, i\}$ $(i \in \boldsymbol{N})$. Then the properties $|x_i| \leqslant \hat{y}_i$, $\hat{y}_{i+1} \geqslant d^{-1}\hat{y}_i$ $(i \in \boldsymbol{N})$ are obvious. In order to show $(\hat{y}_i)_{i \in \boldsymbol{N}} \in s$ we note that for all $k \in \boldsymbol{N}_0$ there is $i_0 \in \boldsymbol{N}$ such that the sequence $(i^k d^{-i})_{i \geqslant i_0}$ is decreasing, and therefore

$$c_k := \sup\left\{\frac{i^k d^{-i}}{j^k d^{-j}};\ 1 \leqslant j \leqslant i\right\} = \max\left\{\frac{i^k d^{-i}}{j^k d^{-j}};\ 1 \leqslant j \leqslant i \leqslant i_0\right\} < \infty.$$

This shows

$$\sup_{i \in \boldsymbol{N}} i^k \hat{y}_i = \sup_{1 \leqslant j \leqslant i} i^k d^{-(i-j)} |x_j| = \sup_{1 \leqslant j \leqslant i} \frac{i^k d^{-i}}{j^k d^{-j}} j^k |x_j| \leqslant c_k q_k(x) < \infty.$$

Finally, if we define $y_i := \max_{j \geqslant i} \hat{y}_j$ $(i \in \boldsymbol{N})$, then it is easy to show that $y = (y_i)_{i \in \boldsymbol{N}}$ has the desired properties. ∎

2.5. Theorem. *s has the compact* SFP. *More precisely: If $K \subset s$ is compact, $\varepsilon > 0$, $k' \in \boldsymbol{N}_0$, $d > 1$, then there exist $z \in s$, $i' \in \boldsymbol{N}$ with the following properties*:

(i) $z_1 = z_2 = \ldots = z_{i'} = 1$, $d^{-1} z_i \leqslant z_{i+1} \leqslant z_i$ $(i \in \boldsymbol{N})$;

(ii) $\sup_{i \geqslant i'} i^{k'} |x_i|/z_i \leqslant \varepsilon$ *for all* $x \in K$;

(iii) *with* $K' := \overline{\operatorname{aco}} K$, *a continuous linear mapping* $T\colon (s_{K'}, p_{K'}) \to s$ *is defined by* $T\big((x_i)_{i \in \boldsymbol{N}}\big) := (x_i/z_i)_{i \in \boldsymbol{N}}$;

(iv) $T|K\colon K \to s$ *is continuous*;

(v) $q_{k'}(Tx - x) \leqslant 2\varepsilon$ *for all* $x \in K$.

Proof. Without loss of generality we assume $K \neq \{0\}$. We define $x' = (x_i')_{i \in \boldsymbol{N}} \in s$ by $x_i' := \sup\{|x_i|;\ x = (x_i)_{i \in \boldsymbol{N}} \in K\}$. Corresponding to x' and d^2 we find $y \in s$ according to Lemma 2.4. There exists $i' \in \boldsymbol{N}$ such that

$$\sup_{i \geqslant i'} i^{2k'} y_i \leqslant \varepsilon^2 q_0(y)^{-1}.$$

We define $z = (z_i)_{i \in \boldsymbol{N}} \in s$ by

$$z_i := \begin{cases} 1 & \text{for } i = 1, \ldots, i', \\ (y_{i'}^{-1} y_i)^{1/2} & \text{for } i > i'. \end{cases}$$

For later use we note $(x_i')^{1/2} \leqslant y_i^{1/2} \leqslant y_1^{1/2} z_i$, $x_i'/z_i \leqslant y_1^{1/2} (x_i')^{1/2}$ $(i \in \boldsymbol{N})$.

Obviously z satisfies (i). Property (ii) holds since for $i \geqslant i'$ we have

$$i^{k'} |x_i|/z_i \leqslant i^{k'} y_i y_{i'}^{1/2} y_i^{-1/2} = y_{i'}^{1/2} (i^{2k'} y_i)^{1/2} \leqslant q_0(y)^{1/2} \big(\varepsilon^2 q_0(y)^{-1}\big)^{1/2} = \varepsilon.$$

(iii): It is obviously sufficient to show that $\{(x_i/z_i)_{i \in \boldsymbol{N}};\ x \in K'\}$ is a bounded subset of s. For $x \in K'$ we have $|x_i| \leqslant x_i'$ $(i \in \boldsymbol{N})$. For $k \in \boldsymbol{N}_0$, $x \in K'$ we therefore have

$$\sup_{i \in \boldsymbol{N}} i^k |x_i|/z_i \leqslant \sup_{i \in \boldsymbol{N}} i^k x_i'/z_i \leqslant y_1^{1/2} \sup_{i \in \boldsymbol{N}} i^k (x_i')^{1/2} < \infty.$$

(iv): For $j \in \boldsymbol{N}$, the mapping $T_j\colon s \to s$, $T_j x := (x_1/z_1, \ldots, x_j/z_j, 0, 0, \ldots)$, is continuous. For $k \in \boldsymbol{N}$, $x \in K$ we have $q_k(Tx - T_j x) \leqslant \sup_{i \geqslant j+1} i^k x_i'/z_i \leqslant y_1^{1/2} \sup_{i \geqslant j+1} (i^{2k} x_i')^{1/2}$. This shows $T_j x \to Tx$ $(j \to \infty)$ uniformly for $x \in K$.

(v): For $x \in K$ we obtain from (i) and (ii)

$$q_{k'}(Tx - x) = \sup_{i \geqslant i'+1} i^{k'} (|x_i|/z_i + |x_i|) \leqslant 2 \sup_{i \geqslant i'} i^{k'} |x_i|/z_i \leqslant 2\varepsilon.$$

In order to show that s has the compact SFP it remains to show (ii) of Definition 1.1. This property follows from the fact that the closed ideal generated by $x \in s$ is given by

$$\{x' \in s;\ \{i \in \boldsymbol{N};\ x_i' \neq 0\} \subset \{i \in \boldsymbol{N};\ x_i \neq 0\}\}. \ ∎$$

After this excursion we return to the situation described at the beginning of this section. For $f \in \mathscr{E}^m(\Omega)$, $\alpha \in \boldsymbol{N}_0^n$ with $|\alpha| \leqslant m$ we define a sequence $x^\alpha(f) = (x_i^\alpha(f))_{i\in\boldsymbol{N}}$ by

$$x_i^\alpha(f) := \sup\{|\partial^\alpha f(\xi)|;\ \xi \in \Omega_i\} \qquad (i \in \boldsymbol{N}).$$

Then the definitions imply immediately

$$\mathscr{B}_\gamma^m(\Omega) = \{f \in \mathscr{E}^m(\Omega):\ x^\alpha(f) \in s \text{ for all } \alpha \in \boldsymbol{N}_0^n,\ |\alpha| \leqslant m\}. \tag{2.2}$$

Moreover, the inequality

$$q_k(x^\alpha(f)) \leqslant 3^k \alpha!\, p_{\max(k,|\alpha|)}(f) \tag{2.3}$$

holds for all $f \in \mathscr{B}_\gamma^m(\Omega)$, $k \in \boldsymbol{N}_0$, $\alpha \in \boldsymbol{N}_0^n$ with $|\alpha| \leqslant m$.

2.6. LEMMA. *Let* (I), (II), (III) *be satisfied, and let* $x \in s$,

$$f(\xi) := \sum_{i\in\boldsymbol{N}} x_i \varphi_i(\xi) \qquad (\xi \in \Omega).$$

Then $f \in \mathscr{B}_\gamma^m(\Omega)$.

We omit the proof which is an immediate consequence of (I), (II), (III) and (2.2).

The construction described in the following proposition contains the main part of the proof of Theorem 2.3.

2.7. PROPOSITION. *Let* (I), (II), (III) *be satisfied. Let* $B \subset \mathscr{B}_\gamma^m(\Omega)$ *be a bounded set. Let further* $k', k'' \in \boldsymbol{N}_0$, $\varepsilon > 0$, $d > 1$. *Then there exist* $i' \in \boldsymbol{N}$, $u \in \mathscr{B}_\gamma^m(\Omega)$, $0 < u(\xi) \leqslant 1$ *for all* $\xi \in \Omega$, *such that:*

(i) $u(\xi) = 1$ *for all* ξ *in a neighbourhood of* $\{\xi \in \Omega;\ \gamma(\xi) \leqslant i'\}$;

(ii) $\dfrac{\varphi_i}{u} \in \mathscr{B}_\gamma^m(\Omega)$ *for all* $i \in \boldsymbol{N}$;

(iii) *for all* $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, *we have* $\left|\dfrac{\partial^\alpha u}{u}\right| \leqslant dK_\alpha \gamma(\cdot)^{k(\alpha)}$ (*with* K_α, $k(\alpha)$ *from* (2.1));

(iv) *for all* $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, $k \in \boldsymbol{N}_0$, $f \in B' := \overline{\operatorname{aco}} B$ *we have* $\gamma(\cdot)^k \left(\dfrac{\partial^\alpha f}{u}\right) \in L_\infty(\Omega)$, *and the mapping* $(\mathscr{B}_\gamma^m(\Omega)_{B'}, p_{B'}) \ni f \mapsto \gamma(\cdot)^k \left(\dfrac{\partial^\alpha f}{u}\right) \in L_\infty(\Omega)$ *is continuous*;

(v) *for all* $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, $k \in \boldsymbol{N}_0$, *the mapping* $B \ni f \mapsto \gamma(\cdot)^k \left(\dfrac{\partial^\alpha f}{u}\right) \in L_\infty(\Omega)$ *is continuous*;

(vi) *for all* $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant \min(k'', m)$, $f \in B$ *we have*

$$\sup\left\{\left|\gamma(\xi)^{k'}\left(\frac{\partial^\alpha f(\xi)}{u(\xi)}\right)\right|;\ \xi \in \bigcup_{i=i'+1}^\infty \Omega_i\right\} \leqslant \varepsilon,$$

$$\left\|\gamma(\cdot)^{k'}\left(\frac{\partial^\alpha f}{u} - \partial^\alpha f\right)\right\|_\infty \leqslant 2\varepsilon.$$

Proof. For each $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, the set $\{x^\alpha(f);\ f \in B\}$ is a bounded subset of s by (2.3). We define $x^\alpha \in s$ by

$$x_i^\alpha := \sup\{x_i^\alpha(f);\ f \in B\} \qquad (i \in \boldsymbol{N}).$$

Since s is a Fréchet space, there exists a family $\{\lambda_\alpha;\ \alpha \in \boldsymbol{N}_0^n,\ |\alpha| \leqslant m\}$ in $(0, \infty)$ such that the series $\sum_{|\alpha|\leqslant m} \lambda_\alpha x^\alpha =: x = (x_i)_{i\in\boldsymbol{N}}$ converges in s; without restriction we assume $\lambda_\alpha = 1$ for all $\alpha \in \boldsymbol{N}_0^n$ with $|\alpha| \leqslant \min(k'', m)$. For the compact subset $\{x' \in s;\ |x_i'| \leqslant x_i\ (i \in \boldsymbol{N})\}$ of s we find $z \in s$, $i' \in \boldsymbol{N}$ according to Theorem 2.5 (where k', ε, d are prescribed as in the hypothesis). We now define

$$u(\xi) := \sum_{i\in\boldsymbol{N}} z_i \varphi_i(\xi) \qquad (\xi \in \Omega)$$

and verify that u and i' have the desired properties. From $z \in s$ and Lemma 2.6 we obtain $u \in \mathscr{B}_\gamma^m(\Omega)$. From property (i) of Theorem 2.5 we obtain $0 < u(\xi) \leqslant 1$ $(\xi \in \Omega)$.

(i) follows from $z_1 = z_2 = \ldots = z_{i'} = 1$ (Theorem 2.5 (i)).

(ii): From the proof of (iii) below it follows that $1/u$ is bounded on Ω_i for all $i \in \boldsymbol{N}$. Let $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$. Since $\partial^\alpha\left(\dfrac{\varphi_i}{u}\right)$ is a linear combination of products of terms $\dfrac{\partial^\beta \varphi_j}{u}$ $(j = i-1, i, i+1)$, we obtain that $\partial^\alpha\left(\dfrac{\varphi_i}{u}\right)$ is bounded. From $\operatorname{supp}\varphi_i \subset \Omega_i$ it follows that $\gamma(\cdot)^k \partial^\alpha\left(\dfrac{\varphi_i}{u}\right)$ is bounded for all $k \in \boldsymbol{N}_0$.

(iii): Let $\xi \in \Omega$, $i \in \boldsymbol{N}$ with $i \leqslant \gamma(\xi) \leqslant i+1$. Then Theorem 2.5 (i) implies $u(\xi) = z_i\varphi_i(\xi) + z_{i+1}\varphi_{i+1}(\xi) \geqslant z_{i+1}$. For $|\alpha| \leqslant m$ we therefore have

$$\frac{|\partial^\alpha u(\xi)|}{u(\xi)} \leqslant \frac{1}{z_{i+1}}\left(z_i|\partial^\alpha\varphi_i(\xi)| + z_{i+1}|\partial^\alpha\varphi_{i+1}(\xi)|\right)$$

$$\leqslant \frac{z_i}{z_{i+1}} K_\alpha \gamma(\xi)^{k(\alpha)} \leqslant dK_\alpha\gamma(\xi)^{k(\alpha)}.$$

(iv): It is sufficient to show that $\left\{\gamma(\cdot)^k \dfrac{\partial^\alpha f}{u};\ f \in B'\right\}$ is bounded in $L_\infty(\Omega)$ for all $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, $k \in \boldsymbol{N}_0$. Now, $f \in B'$ implies $x_i^\alpha(f) \leqslant x_i^\alpha$

$\leqslant \lambda_\alpha^{-1} x_i$ $(i \in \boldsymbol{N})$. Let $\xi \in \Omega$, $i \in \boldsymbol{N}$ with $i-1 < \gamma(\xi) \leqslant i$. Then, for $f \in B'$,

$$\gamma(\xi)^k \frac{|\partial^\alpha f(\xi)|}{u(\xi)} \leqslant i^k \frac{x_i^\alpha(f)}{z_i} \leqslant \lambda_\alpha^{-1} i^k \frac{x_i}{z_i} \leqslant \lambda_\alpha^{-1} q_k\left(\left(\frac{x_i}{z_i}\right)_{i\in\boldsymbol{N}}\right) < \infty.$$

(v): Let $|\alpha| \leqslant m$, $k \in \boldsymbol{N}_0$. For all $i \in \boldsymbol{N}$ the mapping $\mathscr{B}_\gamma^m(\Omega) \ni f \mapsto \gamma(\cdot)^k \varphi_i \frac{\partial^\alpha f}{u} \in L_\infty(\Omega)$ is obviously continuous. From

$$\sup\left\{\left\|\gamma(\cdot)^{k+1}\left(\frac{\partial^\alpha f}{u}\right)\right\|_\infty ; f \in B\right\} < \infty$$

we obtain

$$\left\|\gamma(\cdot)^k\left(\frac{\partial^\alpha f}{u}\right) - \sum_{i=1}^{j} \gamma(\cdot)^k \varphi_i \left(\frac{\partial^\alpha f}{u}\right)\right\|_\infty \to 0$$

$(j\to\infty)$ uniformly for $f \in B$.

(vi): For $|\alpha| \leqslant \min(k'', m)$ we have $\lambda_\alpha = 1$. If $f \in B$, $\xi \in \Omega$ with $i' \leqslant i-1 < \gamma(\xi) \leqslant i$, then as in the proof of (iv),

$$\gamma(\xi)^{k'}\left(\frac{|\partial^\alpha f(\xi)|}{u(\xi)}\right) \leqslant i^{k'}\left(\frac{x_i}{z_i}\right) \leqslant \varepsilon,$$

where the last estimate follows from (ii) of Theorem 2.5. This proves the first estimate. The second estimate follows from this, using property (i) and $u(\xi) \leqslant 1$ for all $\xi \in \Omega$. ■

Proof of Theorem 2.3. Let $B \subset \mathscr{B}_\gamma^m(\Omega)$ be bounded, $B' := \overline{\mathrm{aco}}\, B$, V a neighbourhood of zero in $\mathscr{B}_\gamma^m(\Omega)$. We have to find $u \in \mathscr{B}_\gamma^m(\Omega)$ and a mapping T: $\mathscr{B}_\gamma^m(\Omega)_{B'} \to \mathscr{B}_\gamma^m(\Omega)$ such that the properties of Definition 1.1 are satisfied. There exist $k'' \in \boldsymbol{N}_0$, $\delta > 0$ such that $V \supset \{f \in \mathscr{B}_\gamma^m(\Omega);\ p_{k''}(f) \leqslant \delta\}$.

Let $\varepsilon > 0$ (to be fixed later in order to match δ just introduced), and let $k' \in \boldsymbol{N}_0$ (to be specified according to certain constants occurring in the following proof). Corresponding to B, k', k'', ε, $d := 2$ we obtain $i' \in \boldsymbol{N}$, $u \in \mathscr{B}_\gamma^m(\Omega)$ according to Proposition 2.7. We want to show that a continuous linear mapping T: $\mathscr{B}_\gamma^m(\Omega)_{B'} \to \mathscr{B}_\gamma^m(\Omega)$ can be defined by $Tf := \frac{f}{u}$. In order to show this it is sufficient to show that $\left\{\frac{f}{u};\ f \in B'\right\}$ is a bounded subset of $\mathscr{B}_\gamma^m(\Omega)$.

Let $k \in \boldsymbol{N}_0$, $|\alpha| \leqslant \min(k, m)$. Then $\gamma(\cdot)^k \partial^\alpha\left(\frac{f}{u}\right)$ is a sum of terms of the form

$$\gamma(\cdot)^k \prod_{i=1}^{i_0}\left(\frac{\partial^{\alpha_i} u}{u}\right)\frac{\partial^\beta f}{u}, \tag{2.4}$$

with $\sum_{i=1}^{i_0} \alpha_i + \beta = \alpha$ (proof by induction). Using (iii) of Proposition 2.7 we estimate

$$\left\|\gamma(\cdot)^k \prod_{i=1}^{i_0}\left(\frac{\partial^{\alpha_i} u}{u}\right)\frac{\partial^\beta f}{u}\right\|_\infty \leqslant \left(\prod_{i=1}^{i_0} 2K_{\alpha_i}\right)\left\|\gamma(\cdot)^{k+\Sigma k(\alpha_i)}\left(\frac{\partial^\beta f}{u}\right)\right\|_\infty,$$

and the last quantity is uniformly bounded for $f \in B'$, by (iv) of Proposition 2.7.

Property (i) of Definition 1.1, i.e., $u(Tf) = f$ for all $f \in B$, is obvious. Property (iv) of Definition 1.1, i.e., the continuity of $T|B$: $B \to \mathscr{B}_\gamma^m(\Omega)$, is a rather immediate consequence of (v) and (iii) of Proposition 2.7.

In order to prove property (iii) of Definition 1.1 we let $|\alpha| \leqslant \min(k'', m)$. Then, for $f \in B$, $\gamma(\cdot)^{k''}\left(\partial^\alpha\left(\frac{f}{u}\right) - \partial^\alpha f\right)$ is a sum of the term $\gamma(\cdot)^{k''}\left(\frac{\partial^\alpha f}{u} - \partial^\alpha f\right)$ and of terms of the form (2.4), with $k = k''$, $\beta \neq \alpha$. If k' has been chosen initially such that $k' \geqslant k''$ then (vi) of Proposition 2.7 implies

$$\left\|\gamma(\cdot)^{k''}\left(\frac{\partial^\alpha f}{u} - \partial^\alpha f\right)\right\|_\infty \leqslant 2\varepsilon.$$

The other terms can be estimated, using (i) and (iii) of Proposition 2.7, by

$$\left\|\gamma(\cdot)^{k''} \prod_{i=1}^{i_0}\left(\frac{\partial^{\alpha_i} u}{u}\right)\frac{\partial^\beta f}{u}\right\|_\infty$$

$$\leqslant \prod_{i=1}^{i_0} 2K_{\alpha_i} \sup\left\{\gamma(\xi)^{k''+\Sigma k(\alpha_i)}\left(\frac{|\partial^\beta f(\xi)|}{u(\xi)}\right);\quad \xi \in \bigcup_{i=i'+1}^{\infty} \Omega_i\right\},$$

and this is $\leqslant \prod_{i=1}^{i_0} (2K_{\alpha_i})\varepsilon$ by property (vi) of Proposition 2.7 if $k' \geqslant k'' + \sum_{i=1}^{i_0} k(\alpha_i)$ is required initially. Thus, if we choose initially

$$k' := k'' + \max\left\{\sum_{i=1}^{i_0} k(\alpha_i);\ \alpha_1, \ldots, \alpha_{i_0} \in \boldsymbol{N}_0^n,\ \left|\sum_{i=1}^{i_0} \alpha_i\right| \leqslant \min(k'', m)\right\},$$

then the estimates just stated are satisfied. Therefore $p_{k''}\left(\frac{f}{u} - f\right)$ can be estimated by a finite linear combination of such terms; yielding an estimate $p_{k''}\left(\frac{f}{u} - f\right) \leqslant C_{k''}\varepsilon$, where $C_{k''}$ depends only on k'' and external constants. Choosing $\varepsilon = C_{k''}^{-1}\delta$ initially, we thus obtain $p_{k''}\left(\frac{f}{u} - f\right) \leqslant \delta$, and therefore $Tf - f \in V$.

It remains to show property (ii) of Definition 1.1. Let $f \in B$. Then $\frac{f}{u} \in \mathscr{B}_{\gamma}^{m}(\Omega)$, and therefore $\varphi_F\left(\frac{f}{u}\right) \underset{F}{\to} \frac{f}{u}$ by Remark 2.2(a). From property (ii) above we obtain $\frac{\varphi_F}{u} \in \mathscr{B}_{\gamma}^{m}(\Omega)$ for all $F \in \mathscr{F}$, and therefore $\varphi_F\left(\frac{f}{u}\right) = \left(\frac{\varphi_F}{u}\right) f$ belongs to the ideal generated by f. ■

2.8. Remark. It will be shown in Section 3 that conditions (I), (II), (III) have been chosen general enough to cover the cases of $\mathscr{S}(\boldsymbol{R}^n)$ and $\dot{\mathscr{B}}(\Omega)$. The proof of Theorem 2.3, however, depends on very particular techniques which are only possible for algebras of functions. It would be desirable to find a proof which carries over to a general class of Fréchet algebras.

3. Factorization in $\mathscr{S}(\boldsymbol{R}^n)$ and $\dot{\mathscr{B}}(\Omega)$. As a preparation we are going to supplement condition (I) (of Section 2) by a condition which implies the existence of a partition of unity satisfying (II) and (III).

Let $\emptyset \neq \Omega = \mathring{\Omega} \subset \boldsymbol{R}^n$, $m \in \boldsymbol{N}_0 \cup \{\infty\}$, and let $\gamma \in C(\Omega)$ satisfy (I). The additional condition is:

(IV) $\gamma \in \mathscr{E}^m(\Omega)$, and for all $\alpha \in \boldsymbol{N}_0^n$, $|\alpha| \leqslant m$, there exist $m(\alpha) \in \boldsymbol{N}_0$, $M_\alpha \geqslant 0$ such that

$$|\partial^\alpha \gamma(\xi)| \leqslant M_\alpha \gamma(\xi)^{m(\alpha)} \quad (\xi \in \Omega). \tag{3.1}$$

3.1. Lemma. *Let $\gamma \in C(\Omega)$ satisfy* (I) *and* (IV). *Then there exists a partition of unity $(\varphi_i;\ i \in \boldsymbol{N})$ satisfying* (II) *and* (III).

Proof. There exists a partition of unity $(\psi_i;\ i \in \boldsymbol{Z})$ on $\boldsymbol{R}$, $\psi_i \in \mathscr{D}(\boldsymbol{R})$, $\psi_i \geqslant 0$, $\operatorname{supp}\psi_i \subset (i-1, i+1)$, and such that $\psi_i = \psi_0(\cdot - i)$ holds for all $i \in \boldsymbol{Z}$. For $i \in \boldsymbol{N}$ we define $\varphi_i := \psi_i \circ \gamma$. Then obviously $(\varphi_i;\ i \in \boldsymbol{N})$ is a partition of unity on Ω satisfying (II). For $\alpha = 0$ estimate (2.1) is valid with $K_0 = 1$, $m(0) = 0$. For $0 < |\alpha| \leqslant m$ we have

$$\partial^\alpha \varphi_i = \partial^\alpha(\psi_i \circ \gamma) = \sum_{k=1}^{|\alpha|} (\psi^{(k)} \circ \gamma) \sum_{\beta \in J(\alpha,k)} C(\alpha, k, \beta) \prod_{j=1}^{k} \partial^{\beta(j)} \gamma,$$

where $J(\alpha, k) := \left\{\beta := \left(\beta(1), \ldots, \beta(k)\right) \in (\boldsymbol{N}_0^n)^k;\ \beta(i) \neq 0 \quad (i = 1, \ldots, k),\ \sum_{i=1}^{k} \beta(i) = \alpha\right\}$, and $C(\alpha, k, \beta) \in \boldsymbol{N}_0$ (cf. [5], Satz (15.1), p. 120, [8]). Taking into account (3.1), we obtain

$$|\partial^\alpha \varphi_i(\xi)| = \sum_{k=1}^{|\alpha|} \left|\psi_i^{(k)}\left(\gamma(\xi)\right)\right| \sum_{\beta \in J(\alpha,k)} C(\alpha, k, \beta) \prod_{j=1}^{k} M_{\beta(j)} \gamma(\xi)^{m(\beta(j))}.$$

Since $\left|\psi_i^{(k)}\left(\gamma(\xi)\right)\right| \leqslant \|\psi_0^{(k)}\|_\infty$ $(\xi \in \Omega,\ k = 1, \ldots, |\alpha|)$, and since the sum $\sum_{i \in \boldsymbol{N}} |\partial^\alpha \varphi_i(\xi)|$ contains at most two nonzero terms for each $\xi \in \Omega$, an estimate of the form (2.1) follows. ■

For $\Omega = \boldsymbol{R}^n$, the function $\gamma\colon \boldsymbol{R}^n \to [1, \infty)$,

$$\gamma(\xi) := 1 + |\xi|^2 \quad (\xi \in \boldsymbol{R}^n)$$

obviously satisfies (I) and (IV) for $m = \infty$. For this function γ, $\mathscr{B}_\gamma(\boldsymbol{R}^n) = \mathscr{S}(\boldsymbol{R}^n)$ is the space of rapidly decreasing functions (cf. [10], Ch. 2, § 4, Ex. 14, p. 91). The following result is now a consequence of Lemma 3.1 and Theorem 2.3. (Note that $\mathscr{S}$ is a Montel space, and that for a Fréchet algebra which is a Montel space the compact SLFP is equivalent to the bounded SLFP.) It answers the question of Kamiński mentioned in the introduction.

3.2. Theorem. *$\mathscr{S}(\boldsymbol{R}^n)$ has the compact* SFP.

Since the Fourier transform is a topological isomorphism of $\mathscr{S}(\boldsymbol{R}^n)$ (cf. [10], ch. 4, § 11, Thm. 1, p. 416), transforming pointwise product of two functions into convolution product (denoted by "$*$" we obtain also the following result.

3.3. Corollary. *$\left(\mathscr{S}(\boldsymbol{R}^n), *\right)$ has the compact* SFP.

For $\emptyset \neq \Omega = \mathring{\Omega} \subsetneqq \boldsymbol{R}^n$ we define the boundary distance $r\colon \Omega \to (0, \infty)$, $r(\xi) := \operatorname{dist}(\xi, \complement\Omega)$. There exists a regularized boundary distance, i.e., a function $\tilde{r} \in \mathscr{E}(\Omega)$ with the following properties: There exist $0 < d_1 < d_2$ such that

$$d_1 r(\xi) \leqslant \tilde{r}(\xi) \leqslant d_2 r(\xi) \quad (\xi \in \Omega); \tag{3.2}$$

for each $\alpha \in \boldsymbol{N}_0^n$ there exists $N_\alpha \geqslant 0$ such that

$$|\partial^\alpha \tilde{r}(\xi)| \leqslant N_\alpha r(\xi)^{1-|\alpha|} \quad (\xi \in \Omega) \tag{3.3}$$

(cf. [18], Ch. VI, § 2.1, Thm 2, p. 171). We define $\gamma \in \mathscr{E}(\Omega)$ by

$$\gamma(\xi) := 1 + \tilde{r}(\xi)^{-1} \quad (\xi \in \Omega).$$

For $\alpha \in \boldsymbol{N}_0^n$, $\alpha \neq 0$, we use the formula for the derivative of composite functions mentioned in the proof of Lemma 3.1, in order to obtain

$$|\partial^\alpha \gamma(\xi)| = |\partial^\alpha(\tilde{r}^{-1})(\xi)| \leqslant M_\alpha \tilde{r}(\xi)^{-|\alpha|-1} \leqslant M_\alpha \gamma(\xi)^{|\alpha|+1}$$

with suitable M_α, i.e., property (IV) is satisfied for $m = \infty$. For this function γ, it is shown in [6], Prop. (4.6b, c), p. 75, that

$$\mathscr{B}_\gamma(\Omega) = \check{\mathscr{B}}(\Omega) := \{f \in \mathscr{E}(\Omega)\colon\ \partial^\alpha f \in L_\infty(\Omega) \text{ for all } \alpha \in \boldsymbol{N}_0^n, \text{ and there exists a continuous extension } f_\alpha \in C(\overline{\Omega}) \text{ of } \partial^\alpha f \text{ satisfying } f_\alpha | \partial\Omega = 0\}$$

holds, and that the sequence of norms $(\check{p}_k;\ k \in \boldsymbol{N})$, $\check{p}_k(f) := \sum_{|\alpha| \leqslant k} \frac{1}{\alpha!} \|\partial^\alpha f\|_\infty$, generates the topology on $\mathscr{B}_\gamma(\Omega)$ defined in Section 2. In view of Remark 2.2 (b) this implies $\dot{\mathscr{B}}_\gamma(\Omega) = \overline{\mathscr{D}(\Omega)}^{\check{\mathscr{B}}(\Omega)} = \dot{\mathscr{B}}(\Omega)$ $(= \mathscr{B}^\infty(\Omega)$, defined in Section 1) (cf. [6], Prop. (4.10b), p. 77).

3.4. THEOREM. (a) $\check{\mathscr{B}}(\Omega)$ *has the bounded* SFP.

(b) $\mathring{\mathscr{B}}(\Omega)$ *has the compact* SFP.

Proof. (a): Since γ as defined above satisfies (I), (IV), and since $\check{\mathscr{B}}(\Omega) = \mathscr{B}_\gamma(\Omega)$, the assertion follows from Lemma 3.1 and Theorem 2.3.

(b): Let $K \subset \mathring{\mathscr{B}}(\Omega)$ be compact, without restriction $K = \overline{\text{aco}}\,K$. Let U be a neighbourhood of zero in $\mathring{\mathscr{B}}(\Omega)$; then $U \supset \{f \in \mathring{\mathscr{B}}(\Omega);\ \mathring{p}_k(f) \leqslant \varepsilon\}$ for suitable $k \in \boldsymbol{N}_0$, $\varepsilon > 0$.

Let

$$V := \{f \in \mathring{\mathscr{B}}(\Omega);\ \mathring{p}_k(f) \leqslant \varepsilon/2\}.$$

Since $\check{\mathscr{B}}(\Omega)$ $(\supset \mathring{\mathscr{B}}(\Omega))$ has the bounded SFP, there exist $u \in \check{\mathscr{B}}(\Omega)$ and $T\colon \check{\mathscr{B}}(\Omega)_K \to \check{\mathscr{B}}(\Omega)$ with the properties as in Definition 1.1. In particular, Tf belongs to the closed ideal of $\check{\mathscr{B}}(\Omega)$ generated by f for $f \in K$. Since $f \in \mathring{\mathscr{B}}(\Omega)$, and $\mathring{\mathscr{B}}(\Omega)$ is a closed ideal, we obtain $Tf \in \mathring{\mathscr{B}}(\Omega)$. This shows $\tilde{K} := TK \subset \mathring{\mathscr{B}}(\Omega)$.

$\mathring{\mathscr{B}}(\Omega)$ is continuously embedded in $\mathring{\mathscr{B}}(\boldsymbol{R}^n)$, by extending each function by zero. We thus identify $\mathring{\mathscr{B}}(\Omega)$ with a subspace of $\mathring{\mathscr{B}}(\boldsymbol{R}^n)$.

In Proposition 1.6 it was shown that $\mathring{\mathscr{B}}(\boldsymbol{R}^n)$ has the compact SFP. We apply it to the compact absolutely convex set $\tilde{K}$ and the neighbourhood of zero

$$\tilde{V} := \{f \in \mathring{\mathscr{B}}(\boldsymbol{R}^n);\ \mathring{p}_k(f) \leqslant \varepsilon/2\}.$$

We obtain $\tilde{u} \in \mathring{\mathscr{B}}(\boldsymbol{R}^n)$,

$$\tilde{T}\colon \mathring{\mathscr{B}}(\boldsymbol{R}^n)_{\tilde{K}} \to \mathring{\mathscr{B}}(\boldsymbol{R}^n)$$

with the properties in Definition 1.1.

It remains to show that the function $u\tilde{u}$ and the operator $\tilde{T}T$ have the desired properties. It is easy to verify $u\tilde{u} \in \mathring{\mathscr{B}}(\Omega)$. As above, we obtain $\tilde{T}\tilde{K} \subset \mathring{\mathscr{B}}(\Omega)$. $T\colon \mathring{\mathscr{B}}(\Omega)_K \to \mathring{\mathscr{B}}(\Omega)_{\tilde{K}}$ is a topological isomorphism, and therefore $\tilde{T}T\colon \mathring{\mathscr{B}}(\Omega)_K \to \mathring{\mathscr{B}}(\Omega)$ is continuous.

The properties (i), (iii), (iv) of Definition 1.1 are easily verified. In order to show property (ii) we recall that we know so far that $\tilde{T}Tf$ belongs to the closed ideal of $\mathring{\mathscr{B}}(\boldsymbol{R}^n)$ generated by f for all $f \in K$. It is therefore sufficient to show that if $f \in \mathring{\mathscr{B}}(\Omega)$, $g \in \check{\mathscr{B}}(\boldsymbol{R}^n)$, then gf belongs to the closed ideal of $\mathring{\mathscr{B}}(\Omega)$ generated by f. In [6], Prop. (4.10a), p. 77, it was shown that there exists a sequence $(\eta_j;\ j \in \boldsymbol{N})$ in $\mathscr{D}(\Omega)$ such that $\eta_j f \to f$ $(j \to \infty)$ in $\mathring{\mathscr{B}}(\Omega)$. This implies $g\eta_j f \to gf$ in $\mathring{\mathscr{B}}(\Omega)$. From $g\eta_j \in \mathscr{D}(\Omega) \subset \mathring{\mathscr{B}}(\Omega)$ we therefore obtain the desired statement. ∎

3.5. Remarks. (a) If Ω is such that $\mathring{\mathscr{B}}(\Omega) = \check{\mathscr{B}}(\Omega)$ holds then the statement of (b) of Theorem 3.4 is seemingly weaker than that of (a). The following properties, however, are equivalent ([6], Thm. (4.8), p. 76, and Thm. (4.11), p. 78):

(i) $\mathring{\mathscr{B}}(\Omega) = \check{\mathscr{B}}(\Omega)$;

(ii) $\mathring{\mathscr{B}}(\Omega)$ is a Montel space;

(iii) Ω is quasi-bounded;

(iv) $r(\cdot) \in C_0(\Omega)$ $(\Leftrightarrow 1/\gamma \in C_0(\Omega))$.

Thus, if $\mathring{\mathscr{B}}(\Omega) = \check{\mathscr{B}}(\Omega)$, then the compact and bounded SFP are identical properties for $\mathring{\mathscr{B}}(\Omega)$.

(b) In view of Proposition 1.5 and Theorem 3.4 one might ask whether, for $\emptyset \neq \Omega = \mathring{\Omega} \subsetneqq \boldsymbol{R}^n$, $\mathring{\mathscr{B}}^m(\Omega)$ has a factorization property for numbers $m \neq \infty, 0$ (note that $\mathring{\mathscr{B}}^0(\Omega) = C_0(\Omega)$ has a UBAU (= BAU), and Theorem 1.3 implies the compact SFP for $C_0(\Omega)$). The following example shows that this cannot be expected.

If $f_1, f_2 \in \mathring{\mathscr{B}}^1(0, \infty)$ then $|f_i(t)| \leqslant t\|f_i'\|_\infty$ $(i = 1, 2,\ t > 0)$, and therefore $|(f_1f_2)(t)| \leqslant t^2\|f_1'\|_\infty\|f_2'\|_\infty$. Since it is easy to find $f \in \dot{\mathscr{B}}^1(0, \infty)$ with $t^{-2}|f(t)| \to \infty$ $(t \to 0)$, we obtain $\mathring{\mathscr{B}}^1(0, \infty) \cdot \mathring{\mathscr{B}}^1(0, \infty) \neq \mathring{\mathscr{B}}^1(0, \infty)$.

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Added in proof.

1. A proof of Theorem 1.3 is contained in

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2. The author got to know the reference

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Received October 15, 1982 (1818)

STUDIA MATHEMATICA, T. LXXVII. (1984)

Completeness type properties of locally solid Riesz spaces

by

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Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. The main results, in the terminology of Aliprantis and Burkinshaw [A&B], and Fremlin [F] (this terminology has been changed for the reasons 'intrinsic' to this paper), are as follows. Let (L, τ) be a Hausdorff locally solid Riesz space. It embeds order densely into a Nakano space $(L^{\#}, \tau^{\#})$ if (and only if) τ is Fatou; this embedding is unique. A Dedekind complete (L, τ) embeds order densely into a Hausdorff locally solid Dedekind complete Riesz space $(L^{\#}, \tau^{\#})$ having the Monotone Completeness Property if (and only if) τ is pseudo-Lebesgue.

Let Ω be an extremally disconnected topological space, $C^{\infty}(\Omega)$ the Riesz space of continuous functions, from Ω into the extended real line, which take finite values on dense subsets of Ω.

In the first part of this paper a theory which parallels the one of Banach function spaces by Luxemburg and Zaanen [4], is initiated on $C^{\infty}(\Omega)$. Function filters and their topological vector cores replace function norms and their Banach function spaces. This permits to treat the general locally solid case. In § 1 the (topological) completness properties of vector cores are investigated.

In the second part, to an order dense Riesz subspace of C^{∞} with a locally solid vector topology appropriate function filters are associated.

In the third part the previous results are applied, via the Maeda–Ogasawara representation theorem, to general locally solid Riesz spaces. The main results are as follows.

Let (L, τ) be a Hausdorff topological Riesz space.

(L, τ) embeds order densely into a Hausdorff locally solid-boundedly order-complete $(L^{\#}, \tau^{\#})$ if (and only if) τ is locally solid-order-closed; this embedding is unique up to an isomorphism.

A Dedekind complete (L, τ) embeds order densely into a Hausdorff locally solid Dedekind complete Riesz space $(L^{\#}, \tau^{\#})$ having the Monotone Completeness Property if (and only if) τ is locally solid-pseudo-order-closed.