Hence we have the following formula

$$\frac{1}{1-\lambda K} = 1+\lambda\frac{\Phi(K,\lambda)}{\varphi(\lambda)}, \tag{9}$$

where $\Phi(K,\lambda) = K\Phi_1(\lambda)+ \ldots +K^p\Phi_p(\lambda)$.

Since $\varphi(\lambda)$ is a polynomial of degree $\leqslant p$ and $\frac{1}{1-\lambda K}$ is not defined only at the points at which $\varphi(\lambda) = 0$, then

The spectrum of a vector of rank p consists of p points at most.

We also find from formula (9), in view of (5), the following form of the resolvent

$$K_\lambda = \frac{\Phi(K,\lambda)}{\varphi(\lambda)}.$$

In the application to Fredholm's equation the above way represents a convenient algorithm for finding the resolvent for the kernel with separate variables(14):

$$K(x,y) = \sum_{r=1}^{n} a_r(x)b_r(y);$$

for vector $K = \{K(x,y)\}$ always is of finite rank.

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On type II convergence in the Mikusiński operational calculus

by

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Dedicatad to Professor Jan Mikusiński

Abstract. In the paper it is proved that type II convergence in the field $\mathcal{F}$ of Mikusiński's operators is not topological (Theorem 2.1), which is a solution of the problem posed in [1]. It is given a characterization of type II convergence and, defined in the paper, type II′ convergence. A description of compactness and boundedness in $\mathcal{F}$ with type II′ convergence is given and a sequential completeness of $\mathcal{F}$ is proved.

1. In the field of Mikusiński operators three types of convergence: type I, type I′ and type II are introduced (see [5], p. 144, 147 and [2]). Properties of type I and type I′ convergences are described in [2], [3].

In the paper we shall describe properties of type II convergence. In particular, it will be proved that type II convergence is not topological. This is the negative answer to the problem posed in [1]. Moreover, we shall give some facts about type II convergence, similar to that given in [3] for type I′ convergence.

We shall use terminology and notation from [3].

2. We say that a sequence $\{x_n\}$ of operators is *type* II *convergent to* x (and we write $x_n \xrightarrow{\text{II}} x$) if there exist continuous functions f, g, f_n, g_n $(n = 1, 2, \ldots)$ such that $x_n = f_n/g_n$, $x = f/g$ and $f_n \to f$, $g_n \to g$ almost uniformly.

In the above definition continuous functions can be replaced by locally integrable functions (as in [3] L denotes the set of all such functions) and the almost uniform convergence by the convergence with respect to the following family of pseudonorms:

$$\|f\|_T = \int_0^T |f(t)|\,dt \quad \text{for any } f\in L \text{ and } T>0.$$

The above convergence will be denoted by $\xrightarrow{L}$.

By Corollary 4.2 from [2] (see also Theorem 11 in [3]), it follows that type I′ convergent sequences are type II convergent (to the same limit).

The converse is not true, because the sequence $\{x_n\} = \{1/g_n\}$, where

(1) $$g_n(t) = \begin{cases} 1/n & \text{for } t \in [0, \lambda], \\ 1 & \text{for } t \in (\lambda, \infty) \end{cases}$$

is type II convergent to the operator $x = 1/h^{1}l$ but it is not type I′ convergent.

Note that if a sequence of operators x_n is type II convergent, then the sequence $\Lambda(x_n)$ of the support numbers is bounded from the left and thus the sequence x_n has a common denominator (see [3]).

Let λ be an arbitrary non-negative number.

We say that a sequence of functions $f_n \in L$ $(n = 1, 2, \ldots)$ is *type* II_λ *convergent to* $f \in L$ (and we write $f_n \xrightarrow{\mathrm{II}_\lambda} f$) if there exist functions $g_n, g \in L$ $(n = 1, 2, \ldots)$ such that

$$\Lambda(g) \leqslant \lambda, \quad g_n \xrightarrow{L} g, \quad g_n f_n \xrightarrow{L} g \cdot f.$$

We say that a sequence $\{x_n\}$ of operators is type II_λ convergent to an operator x (and we write $x_n \xrightarrow{\mathrm{II}_\lambda} x$) if there exist functions $f, g, f_n \in L$ $(n = 1, 2, \ldots)$ such that

$$x_n = f_n/g, \quad x = f/g, \quad f_n \xrightarrow{\mathrm{II}_\lambda} f.$$

We have

$$x_n \xrightarrow{\mathrm{II}_\lambda} x \Leftrightarrow x_n - x \xrightarrow{\mathrm{II}_\lambda} 0.$$

If $0 \leqslant \lambda < \mu$, then

$$x_n \xrightarrow{\mathrm{II}_\lambda} x \Rightarrow x_n \xrightarrow{\mathrm{II}_\mu} x.$$

Note that

$$x_n \xrightarrow{\mathrm{II}} x \Rightarrow x_n \xrightarrow{\mathrm{II}_\lambda} x \text{ for some } \lambda \geqslant 0.$$

By Corollary 4.2 in [2] (see also Theorem 11 in [3]) we have

$$x_n \xrightarrow{\mathrm{I}'} x \Leftrightarrow x_n \xrightarrow{\mathrm{II}_0} x,$$

.e., type I′ and type II_0 convergence are equivalent.

LEMMA 2.1. *For any $\lambda > 0$ there are functions $f, f_n \in L$ $(n = 1, 2, \ldots)$ such that $\Lambda(f_n) = \lambda$, $(n = 1, 2, \ldots)$, $\Lambda(f) = 0$, $f_n \xrightarrow{\mathrm{II}_\lambda} f$ and $f_n \overset{\mathrm{II}_\mu}{\not\to} f$ for any $\mu < \lambda$.*

Proof. Let g_n be functions defined by (1). We have

$$g_n(t) \xrightarrow{L} g(t) = \begin{cases} 0, & t \in [0, \lambda], \\ 1, & t \in (\lambda, \infty) \end{cases}$$

and $\Lambda(g) = \lambda$, $\Lambda(g_n) = 0$ $(n = 1, 2, \ldots)$.

By Proposition 2 in [3], there exist functions $f, \varphi_n \in L_0$ $(n = 1, 2, \ldots)$, where L_0 is the set of all $f \in L$ for which $\Lambda(f) = 0$, such that

$$1/g_n = \varphi_n/f \quad (n = 1, 2, \ldots).$$

Let $f_n = g\varphi_n$. We have

$$\Lambda(f_n) = \Lambda(g) + \Lambda(\varphi_n) = \Lambda(g) = \lambda$$

and

$$g_n \cdot f_n = g \cdot (g_n \varphi_n) = gf.$$

Hence

$$f_n \xrightarrow{\mathrm{II}_\lambda} f.$$

Now, suppose that

$$f_n \xrightarrow{\mathrm{II}_\mu} f$$

for some $\mu < \lambda$. Then there exist functions $h, h_n \in L$ $(n = 1, 2, \ldots)$ such that

$$h_n \xrightarrow{L} h, \quad \Lambda(h) \leqslant \mu, \quad h_n f_n \xrightarrow{L} hf.$$

We have

$$\Lambda(h_n f_n) \geqslant \Lambda(f_n) = \lambda \quad (n = 1, 2, \ldots)$$

and thus

$$\Lambda(hf) \geqslant \lambda$$

which leads to the contradiction, because

$$\Lambda(hf) = \Lambda(h) + \Lambda(f) \leqslant \mu.$$

The proof is finished.

THEOREM 2.1. *Type* II *convergence does not satisfy the Urysohn condition, i.e., it is not topological.*

Proof. Let $\{f_{kn}\}$ for $k = 1, 2, \ldots$ be a sequence of functions in L such that

$$f_{kn} \xrightarrow{\mathrm{II}_{k+1}} 0, \quad f_{kn} \overset{\mathrm{II}_k}{\not\to} 0 \quad \text{as} \quad n \to \infty$$

and

$$\Lambda(f_{kn}) \geqslant k.$$

Let $\{f_n\}$ be such a sequence, which contains as subsequences all sequences $\{f_{kn}\}_{n\in N}$ and does not contain others elements than f_{kn} $(k, n \in N)$. The sequence $\{f_n\}$ is not type II convergent to 0, because for any $\lambda > 0$ it contains a subsequence which is not type II_λ convergent to 0.

We shall prove that every subsequence of the sequence $\{f_n\}$ contains a subsequence, which is type II convergent to 0. Let $\{f_{p_n}\}$ be an arbitrary subsequence of the sequence $\{f_n\}$.

Only one of the two following cases can happen:

(a) The sequence $\{f_{p_n}\}$ has infinitely many elements, which belong to the sequence $\{f_{kn}\}$ for some $k = 1, 2, \ldots$ Then the sequence $\{f_{p_n}\}$ contains a sequence type II_{k+1} convergent to 0.

(b) For each $k = 1, 2, \ldots$ the sequence $\{f_{p_n}\}$ contains only a finite number of elements belonging to $\{f_{kn}\}$. Then $\Lambda(f_{p_n}) \to \infty$ which results $f_{p_n} \xrightarrow{L} 0$ and thus also $f_{p_n} \xrightarrow{\mathrm{II}} 0$.

It is known that the set L of all locally integrable functions (and thus all continuous functions) is dense with respect to type I convergence in the subset $\mathscr{F}_0$ of the field $\mathscr{F}$ of the Mikusiński operators (see [4]). It turns out that L is dense with respect to type II convergence in the whole $\mathscr{F}$. Namely, we have

THEOREM 2.2. *For any operator $x \in \mathscr{F}$ there exist functions $\varphi_n \in L$ $(n = 1, 2, \ldots)$ such that*

$$\varphi_n \xrightarrow{\mathrm{II}} x.$$

Proof. Every operator x can be represented in the form $x = h^{-\lambda} x_0$, where $x_0 \in \mathscr{F}_0$ and $\lambda \geqslant 0$.

Since L is dense in $\mathscr{F}_0$ with respect to type I and thus also type II convergence, it suffices to prove that for any $\lambda > 0$ there exist functions $h_n \in L\,(n = 1, 2, \ldots)$ such that

$$h_n \xrightarrow{\mathrm{II}} h^{-\lambda}.$$

Let $f, f_n \in L$ $(n = 1, 2, \ldots)$ and

$$f_n \xrightarrow{\mathrm{II}} f, \quad \Lambda(f) = 0, \quad \Lambda(f_n) = \lambda \quad (n = 1, 2, \ldots).$$

There are in L functions g_n $(n = 1, 2, \ldots)$ such that

$$g_n \cdot f \xrightarrow{\mathrm{I}} 1 = h^0.$$

On the other hand, we have

$$g_n \cdot f_n \xrightarrow{\mathrm{II}} 1$$

and, consequently, the sequence $h_n = h^{-\lambda} g_n f_n$ $(h_n \in L$ for $n = 1, 2, \ldots)$ is convergent to $h^{-\lambda}$.

3. Let for any $\lambda \geqslant 0$ and $f \in L$

$$A_{\lambda,T,\varepsilon}(f) = \inf\Big\{\|fg\|_{\lambda+T}\colon\ g \in L,\ \|g\|_\lambda < \varepsilon,\ \|g\|_{\lambda+T} < 1,\ \int_\lambda^{\lambda+T} |l - g| < \varepsilon\Big\}.$$

In particular, we have

$$A_{0,T,\varepsilon}(f) = B_{T,\varepsilon}(f) = \inf\{\|fg\|_T\colon\ \|g\|_T < 1,\ \|l - lg\|_T < \varepsilon\}$$

(see [3]).

THEOREM 3.1. *A sequence $\{f_n\}$ $(f_n \in L$ for $n = 1, 2, \ldots)$ is type II_λ convergent to 0 if and only if the sequence $A_{\lambda,T,\varepsilon}(f_n)$ is convergent to 0 for any $T, \varepsilon > 0$.*

Proof. Suppose that $f_n \xrightarrow{\mathrm{II}_\lambda} 0$ and take arbitrary $T, \varepsilon > 0$. There are functions $g, g_n \in L$ $(n = 1, 2, \ldots)$ such that $\Lambda(g) = \lambda$, $g_n \xrightarrow{L} g$ and $g_n f_n \xrightarrow{L} 0$. Let $g_0 = h^{-\lambda} g$. By Lemma 2 (see [3]), there exists a function $k \in L$ such that

$$\|kg_0\|_T < 1, \quad \|l - lkg_0\|_T < \varepsilon.$$

We have

$$\|kg\|_\lambda = 0, \quad \|kg\|_{\lambda+T} < 1, \quad \int_\lambda^{\lambda+T} |l - lkg| < \varepsilon$$

and thus

$$\|kg_n\|_\lambda < \varepsilon, \quad \|kg_n\|_{\lambda+T} < 1, \quad \int_\lambda^{\lambda+T} |l - kg_n| < \varepsilon$$

for sufficiently large n. Hence

$$A_{\lambda,T,\varepsilon}(f_n) \leqslant \|kg_n f_n\|_{\lambda+T} \to 0$$

as $n \to \infty$.

Now, suppose that $A_{\lambda,T,\varepsilon}(f_n) \to 0$ for any $T, \varepsilon > 0$. There are indices $\gamma_1 < \gamma_2 < \ldots$ such that

$$A_{\lambda,k,1/k}(f_n) < 1/k$$

for $n \geqslant \gamma_k$ $(k = 1, 2, \ldots)$. That means that there exist functions $g_n \in L$ $(n = 1, 2, \ldots)$ such that

$$\|g_n\|_\lambda < 1/k, \quad \|g_n\|_{\lambda+k} < 1, \quad \int_\lambda^{\lambda+k} |l - lg_n| < 1/k, \quad \|g_n f_n\|_{\lambda+k} < 1/k$$

for $n \geqslant \gamma_k$.

Let $h_n = lg_n$. We have

$$h_n \xrightarrow{L} h^{\lambda}l \quad \text{and} \quad h_n f_n \xrightarrow{L} 0,$$

i.e., $f_n \xrightarrow{\mathrm{II}_\lambda} 0$.

As consequences of Theorem 3.1, we obtain

COROLLARY 3.1. (a) *For any $\lambda > 0$ the convergence II_λ satisfies the Urysohn condition.*

(b) *A sequence $\{f_n\}$, $f_n \in L$ is type II_λ convergent to 0 if and only if there exist functions $h_n \in L$ $(n = 1, 2, \ldots)$ such that $h_n \xrightarrow{L} l_\lambda$ and $h_n \cdot f_n \xrightarrow{L} 0$ $(l_\lambda = h^\lambda l)$.*

(c) *If a sequence x_n of operators is type II_ν convergent to an operator x for any $\nu > \lambda$, then x_n is type II_λ convergent to x; in particular, if for any $\lambda > 0$ the sequence x_n is type II_λ convergent to x, then $x_n \xrightarrow{\Gamma} x$.*

Now, applying (a)–(c), we shall give a characterization of type II convergence in L in terms of the functions $B_{T,\lambda}$ introduced in [3].

THEOREM 3.2. *A sequence $\{f_n\}$, $f_n \in L$ is type II convergent to 0 if and only if there exists $\lambda > 0$ such that for every $T > 0$ we have $B_{T,\lambda}(f_n) \to 0$.*

Proof. Let $f_n \xrightarrow{\mathrm{II}} 0$. Then $f_n \xrightarrow{\mathrm{II}_\nu} 0$ for some $\nu > 0$. We shall show that $B_{T,\lambda}(f_n) \to 0$ for any $T > 0$ and $\lambda > \mu$. It can be assumed that $T > \nu$. There exist functions $g_n, g \in L$ such that $g_n \xrightarrow{L} g$, $\Lambda(g) = \nu$ and $g_n f_n \xrightarrow{L} 0$.

Now, let $g_0 = h^{-\Lambda(g)}g$. By Lemma 2 ([3]), there exists a function $k \in L$ such that

$$\|kg_0\|_{T-\nu} < 1, \quad \|l - lkg_0\|_{T-\nu} < \lambda - \nu.$$

We have $\|kg\|_T < 1$, $\|l - lkg\|_T < \lambda$ and thus $\|kg_n\|_T < 1$, $\|l - lkg_n\|_T < \lambda$ for sufficiently large n. Hence

$$B_{T,\lambda}(f_n) \leqslant \|kg_n f_n\|_T \to 0$$

as $n \to \infty$.

Suppose now that for some $\lambda > 0$ we have $B_{T,\lambda}(f_n) \to 0$ for every $T > 0$. We shall show that $f_n \xrightarrow{\mathrm{II}_\lambda} 0$.

Since type II_λ convergence satisfies the Urysohn condition, it suffices to prove that from the sequence $\{f_n\}$ it can be selected a subsequence, which is type II_λ convergent to 0.

Note that there are indices $\gamma_1 < \gamma_2 < \ldots$ and functions $h_n \in L$ $(n = 1, 2, \ldots)$ such that

$$\|h_n\|_n < 1, \quad \|l - lh_n\|_n < \lambda, \quad \|h_n f_n\|_n < 1/n.$$

By Lemma 1 in [3], there exist a subsequence $\{h_{p_n}\}$ of the sequence $\{h_n\}$ and function $g \in L$ such that $lh_{p_n} \xrightarrow{L} g$. We have $\|l - g\|_T \leqslant \lambda$ for any $T > 0$, so $\Lambda(g) \leqslant \lambda$.

Let $\alpha_n = \gamma_{\beta_n}$, $g_n = lh_{\beta_n}$. We have $g_n f_{\alpha_n} \xrightarrow{L} 0$, $g_n \xrightarrow{L} g$ and $\Lambda(g) \leqslant \lambda$, so

$$f_{\alpha_n} \xrightarrow{\mathrm{II}_\lambda} 0.$$

Now, we give a sufficient condition for a set $A \subset L$ to be type II_λ precompact in $\mathscr{F}$.

THEOREM 3.3. *Let A be an arbitrary subset of L and let $\lambda > 0$. If for every $T > 0$ the set $\{B_{\lambda,T}(f) \colon f \in A\}$ is bounded, then the set A is type II_λ precompact.*

Proof. Let $f_n \in A$ $(n = 1, 2, \ldots)$. We shall select from the sequence $\{f_n\}$ a type II_λ convergent subsequence.

By the assumption, for any $k \in N$ there exist functions $g_{kn} \in L$ $(k, n \in N)$ such that $\|g_{kn}\|_{\lambda+k} < 1$, $\|l - lg_{kn}\|_{\lambda+k} < \lambda$ and the sequence $\|g_{kn} \cdot f_n\|_{\lambda+k}$ $(n = 1, 2, \ldots)$ is bounded.

We can assume that the sequences $\{lg_{kn}\}$, $\{lg_{kn}f_n\}$ $(n = 1, 2, \ldots)$ are convergent for any $k \in N$ in the norm $\|\cdot\|_{\lambda+k}$ because using the diagonal method we can chose a sequence of indices r_n such that the sequences

$$\{lg_{kr_n}\}, \quad \{lg_{kr_n}f_{r_n}\} \quad \text{are convergent for all } k \in N.$$

Let

$$lg_{k_n} \to g_k$$

and

$$lg_{kn}f_n \to \bar{g}_k$$

in the norm $\|\cdot\|_{\lambda+k}$ as $n \to \infty$.

It is clear that $\|l - g_k\| \geqslant \lambda$ and thus $\Lambda(g_k) \leqslant \lambda$. Therefore there are functions $g, h_k \in L$ $(k \in N)$ different from 0, such that

$$\Lambda(g) \leqslant \lambda \quad \text{and} \quad h_k g_k = g \quad \text{for} \quad k = 1, 2, \ldots$$

We have

$$lh_k g_{kn} \to h_k g_k = g$$

and

$$lh_k g_{kn} f_n \to h_k \bar{g}_k$$

in the norm $\|\cdot\|_{\lambda+k}$ as $n \to \infty$.

It suffices to prove that

$$(2) \qquad h_k \bar{g}_k = h_m \bar{g}_m \quad \text{on} \quad [0, k],$$

provided $k < m$. But

$$(lh_m g_{mn} - lh_k g_{kn}) f_n \to h_m \bar{g}_m - h_k \bar{g}_k$$

and

$$(lh_m g_{mn} \cdot lh_k g_{kn}) lg_{kn} f_n \to 0$$

in the norm $\|\cdot\|_{\lambda+k}$ as $n\to\infty$. Since

$$lg_{kn}\to g_k \quad \text{as} \quad n\to\infty$$

in $\|\cdot\|_{\lambda+k}$, we get

$$g_k(h_m\bar{g}_m - h_k\bar{g}_k) = 0 \quad \text{on} \quad [0, \lambda+k].$$

Taking into account that $\Lambda(g_k)\leqslant\lambda$, we obtain (2), in view of the Titchmarsh theorem. Thus the proof is complete.

In particular, Theorem 3.3 and Corollary 3.1 (c) imply Theorem 3 in [3].

4. In Section 2, we have proved that type II convergence in $\mathscr{F}$ is not topological.

By type II′ convergence we mean the weakest topological convergence among the ones stronger than type II convergence. In other words, we have $x_n\xrightarrow{\text{II}'}x$ iff from any subsequence $\{x_{p_n}\}$ of $\{x_n\}$ one can select a subsequence $\{x_{r_n}\}$ such that $x_{r_n}\xrightarrow{\text{II}}x$.

We are going to describe type II′ convergence in L and obtain some properties of type II convergence. For arbitrary sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $S = (T_1, T_2, \ldots)$ of positive numbers and for $f\in L$ let

$$C_{S,\alpha} = \inf\{\alpha_n B_{T_n,n}(f)\colon n\in N\}.$$

THEOREM 4.1. *A sequence $\{f_n\}$, $f_n\in L$ is type II′ convergent to 0 if and only if for arbitrary sequences $S = (T_1, T_2, \ldots)$, $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive numbers $C_{S,\alpha}(f_n)\to 0$.*

Proof. Suppose that $f_n\xrightarrow{\text{II}}0$. Then there exists $k\in N$ such that

$$B_{T,k}(f_n)\to 0 \quad \text{as} \quad n\to\infty$$

for any $T>0$. Consequently, for arbitrary $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $S = (T_1, T_2, \ldots)$ we have

$$C_{S,\alpha}(f_n)\leqslant \alpha_k B_{T_k,k}(f_n)\to 0$$

as $n\to\infty$.

Therefore if $f_n\xrightarrow{\text{II}'}0$, then from any subsequence of the sequence $\{C_{S,\alpha}(f_n)\}$ one can select a subsequence tending to 0, i.e., $C_{S,\alpha}(f_n)\to 0$.

Now, suppose conversely, that the sequence $\{f_n\}$ is not type II′ convergent to 0. Then there is a subsequence $\{f_{p_n}\}$ of the sequence $\{f_n\}$ and there are numbers $T_{k,\varepsilon_k}>0$ $(k = 1, 2, \ldots)$ such that

$$B_{T_k,k}(f_{p_n})\geqslant\varepsilon_k \quad (k, n = 1, 2, \ldots).$$

Putting $S = (T_1, T_2, \ldots)$ and $\alpha = (\varepsilon_1^{-1}, \varepsilon_2^{-1}, \ldots)$ we have

$$C_{S,\alpha}(f_{p_n})\geqslant 1,$$

i.e.,

$$C_{S,\alpha}(f_n)\not\to 0.$$

Before formulating the next theorem, we shall prove the following lemma:

LEMMA 4.1. *Let $f\in L$. If $T>\varepsilon+\Lambda(f)$, then $B_{T,\varepsilon}(f)>0$.*

Proof. Assume that $B_{T,\varepsilon}(f) = 0$. Then there exists a sequence $\{g_n\}$, $g_n\in L$ such that $\|g_n\|_T<1$, $\|l-lg_n\|_T<\varepsilon$ $(n = 1, 2, \ldots)$ and $\|g_nf\|_T\to 0$. By Lemma 1 in [3], there exist a subsequence $\{g_{p_n}\}$ of $\{g_n\}$ and a function $g\in L$ such that

$$lg_{p_n}\to g.$$

Of course, we have $\|lgf\|_T = 0$ and $\Lambda(lg)\leqslant\varepsilon$. By the Titchmarsh theorem, $\Lambda(f)\geqslant T+\varepsilon$, so $T\leqslant\Lambda(f)+\varepsilon$, which contradicts the assumption.

THEOREM 4.2. *A set $A\subset L$ is type II′ precompact (type II precompact) in $\mathscr{F}$ if and only if for arbitrary sequences $S = (T_1, T_2, \ldots)$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ of positive numbers the set $\{C_{S,\alpha}(f);\ f\in A\}$ is bounded.*

Proof. Suppose that the set $A\subset L$ is not type II′ (type II) precompact, i.e., there is a sequence $\{f_n\}\subset A$, which does not contain any subsequence type II convergent in $\mathscr{F}$. In particular, we have for some $\gamma>0$

$$\Lambda(f_n)\leqslant\gamma \quad (n = 1, 2, \ldots).$$

Since type II_λ precompactness for some $\lambda\geqslant 0$ implies type II precompactness, we deduce from Theorem 3.3 that there exists a subsequence $\{f_{p_n}\}$ of $\{f_n\}$ and numbers T_k $(k = 1, 2, \ldots)$ such that

$$T_k>k+\gamma, \quad B_{T_k,k}(f_{p_n})\to\infty$$

as $n\to\infty$ for any $k = 1, 2, \ldots$

By Lemma 4.1, we have $B_{T_k,k}(f_{p_n})>0$ for $n, k = 1, 2, \ldots$ Hence we can select a subsequence $\{f_{r_n}\}$ of $\{f_n\}$ in such a way that

$$B_{T_k,k}(f_{r_n})\geqslant n \quad \text{for} \quad n\geqslant k.$$

Of course, we can find positive numbers $\alpha_1, \alpha_2, \ldots$ such that

$$\alpha_k B_{T_kk}(f_{r_n})\geqslant n$$

for all $k, n = 1, 2, \ldots$ Putting $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $S = (T_1, T_2, \ldots)$, we have

$$C_{S,\alpha}(f_{r_n})\geqslant n\to\infty \quad \text{as} \quad n\to\infty.$$

This means that the set $\{C_{S,\alpha}(f),\ f\in A\}$ is not bounded.

Thus we have proved one of the implications. The second one follows immediately from the preceding theorem, by virtue of homogeneity of the functions $C_{S,\alpha}$.

Note that every sequence of operators which is type II′ bounded has a common denominator. Hence we obtain the following corollary from Theorems 4.1 and 4.2:

THEOREM 4.3. *A set $A \subset \mathscr{F}$ is type* II′ *precompact (or equivalently, type* II *precompact) if and only if A is type* II′ *bounded.*

Finally, we shall prove the following result, concerning completeness of $\mathscr{F}$:

THEOREM 4.4. *Type* II′ *convergence in $\mathscr{F}$ is P-complete and type* II *convergence is Q-complete in $\mathscr{F}$.*

Proof. Note that every type II′ Cauchy sequence in $\mathscr{F}$ has a common denominator (cf. Proposition 2 in [1]). Therefore it suffices to prove that if $f_n \in L$ $(n = 1, 2, \ldots)$ and $\{f_n\}$ is a type II′ Cauchy sequence, then $\{f_n\}$ is type II′ convergent. This means it is enough to show that the set $\{f_n;\ n \in N\}$ is type II precompact.

Suppose that it does not hold, i.e., there exist sequences $S = (T_1, T_2, \ldots)$, $\alpha = (\alpha_1, \alpha_2, \ldots)$ and a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$$C_{S,\alpha}(g_n) \to \infty \quad \text{as} \quad n \to \infty.$$

Then for every $k \in N$ we have

$$B_{T_k,k}(g_n) \to \infty \quad \text{as} \quad n \to \infty$$

and thus there is a subsequence $\{h_n\}$ of $\{g_n\}$ such that

$$B_{T_{2k},2k}(h_{n+1}) \geqslant B_{T_{2k},k}(h_n) + 1$$

for $k, n \in N$, $k \leqslant n$.

Hence, using inequality 3° in Section 2 in [3], we get

$$B_{T_{2k},k}(\varphi_{n+1} - \varphi_n) \geqslant B_{T_{2k},2k}(\varphi_{n+1}) - B_{T_{2k},k}(\varphi_n) \geqslant 1 \tag{3}$$

for $n \geqslant k$.

On the other hand, we have

$$\varphi_{n+1} - \varphi_n \xrightarrow{\text{II}'} 0 \quad \text{as} \quad n \to \infty$$

and consequently there exists index k such that

$$B_{T_k,k}(\varphi_{n+1} - \varphi_n) < 1$$

for infinitely many n, which contradicts (3). The proof is completed.

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