$H_p$-spaces, $p \leq 1$, and spline systems

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Abstract. Using spline systems we construct unconditional bases in spaces $H_p(D)$, $0 < p < 1$. This is used to give a direct isomorphism between $H_p(D)$ and martingale $H_p$ spaces. We also show that our systems are bases in Bergman spaces $A^2_p$, $0 < p < 1$, and we characterize those spaces in terms of coefficients (this gives an explicit isomorphism of $A^2_p$ with $A^1_p$). Applications to complemented subspaces of $H_p$, $p < 1$ and to properties of spline systems in $L_p$, $1 < p < \infty$, are also given.

This paper presents an application of orthonormal spline systems and some related systems of splines to natural $H_p$ spaces. The main emphasis is put on $H_p(D)$, the classical Hardy space of analytic functions on the unit disc in the complex plane. Our methods, however, are mainly those from real variable $H_p$-theory. We use atomic decompositions of $H_p$-functions, as developed in [13] and [31], to prove the continuity of natural operators associated with expansions with respect to spline systems. Our basic result (Theorem 2) gives a construction of an unconditional basis in $H_p(D)$, $p \leq 1$. Moreover, the bases we construct can be used to give explicit isomorphisms between various $H_p$-spaces; most notably we show that $H_p(D)$ is isomorphic to the $H_p$ space of dyadic martingales. This extends to $p < 1$ results of [24], [4] and [32]. Some of these results have been obtained with different proofs by Sjölin and Stromberg [30].

The above-mentioned isomorphisms provide equivalence between some chapters of martingale theory (cf. [31]), constructive function theory (cf. [7]) and $H_p$-spaces. This equivalence allows us to give new proofs for some results of Glaeski's [7] and [8] on spline systems in $L_p$, $p > 1$ (Theorem 11 and Corollary 4).

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As an application of our results we give a characterization of Bergman spaces \(A_q^p, q \leq 1\) on the unit disc (Theorem 6). This characterization is related to the one given in [12]. In particular we get that \(A_q^p\) is isomorphic, as a linear topological space, to \(L_q\). The other application concerns complemented subspaces of \(H_p(D)\). Answering the question from [22] we show that every infinite dimensional complemented subspace of \(H_p(D)\), \(p < 1\), contains a smaller complemented subspace isomorphic to \(L_q\).

Our notation is standard. For the general background in \(H_p\)-theory the reader may consult [13] and [31] and for elements of the theory of splines we suggest [28] and [17] and [9]. In order to make this paper more self-contained, rather lengthy preliminary sections on \(H_p\)-theory and spline systems are added.

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1. Preliminaries, spline systems. Our aim in this section is to construct some systems of splines on the unit circle \(T\). We identify \(T\) with \([-1, 1]\).

If \(m\) is an integer, \(m \geq -1\), and \(V\) is a partition of \(T\) into intervals \(V = \{I_1, I_2, \ldots, I_n\}\) then the spline of order \(m\) with respect to the partition \(V\) is any function \(f\) on \(T\) such that \(f\) is \(m\)-times continuously differentiable and for every \(j = 1, 2, \ldots, n\), \(f|_{I_j}\) is a polynomial of degree at most \(m+1\). The space of all splines of order \(m\) with respect to partition \(V\) will be denoted by \(S^m(V)\).

In this paper we will consider only dyadic partitions. For \(n = 2^h+1\), \(0 \leq h < n\) we define the partition \(V_n\) of \([0, 1]\) by

\[
I_s = \left[ \frac{s-1}{2^h+1}, \frac{s}{2^h+1} \right] \quad \text{if} \quad 1 \leq s \leq 2^h, \\
I_s = \left[ \frac{s-1}{2^h}, \frac{s}{2^h} \right] \quad \text{if} \quad 2^h < s \leq n.
\]

Using this partition we define three partitions of \(T\) as follows

\[
V_n = \{ I = T : I \in V_n \text{ or } -I \in V_n \}, \\
V^1_n = \{ I = T : I \in V_n \text{ or } -I \in V_{n+1} \}, \\
V^2_n = \{ I = T : I \in V_n \text{ or } -I \in V_{n+1} \}.
\]

We order intervals in \(V^1_n\) in such a way that \(I_0\) equals \(I_0\) of partition \(V_n\) if \(s > 0\) and \(I_0\) equals \(-I_{n+1}\) of partition \(V_n\) if \(s \leq 0\). Partitions \(V^2_n\) and \(V^3_n\) are ordered analogously. We define \(t_s\) as the point which is an endpoint of some interval from \(V_{n+1}\), but is not an endpoint of any interval from \(V_n\).

Given a partition \(V = \{I_1, I_2, \ldots, I_n\}\) of the circle \(T\), with intervals \(I_i\) in \(V\) ordered consecutively and \(n > m+2\) we define the basic spline of order \(m\) with respect to partition \(V\), \((b^m_1)^p\), by the following two conditions:

1. for each \(j\) the \((b^m_1)^p\) is a non-zero spline of order \(m\) with respect to \(V\) and the support of \((b^m_1)^p\) equals \(I_j \cup I_{j+1} \cup \ldots \cup I_{j+m+1}\) if \(r > n\) then we interpret \(I_r = I_{n+1}\),

2. for each \(j\) the \((b^m_1)^p\) is 0 and \(\sum_{j=1}^{m+1} (b^m_1)^p = 1\).

Basic splines are investigated in detail in chapter 4 of [28]. In particular, it is shown that these are basic splines and we also have that if the lengths of intervals in \(V\) are comparable (e.g. their ratios are between 1/2 and 2 as is the case for partitions \(V^1_n, V^2_n,\) and \(V^3_n\)) then

\[
\left( \int \frac{|b^m_1|^2}{|f|^2} \right)^{1/2} \sim 1.
\]

Moreover, in this case (cf. Th. 4.41 of [28]) the set of biorthogonal functionals \(\hat{\lambda}_j\) on \(S^m(V)\), i.e., functionals such that \(\langle \hat{\lambda}_j, b^m_1 \rangle = \delta_{ij}, i, j = 1, 2, \ldots, n\), satisfies

\[
|\hat{\lambda}_j(f)| \leq c \left( \int_{supp b^m_1} |f|^2 \right)^{1/2} \quad \text{for every} \quad f \in S^m(V).
\]

It is also true that every spline in \(S^m(V)\) is a linear combination of basic splines.

Lemma 1. Let \((b^m_j)^p\) denote all basic splines of order \(m\) with respect to partition \(V^1_n\) or \(V^2_n\) or \(V^3_n\). There exists a constant \(K_m\) such that for all \(n\) and all \((a_j)\)

\[
K_m \left( \sum_{j=1}^{n-1} |a_j|^2 \right)^{1/2} \leq \left( \int \sum_{j} |a_j b^m_j|^2 \right)^{1/2} \leq K_m \left( \sum_{j} |a_j|^2 \right)^{1/2}.
\]

Proof. We have by (1.7) and (1.8)

\[
\sum_{j} |a_j|^2 \leq c \left( \int \sum_{j} |a_j b^m_j|^2 \right)^{1/2} \leq c \sum_{j} \sum_{k} |a_k b^m_k|^2 \leq c \sum_{j} \sum_{k} |a_j| |b^m_k|^2 \leq c \sum_{j} \sum_{k} |a_j| |b^m_k|^2 .
\]
On the other hand,
\[ \int \left| \sum a_i b_j^m \right|^2 = \sum_i \int |a_i b_j^m|^2 \leq C \sum_i \int \int_{I_i} |a_i b_j^m|^2 \leq C \int \sum_{i,j} |a_i|^2 |b_j^m|^2 \leq C \sum_{i,j} |a_i|^2 |b_j|^2. \]

The above estimates give the lemma.

Let \( h_a \) denote the element of \( S^m(V_2^a) \) which is orthogonal to \( S^m(V_1) \) and has \( L_2(T) \) norm equal 1. This is well defined for \( n > 1 \). We put \( h_a = \text{const} \).

**Lemma 2.** There are constants \( C > 0 \) and \( q, 0 < q < 1 \), independent of \( n \) such that
\[ |h_a^m(t)| \leq C n^{-q} q^{\frac{m(n-1)}{2}}, \]
where \( d(t, t_a) \) denotes the distance on \( T \) between \( t \) and \( t_a \).

**Proof.** Let \( (h_j^m)_{j,m} \) denote basic splines with respect to the partition \( V_2^m \). Let us define the matrix \( \mathbb{M} \) by
\[ a_{ij} = \langle b_i^m, b_j^m \rangle \]
\( \langle \cdot, \cdot \rangle \) denotes the natural scalar product in \( L_2(T) \). Clearly, we have \( a_{ij} = 0 \) if \( \text{supp } b_i^m \cap \text{supp } b_j^m = \emptyset \), and by (1.6) we have \( |a_{ij}| \leq C n^{-1} \). We infer from Lemma 1 that the matrix \( \mathbb{M} \) defines an isomorphism of the space \( \mathbb{M} \) and \( (\mathfrak{M}, \| \cdot \|) \) are bounded independently of \( n \). The result of Domsta [15] or the periodic version of Lemma 2 of [14] gives that the entries \( a_{ij} \) of the matrix \( \mathbb{M} \) satisfy
\[ |a_{ij}| \leq K n^{-q} q^{\frac{m(n-1)}{2}} \]
for some constants \( K > 0 \) and \( q, 0 < q < 1 \). Symbol \( \varphi(i, j) \) denotes the number of intervals from \( V_2^m \) between \( I_i \) and \( I_j \). If we write
\[ h_a^m = \sum_I a_i b_i^m \]
we have
\[ \langle h_a^m, b_i^m \rangle = \sum_I a_i \varphi(i, j) \text{ so } a_i = \frac{1}{\varphi(i, j)} \langle h_a^m, b_i^m \rangle. \]

Let us observe that if \( t_a \notin \text{supp } b_i^m \) then \( b_i^m \in S^m(V_2) \), so \( \langle h_a^m, b_i^m \rangle = 0 \).

This observation, (1.6) and (1.10) gives
\[ |a_i| \leq C n^{-q} q^{\frac{m(n-1)}{2}}, \]
where \( r \) is defined by \( t_a \in I_r \in V_2^a \).

From (1.11) and (1.12) we get
\[ |h_a^m(t)| \leq \sum_I |a_i| \leq C n^{-q} q^{\frac{m(n-1)}{2}}. \]

**Remark 1.** The above proof works only for \( n > m+2 \). For smaller \( n \) we clearly have the desired estimate for some constants.

**Corollary 1.** If \( h_a^{m+k}(t), 0 \leq k \leq m+1 \), denotes the \( k \)-th derivative of \( h_a^m \), then
\[ |h_a^{m+k}(t)| \leq C n^{k+1+q} q^{\frac{m(n-1)}{2}}. \]

or some \( C \) and \( q, 0 < q < 1 \).

**Proof.** We use (1.11) and (1.12) and the expression for the derivative of the basic spline given in Theorem 4.16 of [18].

Let us now introduce two operators acting on functions on \( T \).
\[ Df(t) = f'(t), \]
\[ Hf(t) = \int_0^1 f(s) ds - \int_0^1 f(s) ds dt. \]

The basic relation between those two operators is
\[ \int_T Df(t) Hf(t) dt = -\int_T f(t) h(t) dt. \]

Our next goal is to prove the analog of Corollary 1 for \( H_a^m h_a^m \).

**Lemma 3.** For \( 0 \leq k \leq m+1 \) we have
\[ |H_a^m h_a^m(t)| \leq C n^{-k+q} q^{\frac{m(n-1)}{2}} \]
for some \( C > 0 \) and \( q, 0 < q < 1 \).

**Proof.** Let us start with the following claim:
\[ H_a^m h_a^m \text{ is orthogonal to } S^{m-k}(V_2). \]

Let us take \( f \in S^{m-k}(V_2) \). There exists \( \Phi \in S^m(V_2) \) such that \( D^k \Phi = f \in V_2 \). We have
\[ \int_T H_a^m h_a^m(t) f(t) dt = \int_T H_a^m h_a^m(t) f(t) dt + \int_T f(t) H_a^m h_a^m(t) dt = \int_T H_a^m h_a^m(t) D^k \Phi(t) dt = (-1)^k \int_T h_a^m(t) \Phi(t) dt = 0. \]
Since by (1.14) $H^b_h^m$ is orthogonal to all basic splines in $S^{m-k+1}(V_b^m)$, (1.5) implies that $H^b_h^m$ has a zero in the union of every $m-k+1$ consecutive intervals from $V_b^m$. Let $v_0$ be such a zero for $H^b_h^m$ almost"n" to $t_n$. Then

$$H^b_h^m(t) = \int t H^{b-1}h^m_a(s) ds.$$  

Using this particular representation we can show by induction the desired estimate.

Using functions $h^m_a$ we define the system of even splines on $T$ by

$$g^m_a(t) = \left\{ \begin{array}{ll} h^m_a(t) & \text{if } n > 0 \text{ and } 0 \leq k \leq m+1, \\ (-1)^k H^{b-1} h^m_a(t) & \text{if } n > 0 \text{ and } -m-1 \leq k \leq 0, \\ 0 & \text{if } k \text{ is even}. \end{array} \right.$$  

So $g^m_a(t)$ is indexed by $n = 0, 1, 2, \ldots$ if $k$ is even and by $n = 1, 2, 3, \ldots$ if $k$ is odd. Clearly, for every $m > -1$ and $0 < |k| < m+1$, $(g^m_a, g_{m-k}^a)$ is a biorthogonal system.

Let us put $d(t, t_n) = \min|d(t, t_n), d(t, -t_n)|$. The following omnibus theorem summarizes properties of $(g^m_a)$ for future reference.

**Theorem 1.** Let $m \geq -1$, $0 \leq |k| \leq m+1$. Then

(a) The system $(g^m_a)_{m=0}^\infty$ is an orthonormal system of even functions and it is complete in even functions in $L_2(T)$.

(b) If $k$ is even then $(g^m_{m-k} h^m_a)$ is a system of even functions complete in even functions in $L_2(T)$. If $k$ is odd then $(g^m_{m-k} h^m_a)$ is a system of odd functions complete in odd functions in $L_2(T)$.

(c) $g^m_a$ is orthogonal to $S^{m-k}(V_b^m)$.

(d) $g^m_a(t) = h^m_a(t) + (-1)^k h^m_{m-k}(t)$ where $h^m_{m-k}(t) \in S^{m-k}(V_b^m)$, $h^m_a$ is orthogonal to $S^{m-k}(V_b^m)$ and for some constants $C$ and $q$, $0 < q < 1$ we have $|h^m_a(t)| \leq C m^{1/2+b} q^{m/2} h^m_a(t)$.

(e) For some constants $C$ and $q$, $0 < q < 1$,

$$|g^m_a(t)| \leq C m^{1/2+b} q^{m/2} h^m_a(t).$$

(f) $\left( \int \sum a_n h^m_{m-k}(t) d^2 \right)^{1/2} \sim \left( \sum a_n^2 n^{2k} \right)^{1/2}$.

**Proof.** Everything except (f) follows immediately from previous considerations. The very important condition (f) is a theorem of Hopla [27]. Actually Hopla proved his theorem for systems on the interval, but his proof works in our case, too. The alternative proof can be found in [10].

**Remark 2.** Clearly, we can analogously construct a complete orthonormal system of splines $(G^m_{m-k})_{m=0}^\infty$ on the circle $T$. We can also define functions $G^m_{m-k}$ by $G^m_{m-k} = 1/V_b^m$, $G^m_{m-k} = D^m h^m_a$ if $b \geq 0$, and $G^m_{m-k} = H^{-1} G^m_a$ if $k \leq 0$. For $n = 2b+1$, $0 \leq b < 2b$, let $a_n = -1 + 2n/2b$. (Remember we identify $T$ with $[-1, 1]$.) Then for $G^m_{m-k}$ we have the following analog of Theorem 1:

**Theorem 1.** (a) The system $(G^m_{m-k})_{m=0}^\infty$, $m \geq -1$, $0 \leq |k| \leq m+1$, is a complete system in $L_2(T)$ and

$$\left( \int \sum a_n G^m_{m-k} d^2 \right)^{1/2} \sim \left( \sum a_n^2 n^{2k} \right)^{1/2}.$$

(b) For some constants $C$ and $q$, $0 < q < 1$,

$$|G^m_{m-k}(t)| \leq C m^{1/2+b} q^{m/2} h^m_a(t).$$

**Remark 3.** As far as I know the above material was never presented exactly as above. Nevertheless it is clearly known to the specialists in spline theory. Our construction is a minor variation of the one indicated in [9].

**2. Preliminaries, various $H_p$ spaces, $p \leq 1$ and their relations.** In this section we give precise definitions of various $H_p$ spaces we will be interested in and we summarize their basic properties. There are various closely related $H_p$ spaces on the circle $T$ or on the interval $[0, 1]$. They all fall in the general framework discussed in [13].

For given $p < 1$ by $a$ we will always mean the integer $[1/p - 1]$. We start with the definition of $p$-atom (more precisely, $(p,2)$-atom in the terminology of [31]).

**Definition.** A $p$-atom, $p \leq 1$, on $T$ is either the constant function 1 or a function $a(t)$ such that supp $a$ is contained in some interval $I \not= T$.
and
\[
\|f\|_p^p = \inf \left\{ \left( \sum |a_i|^p \right)^{1/p} : f = \sum a_i \varepsilon_i \right\}.
\]

The following proposition shows the relation between \(H_p(T)\) and \(H_p[0,1]\).

**Proposition 1.** Let \(f \in H_p[0,1]\), \(1 \leq p < 1/2\), and let
\[
F(t) = \begin{cases} f(t) & \text{for } t \in [0,1), \\ f(-1) & \text{for } t \in [-1,0). \end{cases}
\]
Then \(F(t) \in H_p(T)\). Conversely, if \(F(t)\) is an even function in \(H_p(T)\) then \(F\) \(\in H_p[0,1]\).

The standard proof is left to the reader.

This proposition in particular allows us to apply the results for \(H_p[0,1]\) to \(H_p(T)\) or \(H_p(D)\) as was done in [32, Remark 1]. However, it is false for \(p < 1/2\). This fact forced us to develop the system of even splines on \(T: (g_{-k})_{k=0}^{k=2p+1}\).

A \(p\)-molecule on \(T\) centered at 0 (remember \(T = [-1,1]\)) is a function \(M(t)\) such that
\[
\int_{-1}^{1} |M(t)|^p dt = 0, \quad k = 0, 1, 2, \ldots, s
\]
and
\[
\zeta(M) = \left( \int |M(t)|^p dt \right)^{1/p} \left( \int \left| \frac{d}{dt} M(t) \right|^p dt \right)^{1/2} \leq 1,
\]
where \(a = 1 - 1/p + \varepsilon, b = 1/2 + \varepsilon\) for some fixed \(\varepsilon > 1/p - 1\).

A molecule centered at \(t_0 \in T\) is a suitable translation of a molecule centered at 0. The fundamental fact is that each \(p\)-molecule is in \(H_p(T)\) and its norm \(\|M\|_p\) is uniformly bounded. All this is well known, cf. [31].

**Remark 4.** Usually the above-mentioned facts are stated for the real line (or even for \(R^n\)) instead of \(T\). The periodic case (the circle) is fully analogous to \(R\), so all the proofs can be repeated with obvious modifications. The other way to obtain these facts for the circle is to use transference. This says simply that if \(f: R \rightarrow T\) is given by \(f(t) = e^t\) then the induced map \(T_p: H_p(R) \rightarrow H_p(T), T_p(f)(t) = \sum_{k=0}^{+\infty} f(2k\pi + t)\) maps \(H_p(R)\) onto \(H_p(T)\).

The definition of a molecule stated above is not very convenient for our purposes, since the orthogonality relations (2.3) involve functions which are not continuous on the circle. The following proposition remedies this situation.

**Proposition 2.** Let \(M(t)\) be a function on \(T\). Let us fix \(p < 1\) and let \(m+1 \geq s, a, b\) such that
\[
\int_{-1}^{1} |M(t)|^p dt = \left( \int |M(t)|^p dt \right)^{1/p} \left( \int |\frac{d}{dt} M(t) + b(t)|^p dt \right)^{1/2} \leq 1,
\]
where \(a = 1 - 1/p + \varepsilon, b = 1/2 + \varepsilon\) for some fixed \(\varepsilon > 1/p - 1\).

Then \(\|M\|_p \leq C\) for some absolute constant \(C\).

**Proof.** By rotation we may assume that \(t_0 = 0\) and we may identify \(T\) with \([-1,1]\) in this way. Our goal is to find \(b \in L_p(T)\) with \(\|b\|_{L_p} \leq C\) such that \(M - b\) will be a molecule. From our assumptions on \(a, b\), and \(\varepsilon\) we see that there exist splines \(\psi_0, \psi_1, \ldots, \psi_s \in S^m(V)^n\) such that
\[
\psi_i((-1/2, 1/2)) = \alpha_i, \quad i = 0, 1, 2, \ldots, s.
\]
From (2.5) we infer that
\[
\int_{-1}^{1} M(t) b(t) dt = \int_{-1}^{1} M(t) (\psi_i(t) + b(t)) dt.
\]
This implies that there exists a function \(b(t)\) such that
\[
b(t)((-1/2, 1/2)) = 0, \quad \|b\|_{L_p} \leq C \|M((-1, -1/2) \cup (1/2, 1))\|_{L_p} \leq C
\]
and
\[
\int_{-1}^{1} (M(t) - b(t)) b(t) dt = 0 \quad \text{for } i = 0, 1, \ldots, s.
\]
This gives (2.3). In order to check (2.4) we estimate
\[
\int_{-1}^{1} |M + b|^2 \, dt \leq \frac{1}{2} \left( \int_{|t| > 1/2} |M(t)|^2 \right)^{1/2} + \frac{1}{1 - 1/2} \left( \int_{-1/2}^{1/2} |M(t)|^2 \right)^{1/2} \leq C \int_{-1}^{1} |M|^2 \, dt
\]
and
\[
\int_{-1}^{1} ((M + b)(t)) |b|^2 \, dt \leq \int_{-1}^{1} |M(t)|^2 \, dt + \frac{1}{2} \left( \int_{-1/2}^{1/2} |M(t)|^2 \right)^{1/2} \left( \int_{-1}^{1} |b|^2 \right)^{1/2} \leq C \int_{-1}^{1} |M(t)|^2 \, dt.
\]
This gives the proof of the proposition.

The most classical $H_p$ space is the space $H_p(D)$. It is the space of all analytic functions on the unit disc in the complex plane such that
\[
\|f\|_{H_p} = \sup_{r < 1} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
\]

The good general reference about those spaces is [16]. There is a close connection between $H_p(D)$ and $H_p(T)$, cf. [11], [13]. It is given as follows: It is well known that each $f \in H_p(D)$ has a radial limit in the sense of distributions: $f(\theta) = \lim_{r \to 0} f(re^{i\theta})$. Since $f$ is analytic, its real part determines the whole function up to a purely imaginary constant. We have
\[
\|f\|_{H_p(D)} = \lim_{r \to 0} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.
\]

We will also consider martingale $H_p$ spaces. We will limit our attention to dyadic martingales only. The exposition of the theory of those spaces can be found in [3], [19], [21]. By a dyadic interval on $[0, 1]$ we mean an interval of the form $[k/2^n, (k+1)/2^n)$. For function $f$ defined on $[0, 1]$ we define its dyadic maximal function by
\[
f^*(a) = \sup \left\{ \frac{1}{|I|} \int_I |f| : I \text{ is a dyadic interval and } a \in I \right\}.
\]

We say that $f \in H_p(\delta)$, the dyadic $H_p$ space if $\|f\|_p = (\int |f^*(t)|^p \, dt)^{1/p} < \infty$.

It is known (cf. [3], [19]) that a function $f \in H_p(\delta)$ can be represented as a series of Haar functions $f = \sum a_n h_n$ and
\[
\|f\|_p \sim \left( \sum a_n^p \right)^{1/p}.
\]

The symbol $(\cdot)_p$ always denote the orthonormal Haar system. Actually the above facts are true for $0 < p < \infty$. Dyadic $p$-atom, $0 < p \leq 1$, is a function $a(t)$ such that supp $a \subset I$, $I$ dyadic interval, $\int_I a(t) \, dt = 0$ and $|a(t)| \leq |I|^{-1/p}$.

An easy and natural modification of an argument given in [13], p. 611, for $p = 1$ gives

PROPOSITION 3. Every function $f \in H_p(\delta)$, $0 < p < 1$, has a decomposition $f = \sum a_n a_n$, $a_n$ are dyadic $p$-atoms and $\sum |a_n|^p < \infty$. Conversely every dyadic $p$-atom is in $H_p(\delta)$. Moreover,
\[
\|f\|_p \sim \inf \left( \sum |a_n|^p \right)^{1/p} = \sum a_n a_n, a_n \in \text{dyadic p-atoms}.
\]

The above proposition provides the "atomic definition" of $H_p(\delta)$. If we consider the dyadic partition of $T$ identified with $[-1, 1]$ we get the space of dyadic martingales on $T$, denoted by $H_p(T, \delta)$. All properties of this space are clearly identical with properties of $H_p(\delta)$.

To conclude the preliminaries let us make one comment on notation. $\|f\|_p$ will always mean the norm in the $H_p$-space which should be clear from the context. The norm in $L_p(T, \delta)$, $0 < p \leq 1$, will be denoted by $\|f\|_{L_p(T, \delta)}$, with one exception: $\|f\|_1$ will always mean $\|f\|_{L_1(T, \delta)}$.

3. Unconditional bases in $H_p$-spaces, $0 < p < 1$. This section contains our main results. We show that the spline systems on $T$ constructed in Section 1 are unconditional in suitable $H_p(T)$. This allows us to construct unconditional bases in $H_p(\delta)$ and to show that $H_p(\delta)$ and $H_p(\delta)$ are naturally isomorphic as linear topological spaces.

Our basic result is

THEOREM 2. For $m \geq 0$ and $|k| \leq m + 1$ the system $(\phi^{m,k})$ is an unconditional basis in its span in $H_p(T)$ for $1 \leq p < 1/(m + k + 1)$. If $k$ is even this span is the subspace of all even functions and if $k$ is odd the span consists of all odd functions.

We start the proof with two lemmas.

LEMMA 4. Let $a$ be a proper $p$-atom, supp $a \subset I$. Let $m, b, p$ be as in Theorem 2. Then
\[
\int_I |a(t)|^{p-2} \, dt \leq C |I|^{p-1} \int_I a(t)^p \, dt + \int_I a(t)^p \, dt.
\]
and

\[ (3.2) \quad \sup_{a} \left| \int_{I} a(t) g_{n}(t) dt \right| \leq C_{m}^{1/p - 1/2}. \]

**Proof.** Let us identify \( I \) with \([-1, 1]\) in such a way that \( I \) will become a subinterval in \([-1, 1]\) (cf. the definition of \( p \)-atom). From (3.2) we infer that there exists \( A(t) \) such that \( s a(t) + A(t) \) is a \( p \)-atom. An easy estimate yields \( \| A \|_{p} \leq C |I^{1/3}| \| a \|_{p}. \)

Using this we have

\[ \left| \int_{I} a(t) g_{n}(t) dt \right| = \left| \int_{I} D^{1/3} A(t) g_{n}^{(1/3)}(t) dt \right| = \left| \int_{I} A(t) g_{n}^{(1/3)}(t) dt \right| \leq \| A \|_{1} \left( \int |g_{n}^{(1/3)}(t)| dt \right)^{1/3} \leq C |I|^{1/3} \left( \int |g_{n}^{(1/3)}(t)|^{p} dt \right)^{1/3} \leq C |I|^{1/3} \| g_{n} \|_{p}. \]

This gives (3.1).

In order to prove (3.2) we consider two cases:

(a) \( |I| < 1/n \). Since \( s + 1/p > 0 \), \( (3.1) \) gives

\[ (3.3) \quad \left| \int_{I} a(t) g_{n}^{(1/3)}(t) dt \right| \leq C n^{1/p - 1/2}. \]

(b) \( |I| \geq 1/n \). In this case the Hölder inequality gives

\[ (3.4) \quad \left| \int_{I} a(t) g_{n}^{(1/3)}(t) dt \right| \leq \| a \|_{p} \| g_{n}^{(1/3)} \|_{p} \leq C |I|^{1/3} \| g_{n} \|_{p} \leq C n^{1/p - 1/2}. \]

If we put (3.3) and (3.4) together we get (3.2).

**Lemma 5.** Let \( m \geq 0 \) and \( |k| \leq m + 1 \) and \( 1 \geq p \geq 1/(m+k+2) \). There exist constants \( C_{1} \) and \( C_{2} \) depending only on \( m, k, p \) such that

\[ (3.5) \quad C_{1} n^{1/p - 1/2} \leq \| g_{n} \|_{p} \leq C_{2} n^{1/p - 1/2}. \]

Moreover, the right-hand side inequality holds also for \( p = 1/(m+k+2) \).

**Proof.** To show the right-hand side inequality it is enough to show it for \( g_{n}^{(1/3)} \). Using Theorem 1(c) we easily find \( \| g_{n}^{(1/3)} \|_{p} \leq C_{n} \) and

\[ \left| \int_{I} g_{n}^{(1/3)}(t) dt \right| \leq C_{n} \left( \int |g_{n}^{(1/3)}(t)|^{2} dt \right)^{1/2} \leq C_{n} \| g_{n} \|_{p}. \]

These estimates and Proposition 2 give the desired inequality.

The left-hand side inequality follows from (3.2) by duality. We have

\[ 1 = \int_{I} \| g_{n}^{(1/3)} \|_{p} \leq C n^{1/p - 1/2}. \]

This proves Lemma 5.

---

**Proof of Theorem 2.** Let \( a(t) \) be an arbitrary \( p \)-atom. Let us consider the series

\[ (3.6) \quad \sum_{n \geq 2} \pm \int a(t) g_{n}^{(1/3)}(t) dt \cdot g_{n}^{(1/3)}. \]

We have to show that this series represents the function whose \( H_{p} \) norm is bounded independently of the atom \( a(t) \) and of the choice of signs \( \pm \) or \( - \). Let \( I \) be an interval such that \( \sup a \subset I \) and (2.1) holds. Let us fix an integer \( r \) such that \( 2^{r-1} < |I| \leq 2^{r} \). Let us write (3.6) in the form

\[ (3.7) \quad \sum_{n \geq 2} \left( \sum_{n \geq 2} \pm \int a(t) g_{n}^{(1/3)}(t) dt \cdot g_{n}^{(1/3)} \right) = \sum_{n \geq 2} + \sum_{n \geq 2} \pm \int a(t) g_{n}^{(1/3)}(t) dt \cdot g_{n}^{(1/3)}. \]

Using (3.1) and (3.5) we obtain

\[ (3.8) \quad \left\| \sum_{n \geq 2} \right\|_{p} \leq C \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \leq C |I|^{1/3} \left( \int |g_{n}^{(1/3)}(t)|^{p} dt \right)^{1/3} \leq C |I|^{1/3} \| g_{n} \|_{p}. \]

Since \( s + 1/p > 0 \).

Using the Hölder inequality, Theorem 1(c), and (3.5) we have

\[ (3.9) \quad \left\| \sum_{n \geq 2} \right\|_{p} \leq C \sum_{n \geq 2} \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \leq C \sum_{n \geq 2} \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \| g_{n}^{(1/3)} \|_{p} \leq C \sum_{n \geq 2} \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \| g_{n}^{(1/3)} \|_{p} \leq C \sum_{n \geq 2} \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \| g_{n}^{(1/3)} \|_{p} \leq C \sum_{n \geq 2} \left( \int |a(t) g_{n}^{(1/3)}(t) dt|^{p} \right)^{1/p} \| g_{n}^{(1/3)} \|_{p} \leq \text{const}. \]
In order to estimate \( \| \Sigma_{l,t} \|_h \), it is enough to show that

\[
\sum_{l} = \sum_l \pm \int \alpha(t) g^n_{m,k}^{l}(t) dt \leq h^n_{m,k},
\]

has uniformly bounded norm in \( H_p(T) \). By Theorem 1(f) we have

\[
(3.10) \quad \| \sum_l \|_h \leq C |a| \leq C 2^{-|n-\lambda|/p}.
\]

Let \( \alpha \) denote the center of \( I \) and let \( W = 2I \). We write

\[
(3.11) \quad \| \sum_l \|_h \leq |W|^{|n|/2} \int \sum_l \| d(t, a(0) \|^2 dt + \int \sum_l \sum_l \| d(t, a) \|^2 dt.
\]

The first summand does not exceed, by (3.10),

\[
(3.12) \quad \|W|^{|n|/2} \sum_l \| d(t, a) \|^2 dt \leq C 2^{-|n-\lambda|/2p}.
\]

Using Theorem 1(d), (f) and the Hölder inequality we can estimate the second summand of (3.11) as follows:

\[
(3.13) \quad \int \sum_l \sum_l \| d(t, a, b) \|^2 dt \leq C |a| \int \sum_l \sum_l \| a(t) \|^2 dt \leq C |a| \int \sum_l \sum_l \| a(t) \|^2 dt.
\]

The proof of the theorem follows from (3.8), (3.9) and (3.15).

Remark 5. The situation for \( p < 1/(m+k+2) \) is as follows. If \( p < 1/(m+k+2) \) then \( g_{m,k}^{l,n} \) is not a basic sequence since the biorthogonal functionals \( g_{m,k}^{l,n} \) are not continuous on \( H_p(T) \). This follows from the result of Duren-Remlinger-Stein [1] (cf. our Proposition 7).

If \( p = 1/(m+k+2) \) the system \( g_{m,k}^{l,n} \) is a basic sequence. The case \( m = -1, k = 0, p = 1 \) was considered by Billard [2]. The proof in the general case will be given elsewhere.

It is known that \( g_{m,k}^{l,n} \) is not an unconditional basis in \( H_p(T) \) (cf. [23]) and \( g_{m,k}^{l,n} \) is not unconditional in \( H_p(T) \). This fact follows by duality from Example of Ciesielski's [22], p. 316. It seems likely that \( g_{m,k}^{l,n} \) is never unconditional in \( H_p(T) \) for \( p = 1/(m+k+2) \).

For \( g \in L_1(T) \) let \( \vec{g} \) denote the trigonometric conjugate of \( g \) since the trigonometric conjugate operator extends to an isomorphism of \( H_p(T) \) (mod constants) and maps even distributions into odd and vice versa, we obtain

**Corollary 2.** For \( m > 0, |k| \leq m+1 \) the system

\[
(g_{m,k}^{l,n})_{n=1}^{\infty}, \quad \text{if } k \text{ is even}
\]

and the system

\[
(g_{m,k}^{l,n})_{n=1}^{\infty}, \quad \text{if } k \text{ is odd}
\]

is an unconditional basis in \( H_p(T) \) for \( p > 1/(m+k+2) \).

Remark 6. A much more natural basis for \( H_p(T) \) is given by the system \( (g_{m,k}^{l,n})_{n=1}^{\infty} \) (cf. Remark 2). The same proof as the proof of Theorem 2 gives

**Theorem 4.** The system \( (g_{m,k}^{l,n})_{n=1}^{\infty} \) is an unconditional basis in \( H_p(T) \) for \( p > 1/(m+k+2) \).

Now we are in a position to produce an unconditional basis in \( H_p(T) \). For every real function \( f \in L_1(D) \) we define an analytic function on \( D \) (its Cauchy integral) \( C(f) \) by

\[
|C(f)| = C(f(e^t)) = f(t) + \frac{j}{t}, \quad j = \text{constant}.
\]

From Corollary 2 and (2.9) we infer
Theorem 3. For \( m \geq 0, |k| \leq m+1 \), the system \((G_n^{m,k})_{n=0}^\infty\) is an unconditional basis in a complex space \(H_p(D)\) for \( 1 \leq p < 1/(m+k+2) \).

The standard application of the Khintchine inequality and Theorem 3 shows that the following square function type characterization of \(H_p(D)\):

\[
H_p(D) = \{ f \in L_p(D) : \|f\|_{H_p(D)} < \infty \}
\]

Theorem 4. Let \( 1 \leq p < 1/(m+k+2) \), \( m \geq 0, |k| \leq m+1 \). The function \( f(z) = \sum_{n=0}^\infty a_n G_n^{m,k}(z) \) belongs to \(H_p(D)\) if and only if

\[
\left( \frac{1}{\pi} \int \left( \sum_{n=0}^\infty |a_n|^2 |G_n^{m,k}(e^{it})|^2 \frac{dt}{|\theta|^2} \right)^{2p'} dt \right)^{1/2p'} < \infty.
\]

The next theorem establishes a linear topological isomorphism between \(H_p(D)\) and \(H_\infty(\mathbb{D})\).

Theorem 5. Let \( T^{m,k} : H_p(D) \to H_\infty(\mathbb{D}) \) be defined by

\[
T^{m,k} G_n^{m,k} = n^k \chi_n.
\]

The operator \( T^{m,k} \) establishes an isomorphism between \(H_p(D)\) and \(H_\infty(\mathbb{D})\) for \( 1 \leq p < 1/(m+k+2) \).

Proof. We start with the proof that \( T^{m,k} \) is bounded. It is enough to check that \( \|T^{m,k} G_n^{m,k}\|_p \) is uniformly bounded for all atoms \( a \). This reduces to the estimate

\[
\| \sum_n \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \|_p \leq \text{const}.
\]

As in (3.7) we split into three sums \( \sum_1, \sum_2 \) and \( \sum_3 \). Since

\[
\|G_n^{m,k}\|_p \sim n^k \|x_n\|_p,
\]

estimate (3.8) shows that \( \|\sum_1\|_p \leq \text{const} \) and estimate (3.9) shows that \( \|\sum_3\|_p \leq \text{const} \).

The properties of dyadic intervals give that

\[
\left| \text{supp } \sum_2 \right| \leq C 2^{-r}
\]

(we mean \( \text{supp } a \subset I \) with \( 2^{-r-1} < |I| < 2^{-r} \)).

Using (2.10), the Hölder inequality, estimate (3.17) and Theorem 1 (i) together with (3.16) we obtain

\[
\| \sum_1 \|_p \leq C \left( \sum_{n \geq 1} \left| \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \right|^2 \right)^{1/2p'} \cdot \|a\|_{p'} \cdot \|x_n\|_p
\]

\[
\leq C \left( \sum_{n \geq 1} \left| \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \right|^2 \right)^{1/2p'} \cdot \|a\|_{p'} \cdot \|x_n\|_p \leq \text{const}.
\]

So \( T^{m,k} \) is continuous.

In order to show that \( (T^{m,k})^{-1} \) is continuous we use Proposition 3. Let us take a real dyadic atom \( a(t) \), supported on a dyadic interval \( I \). We have

\[
a(s) = \sum_{n \supseteq I} \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \chi_n(s).
\]

So

\[
(T^{m,k})^{-1}(a) = \sum_{n \supseteq I} \left( \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \right) n^k \chi_n.
\]

By (2.9)

\[
\| (T^{m,k})^{-1}(a) \|_{H_\infty(D)} \sim \sum_{n \supseteq I} \left( \int a(t) G_n^{m,k}(t) \frac{dt}{n^k} \right) n^k \chi_n \|_{p'}.
\]

This last norm is estimated exactly as \( \| \sum \|_p \) is estimated in the proof of Theorem 2. This completes the proof.

The fact that \( H_\infty(\mathbb{D}) \) and \( H_p(D) \) are isomorphic was discovered by Maurey in [24] but his proof was not constructive. The constructive proof was given by Carlsson [4] and the author in [32]. The case \( k = 0 \) of Theorem 5 with \( H_\infty(0,1) \) instead of \( H_p(D) \) follows from [30] (cf. also our Theorem 19).

We also have

Theorem 5'. Let \( S^{m,k} : H_p(T) \to H_\infty(\mathbb{D}) \) be defined by

\[
S^{m,k} G_n^{m,k} = n^k \chi_n.
\]

The operator \( S^{m,k} \) establishes an isomorphism between \(H_p(T)\) and \(H_\infty(\mathbb{D})\) for \( 1 \leq p < 1/(m+k+2) \).

The proof is the same as the proof of Theorem 5.
4. Bergman spaces and complemented subspaces of $H_p$, $0 < p < 1$.

This section is more functional analytic in spirit than the rest of the paper. We show how the existence of unconditional bases in $H_p(D)$ formally gives unconditional bases in some Bergman spaces. These bases provide natural isomorphisms of Bergman spaces and $L_p$-spaces. Later we apply this to the proof that every complemented subspace of $H_p$, $0 < p < 1$, contains $f_p$ complemented.

Let us start with some definitions. A $p$-norm on a linear space $X$ is a function $\|\| : X \to \mathbb{R}^+$ such that

$$\|x\| > 0 \text{ for } x \neq 0,$$

$$\|ax\| = |a| \|x\|,$$

$$\|x + y\| \leq \|x\|^p + \|y\|^p.$$

A $p$-Banach space is a linear space $X$ equipped with the $p$-norm $\|\|$ and complete with respect to the metric $\bar{d}(x, y) = \|x - y\|^p$. A $p$-Banach space $X$ has the property that for every bounded sequence $\{x_n\} \subset X$, $\|x_n\| \leq 1$, and for every sequence of scalars $(a_n)$ with $\sum |a_n|^p < \infty$, $\sum a_n x_n \in X$ and $\|\sum a_n x_n\| \leq 1$.

If $X$ is a $p$-Banach space and $Y$ is a $q$-Banach space and $T : X \to Y$ is linear then $\|T\| = \sup \{\|Tx\| : \|x\| \leq 1\}$. $T$ is continuous if and only if $\|T\| < \infty$. Let now $X$ be a $p$-Banach space, $p < 1$ and let $q < q_1$ be given. The $q$-envelope of $X$, denoted $q - X$, is the completion of $X$ with respect to the $q$-quasinorm

$$\|x\|_q = \inf \left\{ \left( \sum |a_n|^q \right)^{1/q} : x = \sum a_n x_n, x_n \in X, \|a_n\|_q \leq 1 \right\}.$$

The proof of the following standard proposition is omitted:

**Proposition 4.** Let $X$ be a $p$-Banach space and $Y$ be a $q$-Banach space, $p < q_1$, and let $T : X \to Y$ be a continuous linear operator. Then there exists a unique extension $T : q - X \to Y$ with $\|T\| = \|T\|_q$.

Now we introduce the weighted Bergman spaces on the unit disc $D$. The space $A^p_q$, $0 < p < \infty$, $-1 < a < \infty$ consists of all functions analytic in $D$ such that

$$\|f\|_{A^p_q} = \left( \int_D |f(x)|^p (1 - |x|^2)^{aq} \, dx \right)^{1/p} < \infty.$$

The following Theorem 6 is essentially known. The case $q = 1$ was proved in [29] and [17]. We give the sketch of the proof for the sake of completeness.

**Theorem 6.** Let $0 < p < q_1$. Then $q - H_p(D) = A^p_{q-1}$ and the norms are equivalent.

We start with the classical lemma due to Hardy–Littlewood [20] (cf. [16], p. 111).

**Lemma 6.** If $0 < p < q < \infty$, $a \geq p$ and $a = 1/p - 1/q$ then for $f \in H_p(D)$ we have

$$\int_D (1 - r)^{a-1} \left( \int_D |f(re^{i\theta})|^q \, d\theta \right)^{1/q} \, dr \leq C \|f\|_p.$$

In particular, $(\lambda - g)$ the identity is a continuous map from $H_p(D)$ into $A^{2\lambda-2}_q$ for $0 < p < q < \infty$.

**Proof of Theorem 6.** In view of Lemma 6 it is enough to represent every $f(x) \in A^{2\lambda-2}_q$, $\|f\|_{A^{2\lambda-2}_q} = 1$ as a sum $f(x) = \sum a_n f_n(x)$ with $a_n \leq \text{const}$ and $\|f_n\|_p \leq \text{const}$. This is done by approximating the $L_p$-representation formula. It can be done by hand (cf. [29], Th. 2) we can use Theorem 2 of [12], which in our special case gives

$$f(x) = \sum a_n \left( \frac{1}{1 - |x|^2} \right)^{1/p} \left( \frac{1}{1 - |e^{i\theta}|^2} \right)^{1/q},$$

where $\zeta \in D$ and $\sum |a_n| < \text{const}$. The $H_p(D)$ norms of functions appearing in (4.1) are uniformly bounded (cf. [16], p. 66).

This completes the proof.

Now let $f(x)$ be an arbitrary unconditional basis for $H_p(D)$ and let $p < q < 1$. Using Proposition 4, Theorem 6 and Lemma 6 we see that there exists a constant $C$ such that for every sequence of scalars $(a_n)$ and every sequence of signs $(a_n)$ we have

$$\left\| \sum a_n f_n \right\|_{A^{2\lambda-2}_q} \leq C \left\| \sum a_n f_n \right\|_{A^{2\lambda-2}_q}.$$

In other words $f_n$ is an unconditional basis in $A^{2\lambda-2}_q$. So using Theorem 3 we obtain

**Theorem 7.** The system $(C_{\lambda} f_n)_{n=1}^\infty$, $m \geq 0$, $|n| \leq m+1$, is a unconditional basis in $A^\infty_{q-1}$ for

$$q > 1/(m+h+2)$$

and

$$-1 < a < q(m+h+2) - 2.$$

Our goal now is to characterize $A^p_q$ in terms of coefficients with respect to the system $(C_{\lambda} f_n)_{n=1}^\infty$. We start with

**Proposition 5.** Let $f = \sum a_n C_{\lambda} f_n$ be $H_p$, $p > 1/(m+h+2)$, and let $q > p$. Then for some constant $C = C(m, h, p, q)$,

$$\left( \sum_n |a_n|^q (1 - |e^{i\theta}|^2)^{aq} \right)^{1/q} \leq C \|f\|_p.$$
Proof. As usual, it is enough to show that for every \( p \)-atom on \( T \) we have

\[
\left( \sum_{n} \left| \int a(t) g_{n}^{m-k}(t) dt \right|^{p} n^{(\frac{1}{2} + \frac{k}{p} - 1)q} \right)^{\frac{1}{p}} \leq \text{const.}
\]

We proceed analogously as in the proof of Theorem 2. Let \( \|a\|_{q} \leq 1 \), \( \|a\|_{q} \leq 1 \), and \( 2^{-m} < \|a\|_{q} \leq 2^{\frac{1}{2}} \). We write

\[
\sum_{n} \left| \int a(t) g_{n}^{m-k}(t) dt \right|^{p} n^{(\frac{1}{2} + \frac{k}{p} - 1)q} = \sum_{n} \sum_{\sum t_{n} \neq \gamma_{n} \neq \delta_{n}} \int a(t) g_{n}^{m-k}(t) dt \left( n^{\frac{1}{2} + \frac{k}{p} - 1} \right)^{\frac{1}{q}}
\]

Estimates (3.8) and (3.9) give that \( \sum_{1} \) and \( \sum_{2} \) are finite if \( q \) is replaced by \( p \). Since we have \( p < q \), we have the desired inequality for \( \sum_{1} \) and \( \sum_{2} \).

To estimate \( \sum_{3} \) we use Theorem 1 (f) and the Hölder inequality to obtain

\[
\sum_{3} \leq \left( \sum_{n} \left( \int a(t) g_{n}^{m-k}(t) dt \right)^{p} n^{2q} \right)^{\frac{1}{p}} \left( \sum_{n} n^{\frac{1}{2} + \frac{k}{p} - 1} \right)^{\frac{1}{q}}
\]

\[
\leq C \|a\|_{q} \sum_{n} \int a(t) g_{n}^{m-k}(t) dt \left( n^{\frac{1}{2} + \frac{k}{p} - 1} \right)^{\frac{1}{q}}
\]

\[
= C \|a\|_{q} \sum_{n} \int a(t) g_{n}^{m-k}(t) dt \left( n^{\frac{1}{2} + \frac{k}{p} - 1} \right)^{\frac{1}{q}}
\]

The last inequality uses (3.1) and the fact that \( g > p \).

Lemma 5 and (3.9) give \( \|C_{n} g_{n,k}\|_{q} \sim n^{\frac{1}{2} + \frac{k}{p} - 1} \). This observation and Proposition 5 imply that \( q - H_{p}(D) \) for \( 1 > q > p \) consists of all sums

\[
\sum_{n=0}^{\infty} a_{n} C_{n} g_{n,k} \text{ such that }
\sum_{n=0}^{\infty} |a_{n}|^{2} n^{\frac{1}{2} + \frac{k}{p} - 1} q < \infty,
\]

where \( p, m \) and \( k \) are related as in Theorem 2. This fact and Theorem 6 yield the following characterisation of \( A_{r}^{q} \).

Theorem 8. Let \( 1/(m+k+2) < q < 1, \quad -1 < a < q(m+k+2)-2 \).

The function \( f(z) \) is in \( A_{r}^{q} \) if and only if

\[
f(z) = \sum_{n=0}^{\infty} a_{n} C_{n} g_{n,k}(z)
\]

and

\[
\sum_{n=0}^{\infty} |a_{n}|^{2} n^{\frac{1}{2} + \frac{k}{p} - 1} q < \infty.
\]

This theorem can be compared (in the special case of the unit disc) with Theorem 2 of [12]. Our decomposition has the advantage over decomposition from [12] in being unique. On the other hand the functions \( C_{n} g_{n,k} \) we use are much less natural than the functions associated with the Bergman kernel as used in [12] (cf. (4.1)).

In the language of functional analysis Theorem 8 means that the basis \( n^{(\frac{1}{2} + \frac{k}{p} - 1)} \) \( C_{n} g_{n,k} \) for \( A_{r}^{q} \), with \( p, q, m, k \), and as in Theorem 8, is equivalent to the unit vector basis in \( \ell_{q} \). In particular we get

Corollary 3. The space \( A_{r}^{q} \), \( 0 < q \leq 1, \quad -1 < a < \infty \), is isomorphic to \( \ell_{q} \).

This corollary is known (cf. [22], Th. 2.4 and 3.3) but as far as we know Theorem 8 gives the first explicit construction of a basis in \( A_{r}^{q} \).

Remark 7. Since the dual of \( H_{p}(D), \quad p < 1 \), is clearly the same as the dual of \( A_{r}^{q} \), (cf. [29]) Theorem 8 allows us to give the formal description of this dual.

Proposition 6. The dual of \( H_{p}(D), \quad p < 1 \), can be identified with all infinite series \( \sum_{n=0}^{\infty} a_{n} C_{n} g_{n,k} \), \( p > 1/(m+k+2) \) such that \( \sum_{n=0}^{\infty} |a_{n}|^{2} n^{(-\frac{1}{2} + \frac{k}{p} - 1)} \)

\[
< \infty.
\]

On the other hand, the dual of \( H_{p}(D), \quad p < 1 \), has been described in [17] (cf. also [29] and [12]).

A function \( f \in C(T) \) belongs to \( A_{r} \), \( 0 < a < 1 \), if

\[
f(t_{1}) - f(t_{2}) \leq C d(t_{1}, t_{2})^{\gamma}, \quad t_{1}, t_{2} \in T
\]

and is said to belong to \( A_{r} \), if

\[
|f(t+h) - f(t)| < C |h|
\]

for all \( h \) and \( t \in T \).

The result of Duren–Romeberg–Shields [17] is as follows

Proposition 7. The dual of \( H_{p}(D), \quad p < 1 \), can be identified with the space of all functions \( f \) continuous in \( D \) and analytic \( \in D \) such that if

\[
1/2 - 1/(m+k+2), \quad h \in \mathbb{R}
\]

\[
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for all \( h \) and \( t \in T \).
The proof of Theorem 9 reduces to the following two propositions:

**Proposition 8.** Let \( \varphi_r \) satisfy (4.2), (4.3) and (4.4). Then a certain subsequence of \( \varphi_r \)'s is equivalent in \( H_p(D) \) to a unit vector basis in \( \ell_p \).

**Proposition 9.** Let \( \psi_r = \sum_{j=1}^{k_r} \beta_j C_{\psi_r}^p \) be such that

\[
\frac{1}{(n+1)(n+2)} < p < 1/n \quad (n = 1, 2, 3, \ldots) \text{ then } \quad \mathbf{D}^{(n)} f(\psi_r^p) \in A_{\psi_r}.
\]

if \( p = 1/(n+1) \quad (n = 1, 2, 3, \ldots) \text{ then } \quad \mathbf{D} f(\psi_r^p) \in A_{\psi_r}.
\]

If we put together Propositions 6 and 7 and take into account that the pairing is the same in both Propositions we get a constructive characterization of smooth analytic functions in terms of coefficients of expansions with respect to systems \( C_p^{(n)} \). We also see that the dual of \( H_p(D) \), \( p < 1 \), i.e., the space of smooth analytic functions is isomorphic to \( L_p \). The non-analytic version of those results has been obtained by Cieliecki (cf. [7]).

Now we intend to apply our results to the investigation of complemented subspaces of \( H_p(D) \). Our main result in this direction is the following.

**Theorem 9.** Let \( X \) be an infinite dimensional complemented subspace of \( H_p(D) \), \( 0 < p < 1 \). Then \( X \) contains a complemented subspace isomorphic to \( \ell_p \).

This theorem answers the question asked in [22]. Let us remark that for \( \infty > p > 1 \) the analogous statement is not true.

**Lemma 7.** If a p-Banach space \( X = X_1 \oplus X_2 \) is a direct sum of its subspaces \( X_1 \) and \( X_2 \) and if \( p < q < 1 \) then

\[
g - X = (g - X_1) + (g - X_2).
\]

The standard proof of this lemma is omitted.

**Proof of Theorem 9.** Let \( X \) be an infinite dimensional complemented subspace of \( H_p(D) \), i.e., \( H_p(D) = X + Y \). By Lemma 7 and Theorem 6

\[
1 - H_p(D) = A_{\psi_r} - (1 - X) + (1 - Y).
\]

Since \( X \) is infinite dimensional, we infer that \( 1 - X \) is an infinite dimensional Banach space. Moreover, by the definition of \( \mathbf{D} \)-envelopes of \( X \) \( \|x\|_p = 1 \) is a non-compact subset of \( 1 - X \). Let us take \( s \) such that \( p > 1/(n+2) \) and let us consider \( C_{\psi_r}^{(n)} \), the unconditional basis in \( H_p(D) \) and simultaneously in \( A_{\psi_r} \). By a standard perturbation argument we may assume that there are functions \( \varphi_r \in X \) such that

\[
\varphi_r = \sum_{j=1}^{k_r} a_j C_{\psi_r}^p,
\]

where every \( k_r = 2^{(r)} \) and \( s(r) \) is a strictly increasing sequence of integers

\[
\|\varphi_r\|_p = 1,
\]

\[
\|\varphi_r\|_{1,\ell_{p-2}} \geq C > 0 \quad \text{ for } r = 1, 2, \ldots
\]

In order to show that \( P \) is continuous it is enough to show that for every \( f \in H_p(D) \)

\[
\sum_{r=1}^{k_r} \left| \left| \mathbf{D} f(\psi_r) \right|_p \right|^p < C \|f\|_p.
\]

But for \( f \) such that \( f(0) = 0 \) we have

\[
\mathbf{D} f(\psi_r)(t) \|_p = 2 \int \mathbf{D} f(\psi_r)(t) \|_p < C \|f\|_p.
\]

so \( P \) is continuous by (2.9) and (4.7).

**Proof of Proposition 8.** We will use some facts about uniform integrability of functions. The facts we need are summarised in the following

**Lemma 8.** Let \( 1 \leq a < 2, C > 1 \) and let \( \mu \) be a probability measure. Let \( f_1, f_2, \ldots \) in \( L_a(\mu) \) be such that \( \int |f_n|^a d\mu = 1, n = 1, 2, \ldots \), and for every sequence of scalars \( c_1, c_2, \ldots \) we have

\[
C \left( \int \sum |c_n f_n^a| d\mu \right)^{1/a} \geq \left( \sum |c_n|^a \right)^{1/a}.
\]
Then there exists a constant $\gamma$ and a sequence of disjoint sets $A_1, A_2, \ldots$ such that for some subsequence $(f_{n_k})$ we have

$$\left( \int_{A_k} |f_{n_k}|^\alpha \, dx \right)^\beta \geq \gamma.$$  

The proof of this lemma can be found in [18]; the proof of the condition (d) in the proof of Theorem 3.1.

Let now $p$ denote any of the $p_i$'s. Let us factor $v_i(x) = v(x) \cdot h(x)$ in such a way that $g \in H_i(D)$, $1 < a < 2$ and $h \in H_i(D)$, $0 < \beta < 2$, $1/\alpha - 1/b = 1/p$ and on the unit circle $T$ we have $|g| = |g|^\alpha$ and $|h| = |h|^\beta$.

We have by the Hölder inequality

$$\left( \int_{A_k} |f_{n_k}|^\alpha \, dx \right)^\beta \leq \left( \int_{A_k} |v_{n_k}(x)|^\alpha \, dx \right)^\beta \leq \left( \int_{A_k} |h(x)|^\beta \, dx \right)^\beta.$$  

We use the Hölder inequality

$$\left( \int_{A_k} |f_{n_k}|^\alpha \, dx \right)^\beta \leq C \left( \int_{A_k} |v(x)|^\alpha \, dx \right)^\beta \leq C \left( \int_{A_k} |h(x)|^\beta \, dx \right)^\beta.$$  

Lemma 6 applied for $q = 2$, $p = \beta$, $\lambda = 2$ gives

$$\left( \int_{A_k} |f_{n_k}|^\alpha \, dx \right)^\beta \leq C \|h\|_\beta = C \|v\|_\alpha = C.$$  

From (4.10) and (4.11) we get

$$\int_{A_k} |f_{n_k}|^\alpha \, dx \geq C.$$

If we write $g(x) = \sum_{n=0}^\infty b_n x^n$ the direct evaluation of the integral in (4.12) gives

$$\left( \int_{A_k} |f_{n_k}|^\alpha \, dx \right)^\beta \geq \gamma.$$  

Application of Hölder's inequality and the Hausdorff–Young theorem in (4.13) gives $(1/\alpha + 1/\mu' = 1)$

$$C \left( \sum_{n=0}^\infty |b_n|^\beta \right)^{1/\beta} \left( \sum_{n=0}^\infty |b_n|^\alpha \right)^{1/\alpha} \leq C \left( \sum_{n=0}^\infty |b_n|^\beta \right)^{1/\beta} \left( \sum_{n=0}^\infty |b_n|^\alpha \right)^{1/\alpha}.$$  

Let us now consider the operator $T: H_\alpha(D) \to L_\beta$ given by

$$T(a) = \sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty a_n x^n + \sum_{n=0}^\infty b_n x^n.$$

By [20], Theorem 6.2, it is a continuous linear operator. If $v_i = v_i - \delta$ we have by (4.14) $\|T v_i\|_\beta \geq C$. Since $v_i(x)$ converges pointwise to zero in $D$, the same is true about $v_i(x)$. This implies that $T(v_i)$ converges to zero coordinatewise in $L_\beta$. Using the standard shrinking argument (cf. [20]) we infer that some subsequence of $T(v_i)$ is equivalent to the unit vector basis in $L_\beta$. This means that the subsequence of $v_i$ satisfies the assumptions of Lemma 8. So for further subsequence we have a sequence of disjoint $A_k \subset T$ such that

$$\int_{A_k} |f_{n_k}|^\alpha \, dx \geq C.$$  

We have

$$\int \sum_{n=0}^\infty |a_n v_i| \left| v_i \right| \, dx \leq \sum_{n=0}^\infty |a_n v_i| \left| v_i \right| \leq \sum_{n=0}^\infty |a_n|.$$  

To prove the other inequality we use the fact that $v_i$ is an unconditional basic sequence, so by the Khintchine inequality and (4.15) we have

$$\int \sum_{n=0}^\infty |a_n v_i| \left| v_i \right| \, dx \geq C \int \sum_{n=0}^\infty |a_n| \left| v_i \right| \, dx \geq C \int \sum_{n=0}^\infty |a_n| \left| v_i \right| \, dx \geq C \int \sum_{n=0}^\infty |a_n| \left| v_i \right| \, dx.$$  

Inequalities (4.16) and (4.17) complete the proof of Proposition 8.

Proof of Proposition 9. It is enough to show (4.7) with $\psi(t)$ replaced by $\Re \psi(t)$. Let us denote

$$\Re \psi(t) = \eta(t) = \sum_{n=0}^\infty \beta_n g_n(t).$$
5. Unconditionality in $L_p$, $1 < p < \infty$. In this section we will concentrate our attention on $T$ and $[0,1]$ and $p \geq 1$. Our main tool will be dyadic $H_1$ and interpolation theorems. On $T$ we will work with systems $(G_{n,k}^{m,k})_{n,k}$ and on $[0,1]$ we will work with Ciesielski’s systems $(f_{n,0}^m)_{n \in \mathbb{N}}$.

For the definition and detailed investigation of these systems we refer to [7]. Let us only remark that $(f_{n,0}^m) = (f_n^m)$ is a system of orthonormal splines on $[0,1]$, i.e., for $-m \leq n < 0$ $(f_n^m)$ is the orthonormalisation of the monomials $1, t, \ldots, t^n$ and next we have an orthonormal system of splines of order $m$ corresponding to a natural dyadic partition of $[0,1]$. $f_n^m$ denotes the $n$th derivative of $f_n^m$ if $k \geq 0$ and $(-k)^{th}$ antiderivative if $k < 0$.

The difference between $(G_{n,k}^{m,k})$ considered on $[-1,1]$ and Ciesielski’s systems lies in different behaviour at the endpoints. This brings in certain asymmetry.

The interpolation theorem we shall be using is the following special case of Theorem D of [33].

**Proposition 10.** Let $T$ be a continuous linear operator from $H_1$ into $L_1$ (in particular, from $H_1$ into $H_1$; any $H_1$-space we are considering works) and from $L_1$ into $L_2$. Then $T$ is a continuous linear operator from $L_2$ into $L_2$, for $1 < p < 2$.

A look at Theorem 2 gives that $(G_{n,k}^{m,k})_{n,k}$ is an unconditional basis in $H_1(T)$ for $m > 0$ and $k = -m - 1$. Since $(G_{n,k}^{m,k})_{n,k}$ is also an unconditional basis in $L_1(T)$, Proposition 10 gives that the system $(G_{n,k}^{m,k})_{n,k}$, $m > 0$, $|k| \leq m + 1$, $k \neq -m - 1$, is an unconditional basis in $L_2(T)$, $1 < p < 2$.

In this range of $m$ and $k$ we can interpolate operators $S_n^m$ (cf. Theorem 3 and Theorem 1 (a)). The results are summarised in

**Proposition 11.** The system $(G_{n,k}^{m,k})_{n,k}$, $m > 0$, $|k| \leq m + 1$, $k \neq -m - 1$ is an unconditional basis in $L_p$, $1 < p < 2$, equivalent to $n^q X_a$.

By duality we get

**Proposition 12.** The system $(G_{n,k}^{m,k})_{n,k}$, $m > 0$, $|k| \leq m + 1$, $k \neq -m - 1$ is an unconditional basis in $L_p$, $1 < p < \infty$, equivalent to $n^q X_a$.

The next theorem allows us to consider also the exceptional cases $k = -m - 1$ and $k = m + 1$.

**Theorem 10.** For every sequence $e = (e_m^0)_{m \in \mathbb{N}}$ we define an operator $T^+_{e^0}$: $H_1(T) \rightarrow L_2(T)$ by

$$T^+_{e^0}(\sum_{m=0}^n a_m G_m^h) = \sum_{m=0}^n a_m e_m G_m^h.$$
Proof. Once more the proof is patterned after the proof of Theorem 2. Let \( a \) denote the dyadic 1-atom and let \( I \) be a dyadic interval such that \( \sup \{ a \} \subset I \), \( |a| \leq |I|^{-1/4} \). Let us denote \( |I| = 2^{-n} \). Clearly, we have to show

\[
\sup_{\varepsilon > 0} \sup_{I} (\mathcal{T}^{\varepsilon,k}(a)|_{\varepsilon I}) \leq \text{const.}
\]

Let us write

\[
\mathcal{T}^{\varepsilon,k}(a) = \sum_{n \in \mathbb{Z}^d} + \sum_{n \in 2^n I} + \sum_{n \in \mathbb{Z}^d} \int a(t)G_n^{\varepsilon,k}(t)dtG_n^{\varepsilon,k}
\]

\[
= \sum_1 + \sum_2 + \sum_3.
\]

To estimate \( \sum_3 \), we observe that because \( I \) is dyadic \( G_n^{\varepsilon,k}|I \) is a polynomial, so if \( \mathcal{A} \) is such that \( \sup \mathcal{A} \subset I \) and \( \mathcal{A}' = a \omega \) we have

\[
\left| \int a(t)G_n^{\varepsilon,k}(t)dt \right| = \left| \int \mathcal{A}(t)G_n^{\varepsilon,k}(t)dt \right| 
\leq |A| \left( \int |G_n^{\varepsilon,k}(t)|^4 dt \right)^{1/4} \leq C 2^{-N/2} n^{-k/2} G^{2}(I\mathcal{A}).
\]

Using this and the estimate \( |G_n^{\varepsilon,k}|_{L^4(I\mathcal{A})} \leq k^{-1/2} \) we get that \( |\sum_1|_{L^1(I)} \leq \text{const.} \)

The sum \( \sum_2 \) is estimated exactly like (3.9). To estimate \( |\sum_2|_p \), we may observe that dyadic 1-atom is 1-atom, so like in the proof of Theorem 2 we can show that \( \sum_2 \) is a 1-molecule, so its norm in \( L^r(I) \) is uniformly bounded. We can also give a direct estimate as follows:

\[
\int |\sum_2|_p \int a(t)G_n^{\varepsilon,k}(t)dtG_n^{\varepsilon,k}(s)ds 
\leq \int \left( \sum_{n \in \mathbb{Z}^d} |a(t)G_n^{\varepsilon,k}(t)|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}^d} |G_n^{\varepsilon,k}(s)|^{1/2} \right)^{1/2} ds 
\leq C |a||s| \int \left( \sum_{n \in \mathbb{Z}^d} n^{-2d|\omega|} |\omega|^2 n^{-k} \right)^{1/2} ds 
\leq C |a||s| \int \left( \sum_{n \in \mathbb{Z}^d} 2^{-n}|\omega|^2 n^{-2d|\omega|} \right)^{1/2} ds
\]

Also,

\[
\int |\sum_1|_p 
\int a(t)G_n^{\varepsilon,k}(t)dtG_n^{\varepsilon,k}(s)ds 
\leq C |a||s| \int 2^{-n/2} \int \sum_{n \in \mathbb{Z}^d} |\omega|^2 n^{-2d|\omega|} \leq \text{const.}
\]

This completes the proof of the theorem.

We can summarise our consideration as follows:

**Theorem 11.** The system \( G_n^{\varepsilon,k} \), \( m \geq -1 \), \( |\omega| \leq m \), is an unconditional basis in \( L^p(I) \), \( 1 < p < \infty \). If \( |\omega| < m \) then this basis is equivalent to \( n^{-k} \).

The above theorem is a periodic analog of results of Ciesielski [8]. Unfortunately our method does not give the case \( |\omega| = m + 1 \) in the equivalence result.

**Remark 8.** Despite Theorem 10 the system \( G_n^{\varepsilon,k} \) need not be a basis for \( H_f(I, \varepsilon) \). The trouble is that the norm of \( G_n^{\varepsilon,k} \) in \( H_f(I, \varepsilon) \) can be substantially bigger than \( n^{-k/2} \).

Now we will briefly describe the situation for Ciesielski’s systems. The argument fully analogous to the proofs of Theorem 2 and Theorem 5 gives

**Theorem 12.** The system \( f_n^{\omega,k} \), \( n \geq -1 \), \( m \leq k \), \( n \geq -1 \), \( -m \leq k \leq 0 \), is an unconditional basis in \( H_f(I, \varepsilon) \), \( \varepsilon > 0 \), for \( p > 1/(m+k+2) \). Moreover, the basis \( f_n^{\omega,k} \), \( n \geq -1 \), \( m \leq k \leq 0 \), is equivalent to the basis \( n^{-k/2} \) in \( H_f(I, \varepsilon) \) for \( p > 1/(m+k+2) \).

For \( k = 0 \) this theorem was established in a different way in [30].

The trouble with the derivative (i.e., the case \( k > 0 \)) is that its norm in \( H_f(I, \varepsilon) \) may be bigger than it should be. To indicate this we will prove

**Proposition 13.** Let \( f_n \) denote \( f_n^{\varepsilon,k} \). The norm of \( f_n \) in \( H_f(I, \varepsilon) \) is greater than or equal to \( C \cdot n^{-2d/3} \).

**Proof.** The system \( f_n \) is the classical Franklin system investigated
in detail in [5] and [6]. Let us define
\[ \varphi(t) = \begin{cases} n & \text{if } 0 \leq t < 2^{-n}, \\ -\log_2 t & \text{if } 2^{-n} \leq t \leq 1. \end{cases} \]

It is an easy and well known exercise that \( \| \varphi \|_{\text{MO}} \leq \text{const} \) (cf. [25], Ex. 2.4). On the other hand,
\[ \int_0^1 f_n(t) \varphi(t) dt = f_n(t) \varphi_n(t) - \int_0^1 f_n(t) \varphi'(t) dt. \]

Using Lemma 3 of [5] and exponential inequalities for Franklin functions we infer that the first summand is of the order of magnitude \( n^{-\alpha} \). The second summand is estimated as
\[ \int_0^1 f_n(t) \varphi'(t) dt \lesssim \left( \int_0^1 f_n(t) dt \cdot \sup |\varphi'(t)| \right) \lesssim n^{-\alpha}. \]

Since \( H_p(0,1] = \text{BMO} \), these inequalities prove the proposition.

Nevertheless we can repeat the proof of Theorem 10 to get

**Theorem 13.** The operator \( T_{n}\ast \), \( s = (a_k), n = \pm 1, n \geq -1, |k| \leq n+1, \) defined by
\[ T_{n}\ast (f_{n}) = a_n b_{n-k}, \]

is a continuous map from \( H_p(0,1] \) into \( L_1 \), and \( \sup \| T_{n}\ast \| < \infty. \)

This theorem and interpolation give

**Corollary 4.** The system \( (f_{n})_{n=1}^{\infty} \) is an unconditional basis in \( H_p(0,1] \), \( 1 < p < \infty \).

**Remark 9.** Theorems 10 and 13 can be extended to \( p < 1 \) also. The extension is obvious. It was not given because we do not see interesting applications.

**Remark 10.** Like in Remark 7 and Propositions 6 and 7 there is a connection between smooth functions on \( [0,1] \) and \( H_p(0,1] \). We can also get the characteristics of smooth functions in terms of coefficients with respect to the systems \( (f_{n})_{n=1}^{\infty} \). Those results have been obtained in [7].

The \( p \)-envelopes of \( H_p(0,1] \), \( p < q < \infty \), have been described by Aleksandrov [1].
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