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A remark on approximate solving of a class of initial-boundary value problems

by

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Abstract. The paper is a continuation of the author's earlier work [2]. It deals with a class of initial-boundary value problems for the equation $Au+u_t=f$, where A is a linear elliptic operator of order $2m$ in space variables with time-depending coefficients. For approximate solving these problems we propose a finite element method based on a suitable family of triangulations of the space-time domain. An estimate of the error is given.

Let Ω be a polyeder in R^n. We are dealing in this paper with approximate solutions of the initial-boundary value problem for the equation

$$Au+u_t=f, \tag{1}$$

where

$$Au=\sum_{|\alpha|,|\beta|\leqslant m}(-1)^{|\alpha|}D_x^\alpha\big(a_{\alpha\beta}(x,t)D_x^\beta u\big) \tag{2}$$

is an elliptic differential operator defined for $x\in\Omega$, $t\in(0,T)$. Assuming that the coefficients are measurable functions bounded in $D_T=\Omega\times(0,T)$ and that all the derivatives occurring in (2) (understood in the weak sense) are square summable, we can associate with A its Dirichlet bilinear form

$$a(w,v)=\sum_{|\alpha|,|\beta|\leqslant m}(a_{\alpha\beta}D_x^\beta w, D_x^\alpha v)_{L^2(D_T)}. \tag{3}$$

To formulate the initial-boundary value problem in a weak form we introduce the Hilbert spaces

$$H_{m,0}=\{v\in L^2(D_T):\ D_x^\alpha v\in L^2(D_T),\ |\alpha|\leqslant m\},$$

$$H_{m,1}=\{v\in H_{m,0}:\ v_t\in L^2(D_T)\}$$

with the corresponding norms $\|\ \|_{m,0}$ and $\|\ \|_{m,1}$ defined by the scalar products

$$(w,v)_{m,0}=\sum_{|\alpha|\leqslant m}(D_x^\alpha w, D_x^\alpha v)_{L^2(D_T)}$$

and

$$(w, v)_{m,1} = (w, v)_{m,0} + (w_t, v_t)_{L^2(D_T)},$$

respectively. In $H_{m,1}$ one more norm is introduced, namely

$$|||v|||_m^2 = \|v\|_{m,0}^2 + \|v(\cdot, 0)\|_{L^2(\Omega)}^2 + \|v(\cdot, T)\|_{L^2(\Omega)}^2$$

(the value of v for $t = 0$ and $t = T$ in the sense of trace). The boundary condition is defined by a linear subspace V of $H_{m,0}$, which is supposed to contain the set $C_{0,x}^{\infty}(D_T)$ of all functions infinitely differentiable in $\overline{D_T}$, vanishing in some neighbourhood of the set $\partial\Omega \times [0, T]$. We suppose that $a(\cdot, \cdot)$ is V-elliptic. This means that there exists a constant $d > 0$ such that

$$a(v, v) \geqslant d\|v\|_{m,0}^2 \tag{4}$$

for all $v \in V$. Introducing on $H_{m,1} \times H_{m,1}$ the bilinear form

$$B(w, v) = a(w, v) - (w, v_t)_{L^2(D_T)} + \big(w(\cdot, T), v(\cdot, T)\big)_{L^2(\Omega)}$$

and denoting

$$l_{f,u_0}(v) = \big(u_0, v(\cdot, 0)\big)_{L^2(\Omega)} + (f, v)_{L^2(D_T)},$$

we give the following weak formulation of our initial-boundary value problem:

(P) given $u_0 \in L^2(\Omega)$ and $f \in L^2(D_T)$, find a $u \in V$ satisfying

$$B(u, v) = l_{f,u_0}(v) \tag{5}$$

for all $v \in V \cap H_{m,1}$ vanishing for $t = T$.

It has been proved in [2] that (P) is solvable and its solution is unique in the space $H_{m,1}$. Moreover, if $u \in H_{m,1}$, it has the following properties:

I. u is a solution of (P) if and only if it satisfies in D_T the differential equation (1) together with the initial condition

$$u(\cdot, 0) = u_0 \tag{6}$$

and boundary conditions

(b_1) $u \in V$,

(b_2) $(Au, v) = a(u, v)$

for all $v \in H_{m,1}$, $v(\cdot, T) = 0$.

II. Identity (5) holds for all $v \in V \cap H_{m,1}$.

Note that (b_2) implies some "natural" boundary conditions on $\partial\Omega \times (0, T)$.

If W is a finite dimensional subspace of $V \cap H_{m,1}$, the approximate problem is formulated as follows:

($\tilde{\mathrm{P}}$) find a $\tilde{u} \in W$ such that

$$B(\tilde{u}, \varphi) = l_{f,u_0}(\varphi)$$

holds for all $\varphi \in W$.

Problem ($\tilde{\mathrm{P}}$) is a kind of Galerkin approximation of the initial-boundary value problem in question, where the basic functions depend on x, t. Particularly, if W consists of spline functions, we are led to a finite element method based on a triangulation of the space-time domain D_T.

It was proved in [2] that

$$|||u - \tilde{u}|||_m \leqslant cd^{-1} \inf_{\varphi \in W} \|u - \varphi\|_{m,1}, \tag{7}$$

where c is a positive constant depending on the operator A and the domain D_T. Using (7), we are going to obtain further estimates of the error $e = u - \tilde{u}$, assuming a special form of approximating functions and some regularity of the exact solution u. Namely, let us consider a triangulation T_h of the domain Ω (see [1]) with $h = \max_{\varkappa \in T_h} \operatorname{diam} \varkappa$ and a partition S_τ: $0 = t_0 < t_1 < \ldots < t_\nu = T$ of the segment $(0, T)$ with $\tau = \max_{1 \leqslant j \leqslant \nu} (t_j - t_{j-1})$. The finite family of cylinders $\varkappa \times (t_{j-1}, t_j)$ with $\varkappa \in T_h$ and $j = 1, 2, \ldots, \nu$ is obviously a triangulation of D_T, which we denote by $T_h \times S_\tau$. The approximating space W is now constructed as the finite element space $W_{h,\tau}$ (see [1]) corresponding to $T_h \times S_\tau$. Given a family of triangulations $\{T_h \times S_\tau\}$, we shall assume that each finite element (K, P, Σ) with $K \in T_h \times S_\tau$ may be obtained from a pattern one $(\hat{K}, \hat{P}, \hat{\Sigma})$ by an affine transformation of the form

$$x = A\hat{x} + a, \quad t = \alpha\hat{t} + \beta \quad \big((\hat{x}, \hat{t}) \in \hat{K}\big) \tag{8}$$

with a non-singular $n \times n$ matrix A and $\alpha \neq 0$. To formulate our approximation result we use the following notation:

ϱ for the upper bound of diameters of balls contained in a fixed $\varkappa \in T_h$;

$H_p(\Xi)$ for the Sobolev space of order $p = 0, 1, 2, \ldots$ over a domain $\Xi \subset R^{n+1}$ with the norm denoted by $\| \ \|_{p,\Xi}$;

$$|v|_{p,\Xi} = \Big(\sum_{|\alpha| = p} \|D^\alpha v\|_{L^2(\Xi)}^2\Big)^{1/2} \quad \text{for} \quad v \in H_p(\Xi);$$

P_r for the set of all polynomials in variables x, t of order $\leqslant r$.

All the quantities corresponding to the pattern finite element will be marked with ^.

THEOREM. *Suppose that*

(i) $u \in H_{r+1}(D_T)$ *with* $r > m+(n-1)/2$;

(ii) $P_r \subset \hat{P} \subset H_m(\hat{K})$;

(iii) $W = W_{h,\tau} \subset H_m(D_T)$;

(iv) *there are two positive constants* σ_1, σ_2 *such that*

$$\sigma_1 \tau \leqslant \varrho \leqslant h \leqslant \sigma_2 \tau .$$

Then

$$|||e|||_m \leqslant \gamma \hat{c} h^{r+1-m} |u|_{r+1,D_T} \tag{9}$$

with constant $\hat{c}$ *depending on the pattern element and*

$$\gamma = cd^{-1}(\sigma_2/\sigma_1)^{m+r+1}.$$

The proof is quite similar to the proof in elliptic problems (see [1]). In view of (7) it is sufficient to estimate $\|u - \Pi_{h,\tau} u\|_{m,D_T}$, where $\Pi_{h,\tau}$ is the interpolation operator connected with the space $W_{h,\tau}$. Note that in view of (iii) the interpolate $\Pi_{h,\tau} u$ is defined by the values at the knots of the derivatives $D^\alpha u$, $|\alpha| \leqslant m-1$, but (i) ensures that $u \in C^{m-1}(\overline{D_T})$ according to the Sobolev lemma, and so these values are well defined. Denoting by B the matrix of the affine transformation (8) we have for $K \in T_h \times S_\tau$ and $l = 0, 1, \ldots, m$

$$|u - \Pi_{h,\tau} u|_{l,K} \leqslant \hat{c}_1 \|B^{-1}\|^l \|B\|^{r+1} |u|_{r+1,K},$$

where $\| \ \|$ denotes the spectral norm of the matrix in question. But

$$\|B\| \leqslant \sqrt{\|A\|^2 + \alpha^2}, \qquad \|B^{-1}\| \leqslant \sqrt{\|A^{-1}\| + \alpha^{-2}}$$

and (see [1], Theorem 3.1.3)

$$\|A\| \leqslant h/\hat{\varrho}, \qquad \|A^{-1}\| \leqslant \hat{h}/\varrho;$$

thus

$$\|B\| \leqslant \sqrt{h^2/\hat{\varrho}^2 + \tau^2/\hat{\tau}^2}$$

and similarly

$$\|B^{-1}\| \leqslant \sqrt{\hat{h}^2/\varrho^2 + \hat{\tau}^2/\tau^2}.$$

Elementary calculation yields

$$|u - \Pi_{h,\tau} u|_{l,K} \leqslant \hat{c}_2 (\sigma_2/\sigma_1)^{l+r+1} h^{r+1-l} |u|_{r+1,K}. \tag{10}$$

We obtain (9) after summing both sides of (10) over all $K \in T_h \times S_\tau$ and $l = 0, 1, \ldots, m$.

References

[1] P. Ciarlet, *The finite element method for elliptic problems*, New York 1978.

[2] H. Marcinkowska, *On internal approximations of parabolic problems*, Ann. Polon. Math. 42(1983), 173–180.

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