A remark on approximate solving of a class of initial-boundary value problems

by

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Abstract. The paper is a continuation of the author's earlier work [2]. It deals with a class of initial-boundary value problems for the equation $Au + u_t = f$, where $A$ is a linear elliptic operator of order $2m$ in space variables with time-dependent coefficients. For approximate solving these problems we propose a finite element method based on a suitable family of triangulations of the space-time domain. An estimate of the error is given.

Let $\Omega$ be a polyeder in $\mathbb{R}^n$. We are dealing in this paper with approximate solutions of the initial-boundary value problem for the equation

\begin{equation}
Au + u_t = f,
\end{equation}

where

\begin{equation}
Au = \sum_{|\alpha| + |\beta| \leq m} (-1)^{|\alpha|} D_\alpha^\beta \left[ a_{\alpha \beta}(x, t) D_\alpha^\beta u \right]
\end{equation}

is an elliptic differential operator defined for $x \in \Omega, t \in (0, T)$. Assuming that the coefficients are measurable functions bounded in $\Omega \times (0, T)$ and that all the derivatives occurring in (2) (understood in the weak sense) are square summable, we can associate with $A$ its Dirichlet bilinear form

\begin{equation}
\sigma(w, \nu) = \sum_{|\alpha| + |\beta| \leq m} \langle a_{\alpha \beta} D_\alpha^\beta w, D_\beta^\nu \nu \rangle_{L^2(\Omega)}.
\end{equation}

To formulate the initial-boundary value problem in a weak form we introduce the Hilbert spaces

\[ H_{m,0} = \{ v \in L^2(\Omega) : D_\alpha^\beta v \in L^2(\Omega), |\alpha| \leq m \}, \]

\[ H_{m,1} = \{ v \in H_{m,0} : \nu \in L^1(\Omega) \} \]

with the corresponding norms $\| \cdot \|_{m,0}$ and $\| \cdot \|_{m,1}$ defined by the scalar products

\[ (w, \nu)_{m,0} = \sum_{|\alpha| \leq m} \langle D_\alpha^\beta w, D_\beta^\nu \nu \rangle_{L^2(\Omega)} \]
and
\[ (w, \psi)_{H^{1,1}} = (w, \psi)_{H^{1,1}} + (w, \psi)_{L^2(D_0)} \]
respectively. In \( H^{1,1} \) one more norm is introduced, namely
\[ \|w\|_{H^{1,1}} = \|w\|_{H^{1,1}} + \|w(\cdot, 0)\|_{L^2(\Omega)} + \|w(\cdot, T)\|_{L^2(\Omega)} \]
(the value of \( w \) for \( t = 0 \) and \( t = T \) in the sense of trace). The boundary condition is defined by a linear subspace \( V \) of \( H^{1,1} \), which is supposed to contain the set \( C^0_0(D_0) \) of all functions infinitely differentiable in \( D_0 \), vanishing in some neighborhood of the set \( \partial \Omega \times (0, T) \). We suppose that \( a(\cdot, \cdot) \) is \( V \)-elliptic. This means that there exists a constant \( d > 0 \) such that
\[ a(\cdot, \cdot) \geq d \|w\|_{H^{1,1}} \]
for all \( w \in V \). Introducing on \( H^{1,1} \times H^{1,1} \) the bilinear form
\[ B(w, \psi) = a(w, \psi) - (w, \psi)_{L^2(D_0)} + (w(\cdot, 0), \psi(\cdot, T))_{L^2(\Omega)} \]
and denoting
\[ I_{H^{1,1}}(w) = (w, \psi(\cdot, 0))_{L^2(\Omega)} + (f, \psi)_{L^2(D_0)} \]
we give the following weak formulation of our initial-boundary value problem:
\[ (P) \] given \( u \in L^2(\Omega) \) and \( f \in L^2(D_0) \), find a \( u \in V \) satisfying
\[ B(u, \psi) = I_{H^{1,1}}(\psi) \]
for all \( \psi \in V \cap H^{1,1} \), vanishing for \( t = T \).
It has been proved in [2] that \( (P) \) is solvable and its solution is unique in the space \( H^{1,1} \). Moreover, if \( u \in H^{1,1} \), it has the following properties:
1. \( u \) is a solution of \( (P) \) if and only if it satisfies in \( D_0 \) the differential equation (1) together with the initial condition (6)
2. Boundary conditions
   \begin{align*}
   & (b_1) \quad u \in V, \\
   & (b_2) \quad (Au, \psi) = a(u, \psi)
   \end{align*}
for all \( \psi \in H^{1,1}, \psi(\cdot, T) = 0 \).
III. Identity (6) holds for all \( \psi \in V \cap H^{1,1} \).
Note that \( (b_2) \) implies some “natural” boundary conditions on \( \partial \Omega \times (0, T) \).

If \( W \) is a finite dimensional subspace of \( V \cap H^{1,1} \), the approximate problem is formulated as follows:
\[ (P) \] find a \( \tilde{u} \in W \) such that
\[ B(\tilde{u}, \psi) = I_{H^{1,1}}(\psi) \]
holds for all \( \psi \in W \).
Problem \( (P) \) is a kind of Galerkin approximation of the initial-boundary value problem in question, where the basic functions depend on \( x, t \). Particularly, if \( W \) consists of spline functions, we are led to a finite element method based on a triangulation of the space-time domain \( D_0 \).
It is proved in [2] that
\[ \|u - \tilde{u}\|_{H^{1,1}} \leq c d^{-1} \inf_{\psi \in W} \|u - \psi\|_{H^{1,1}} \]
where \( c \) is a positive constant depending on the operator \( A \) and the domain \( D_0 \). Using (7), we are going to obtain further estimates of the error \( e = u - \tilde{u} \), assuming a special form of approximating functions and some regularity of the exact solution \( u \). Namely, let us consider a triangulation \( T_\delta \) of the domain \( \Omega \) (see [1]) with \( h = \max \text{diam} \delta \) and a partition \( S_\gamma \): \( 0 = t_0 < t_1 < \ldots < t_\gamma = T \) of the segment \( (0, T) \) with \( \tau = \max (t_j - t_{j-1}) \).
The finite family of cylinders \( \times \times T_\delta, j \in \{1, 2, \ldots, \} \) is obviously a triangulation of \( D_0 \), which we denote by \( D_\delta \). The approximating space \( W \) is now constructed as the finite element space \( W_{\delta, \tau} \) (see [1]), corresponding to \( T_\delta \times S_\tau \). Given a family of triangulations \( (T_\delta, \tau) \), we shall assume that each finite element \( (K, F, \Sigma) \) with \( K \in T_\delta \times S_\tau \) may be obtained from a pattern one \( (\tilde{K}, \tilde{F}, \tilde{\Sigma}) \) by an affine transformation of the form
\[ x = A \tilde{x} + a, \quad t = \tilde{t} + \beta \quad (\tilde{x}, \tilde{t}) \in \tilde{K} \]
with a non-singular \( n \times n \) matrix \( A \) and \( a \neq 0 \). To formulate our approximation result we use the following notation:
\( \epsilon \) for the upper bound of diameters of balls contained in a fixed \( x \in T_\delta \);
\( H_p(\mathcal{E}) \) for the Sobolev space of order \( p = 0, 1, 2, \ldots \) over a domain \( \mathcal{E} \in H^{p+1} \) with the norm denoted by \( \| \cdot \|_{H_p(\mathcal{E})} \)
\[ \|w\|_{H_p(\mathcal{E})} = \left( \sum_{|\alpha| = p} \|D^\alpha w\|_{L^2(\mathcal{E})}^2 \right)^{1/2} \quad \text{for} \quad w \in H_p(\mathcal{E}) \]
\( P_r \) for the set of all polynomials in variables \( x, t \) of order \( r \).
All the quantities corresponding to the pattern finite element will be marked with \( \tilde{\cdot} \).
THEOREM. Suppose that
(i) \( u \in H_{r+1}(D_{2}) \) with \( r > m + (n-1)/2 \);
(ii) \( P \subseteq \tilde{P} \subseteq H_{m}(K) \);
(iii) \( W = W_{K} \subseteq H_{n}(D_{2}) \);
(iv) there are two positive constants \( \sigma_{1}, \sigma_{2} \) such that
\[ \sigma_{1} r \leq q \leq \sigma_{2} r. \]
Then
\[ |||\delta|||_{\infty} \leq \gamma \delta^{m+1-m} |||u|||_{r+1,D_{2}} \]
with constant \( \gamma \) depending on the pattern element and
\[ \gamma = c \delta^{-1}(\sigma_{2}/\sigma_{1})^{m+1}. \]

The proof is quite similar to the proof in elliptic problems (see [1]). In view of (7) it is sufficient to estimate \( |||u - P_{h} u|||_{\infty,D_{2}} \), where \( P_{h} \) is the interpolation operator connected with the space \( W_{h} \). Note that in view of (iii) the interpolates \( P_{h} u \) is defined by the values at the knots of the derivatives \( D^{m} u \), \( |||u|||_{r} \leq m - 1 \), but (i) ensures that \( u \in C^{m-1}(D_{2}) \) according to the Sobolev lemma, and so these values are well defined. Denoting by \( B \) the matrix of the affine transformation (8) we have for \( K \in T_{h} \times S_{h} \) and \( l = 0, 1, \ldots, m \)
\[ |||u - P_{h} u|||_{l,K} \leq \delta_{l} |||B^{-1}|||^{l} |||B|||^{l+1} |||u|||_{r+1,K}, \]
where \(||| \cdot |||\) denotes the spectral norm of the matrix in question. But
\[ |||B||| \leq \sqrt{|||A|||^{2} + \delta^{2}}, \quad |||B^{-1}||| \leq \sqrt{|||A^{-1}|||^{2} + \delta^{-2}} \]
and (see [1], Theorem 3.1.3)
\[ |||A||| \leq \delta^{1/2}, \quad |||A^{-1}||| \leq \delta^{1/2}, \]
thus
\[ |||B||| \leq \sqrt{\delta^{1/2} + \delta^{1/2}}, \quad |||B^{-1}||| \leq \sqrt{\delta^{1/2} + \delta^{1/2}}. \]
and similarly
\[ |||B|||^{-1} \leq \sqrt{\delta^{1/2} + \delta^{1/2}}. \]
Elementary calculation yields
\[ |||u - P_{h} u|||_{l,K} \leq \delta_{l}(\sigma_{2}/\sigma_{1})^{l+1} |||u|||_{r+1,K}. \]