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Topologically maximal pretopologies

by

SZYMON DOLECKI* (Warszawa) and GABRIELE H. GRECO (Trento)

Dedicated to Professor Jan Mikusiński on his 70th birthday

Abstract. The topologization $\mathfrak{T}\theta$ of a pretopology θ is the finest topology coarser than θ. A necessary and sufficient condition is given for a pretopology θ to be maximal in $\mathfrak{T}^{-1}\mathfrak{T}\theta$. It is proved that Fréchet pretopologies θ with the unicity of sequential limits are the greatest elements of $\mathfrak{T}^{-1}\mathfrak{T}\theta$. Examples of a topology τ without maximal pretopologies in $\mathfrak{T}^{-1}\tau$ and of a non-maximal pretopology which is not a power of any other pretopology are given. Application to elementary (called also sequential) non-Fréchet topologies is discussed.

1. Introduction. Pretopologies are the convergences which are determined by closure operations. Topologies are those pretopologies for which the corresponding closures are idempotent. To every pretopology π there corresponds the finest topology coarser than π, called the *topologization* of π (e.g. Hausdorff [10], Choquet [3], Čech [2]).

The question that imposes itself is for what topology τ there is no other pretopology whose topologization is τ. More generally, which pretopologies are topologically maximal (such that no other pretopology with the same topologization is finer)? In this paper we characterize such pretopologies (Theorem 6.1, Corollary 6.2). Moreover, we show that Fréchet pretopologies with the unicity of sequential limits are the finest among the pretopologies with the same topologization (Theorem 6.4).

The topologization of a given pretopology π may be obtained as a power (of some ordinal order) of π (e.g. Hausdorff [10], Čech [2], Hammer [9], Kent and Richardson [11], Novák [12]). We show that there are topologically non-maximal pretopologies which are not powers of any other pretopology (Example 6.6).

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We also observe that there are ordered sets of all pretopologies having the same topologization, without any maximal pretopology (Example 6.5).

Besides powers, we consider (topological) products of pretopologies. Their use enables one to confront various problems, for instance the question of whether the infimum of two topologies (in the lattice of pretopologies) is a topology.

We furnish some new examples of pretopologies and recall the pretopology of Féron [6]. They are of importance since the topologization of properly chosen pretopologies yields new topologies with some prescribed properties. We give an example of application of this method in obtaining elementary (traditionally called sequential) topologies which are not Fréchet (at any point) (Section 7 and [4], Theorem 15.4).

2. Pretopologies. This section is devoted to a recollection of some basic facts concerning pretopologies. Most of these results are due to Choquet [3]. They may be found dispersed in the book [2] of Čech. A new side-light has been shed on them in [4].

Denote by φX the collection of all the filters on X. Every relation in $\varphi X \times X$ is called a *convergence* ([4]).

Convergences will be denoted by small Greek letters $(\pi, \sigma, \theta, \tau, \ldots)$ and also by $\mathrm{Lim}^{\pi}, \mathrm{Lim}^{\sigma}, \mathrm{Lim}^{\theta}, \ldots$ The value of a convergence π at a filter $\mathscr{F}$ will be always denoted by $\mathrm{Lim}^{\pi}\mathscr{F}$ and called the *limit of* $\mathscr{F}$ (in π). We say that $\mathscr{F}$ *converges to* x (in π) whenever $x \in \mathrm{Lim}^{\pi}\mathscr{F}$. Of course, $(\mathrm{Lim}^{\pi})^{-1}x$ is the set of all the filters that converge to x. The domain $D(\pi)$ is $\{\mathscr{F}: \mathrm{Lim}^{\pi}\mathscr{F} \neq \emptyset\}$.

We say that θ is *coarser* than τ (τ *finer* than θ: $\tau \geqslant \theta$) if $\mathrm{Lim}^{\theta} \supset \mathrm{Lim}^{\tau}$. A convergence ξ is called *constants-preserving* if

$$\mathrm{Lim}^{\xi}\mathscr{N}_{\iota}(x) \ni x \quad \text{for each } x \in X, \tag{2.1}$$

where $\mathscr{N}_{\iota}(x) = \{A \subset X: x \in A\}$ is the *discrete filter* of x. Constants-preserving convergences form a complete lattice (closed sublattice of all convergences), the finest element of which is the discrete convergence ι $((\mathrm{Lim}^{\iota})^{-1}x = \{\mathscr{N}_{\iota}(x)\})$ and the coarsest is the chaotic convergence o ($\mathrm{Lim}^{o}\mathscr{F} = X$ for each $\mathscr{F}$).

A convergence ξ is called *stable* if for every collection $\mathfrak{F}$ of filters

$$\mathrm{Lim}^{\xi} \wedge \mathfrak{F} = \bigcap_{\mathscr{F} \in \mathfrak{F}} \mathrm{Lim}^{\xi}\mathscr{F}. \tag{2.2}$$

Pretopologies are the constants-preserving stable convergences [3] (in [4] we do not require (2.1)). It follows that each pretopology π is *isotone*:

$$\mathscr{F} \subset \mathscr{G} \Rightarrow \mathrm{Lim}^{\pi}\mathscr{F} \subset \mathrm{Lim}^{\pi}\mathscr{G}. \tag{2.3}$$

A *coilology* Ξ of a complete lattice Π is a subset of Π such that all the suprema coincide in Π and Ξ. The map $\boldsymbol{P}_{\Xi}$ which to every element π of Π assigns the greatest element of Ξ less than π is called the (*coilological*) *projection* on Ξ. Coilological projections are precisely isotone idempotent maps less than the identity (see [4]). The set of all pretopologies is a coilology in the lattice of convergences. The corresponding projection $\mathfrak{P}$ is called the *pretopologization*.

The infimum of all filters convergent to x (in ξ) is called the *neighborhood filter* of x (in ξ):

$$\mathscr{N}_{\xi}(x) = \bigwedge (\mathrm{Lim}^{\xi})^{-1}x. \tag{2.4}$$

Constants-preserving convergences have the property that

$$x \in Q \quad \text{for every } Q \in \mathscr{N}(x). \tag{2.5}$$

A convergence is stable if and only if

$$x \in \mathrm{Lim}\,\mathscr{F} \Leftrightarrow \mathscr{F} \supset \mathscr{N}(x), \tag{2.6}$$

so that pretopologies are the convergences characterized by (2.5), (2.6). Accordingly, every pretopology π is determined by its neighborhood system $\mathscr{N}_{\pi}: X \to \varphi X$ which satisfies (2.5).

The *interior* $\mathrm{int}^{\pi}A$ for a pretopology π is the set of those points for which A constitutes a *neighborhood*:

$$x \in \mathrm{int}^{\pi}A \Leftrightarrow A \in \mathscr{N}_{\pi}(x). \tag{2.7}$$

The *closure* of a set A is defined by

$$\mathrm{cl}^{\pi}A = (\mathrm{int}^{\pi}A^{c})^{c}. \tag{2.8}$$

Interior and closure are operations from 2^X to 2^X. They satisfy

$$\begin{array}{ll} \mathrm{int}\,X = X, & \mathrm{cl}\,\emptyset = \emptyset, \\ A \supset \mathrm{int}\,A, & A \subset \mathrm{cl}\,A, \\ \mathrm{int}(A \cap B) = \mathrm{int}\,A \cap \mathrm{int}\,B; & \mathrm{cl}(A \cup B) = \mathrm{cl}\,A \cup \mathrm{cl}\,B; \end{array} \tag{2.9}$$

already Hausdorff [10] studies spaces with closure operations satisfying (2.9).

Interior and closure constitute also relations ($\mathrm{cl}, \mathrm{int}: 2^X \rightrightarrows X$). Thus, the inverse relations $\mathrm{int}^{-1}, \mathrm{cl}^{-1}$ satisfy $\mathrm{int}^{-1}x = \{A: x \in \mathrm{int}\,A\}$, $\mathrm{cl}^{-1}x = \{A: x \in \mathrm{cl}\,A\}$.

Every interior satisfying (2.9) has the property that for each x, $\mathrm{int}^{-1}x$ is a filter such that each $Q \in \mathrm{int}^{-1}x$ contains x. Therefore there is equivalence between pretopologies and interior operations:

(2.10) $$\mathcal{N}_\pi(x) = (\mathrm{int}^\pi)^{-1}x.$$

The *conjugate* $\mathcal{A}^*$ of a family $\mathcal{A}$ of subset of X is the family of the complements of those subsets of X which are not in $\mathcal{A}$. In view of (2.8), for each convergence,

(2.11) $$\mathrm{cl}^{-1}x = [\mathrm{int}^{-1}x]^*.$$

The *grill* $\mathcal{A}^\#$ of $\mathcal{A}$ is the family of all those subsets of X which intersect every set in $\mathcal{A}$. If $\mathcal{A}$ is *isotone* ($B \subset A$ and $B \in \mathcal{A}$ imply $A \in \mathcal{A}$), then $\mathcal{A}^* = \mathcal{A}^\#$. Therefore, by (2.10) and (2.11),

(2.12) $$x \in \mathrm{cl}\,A \Leftrightarrow Q \cap A \neq \emptyset \quad \text{for every } Q \in \mathcal{N}(x).$$

As a consequence of (2.12), one has

(2.13) $$\mathrm{cl}^\xi A = \bigcup_{A \in \mathcal{F}^\#} \mathrm{Lim}^\xi \mathcal{F},$$

and one notes that if π is a pretopology, then

(2.14) $$\mathrm{Lim}^\pi \mathcal{F} = \bigcap_{A \in \mathcal{F}^\#} \mathrm{cl}^\pi A.$$

Clearly the supremum of $\mathcal{F}, \mathcal{G} \in \varphi X$ exists (in φX) if and only if $\mathcal{F} \subset \mathcal{G}^\#$ ($\mathcal{G} \subset \mathcal{F}^\#$). We shall denote by $\mathcal{F} \vee A$ the supremum of $\mathcal{F}$ and of the discrete filter $\mathcal{N}_\iota(A) = \{C \subset X\colon A \subset C\}$; we shall call it the *trace* of $\mathcal{F}$ on A.

We include the following formulae for neighborhoods, interiors and closures of suprema and infima because they turn out to be instrumental on various occasions.

(2.15) $$\mathcal{N}_{\vee\Sigma}(x) = \bigvee_{\sigma \in \Sigma} \mathcal{N}_\sigma(x) \quad \text{and} \quad \mathcal{N}_{\wedge\Sigma}(x) = \bigwedge_{\sigma \in \Sigma} \mathcal{N}_\sigma(x).$$

The above serves to compute

(2.16) $$\mathrm{int}^{\vee\Sigma} Q = \bigcup_{\substack{\Sigma' \supset \Sigma \\ \text{finite}}} \bigcup_{\bigcap_{\sigma \in \Sigma'} Q_\sigma \subset Q} \bigcap_{\sigma \in \Sigma'} \mathrm{int}^\sigma Q \quad \text{and} \quad \mathrm{int}^{\wedge\Sigma} Q = \bigcap_{\sigma \in \Sigma} \mathrm{int}^\sigma Q,$$

(2.17) $$\mathrm{cl}^{\vee\Sigma} A = \bigcap_{\substack{\Sigma' \subset \Sigma \\ \text{finite}}} \bigcap_{\bigcup_{\sigma \in \Sigma'} A_\sigma \supset A} \bigcup_{\sigma \in \Sigma} \mathrm{cl}^\sigma A_\sigma \quad \text{and} \quad \mathrm{cl}^{\wedge\Sigma} A = \bigcup_{\sigma \in \Sigma} \mathrm{cl}^\sigma A.$$

3. Composition of pretopologies. Our first objective is to define a composition of pretopologies. To this end we shall need a set-theoretical notion of limit.

Let $\mathcal{M}$ be a relation from a set Y to the space 2^X of all subsets of X ($\mathcal{M} \subset Y \times 2^X$) and let $\mathcal{A}$ be a family of subsets of Y. The *lower limit of $\mathcal{M}$ along $\mathcal{A}$* is defined by ([4], [8])

(3.1) $$\mathcal{M}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \bigcap_{y \in A} \mathcal{M}(y).$$

We have that

(3.2) $$\begin{gathered} \mathcal{M} \subset \mathcal{N} \Rightarrow \mathcal{M}(\mathcal{A}) \subset \mathcal{N}(\mathcal{A}), \\ \mathcal{A} \leqslant \mathcal{B} \Rightarrow \mathcal{M}(\mathcal{A}) \subset \mathcal{M}(\mathcal{B}), \end{gathered}$$

where $\mathcal{B} \geqslant \mathcal{A}$ means that for every $A \in \mathcal{A}$ there is $B \in \mathcal{B}$ such that $B \subset A$.

In particular, $\mathcal{N}_\iota(\mathcal{A})$ is the family of all the sets which include an element of $\mathcal{A}$; $\mathcal{M}(A)$ ($= \mathcal{M}\{A\}$, by definition) is the family of all those Q which belong to $\mathcal{M}(y)$ for each $y \in A$. If $\{\mathcal{A}_i\}_{i \in I}$ is a collection of isotone families of subsets of X, then

(3.3) $$\mathcal{M}\Big(\bigcap_{i \in I} \mathcal{A}_i\Big) = \bigcap_{i \in I} \mathcal{M}(\mathcal{A}_i).$$

If $\mathcal{A}$ is a filter and for each y, $\mathcal{M}(y)$ is a filter, then $\mathcal{M}(\mathcal{A})$ is a filter. Then one has

$$\mathcal{M}(\mathcal{A}) = \Big\{\bigcup_{y \in A} M_y, \text{ where } M_y \in \mathcal{M}(y),\ A \in \mathcal{A}\Big\}.$$

The *composition* $\pi\sigma$ of pretopologies π and σ is the pretopology whose neighborhood filters are

(3.4) $$\mathcal{N}_{\pi\sigma}(x) = \mathcal{N}_\pi\big(\mathcal{N}_\sigma(x)\big);$$

in other words, $Q \in \mathcal{N}_{\pi\sigma}(x)$ if and only if $\mathrm{int}^\pi Q \in \mathcal{N}_\sigma(x)$. On rewriting (3.4), one gets

(3.5) $$\mathrm{int}^{\pi\sigma} = \mathrm{int}^\sigma \mathrm{int}^\pi; \quad \mathrm{cl}^{\pi\sigma} = \mathrm{cl}^\sigma \mathrm{cl}^\pi.$$

The composition is associative. The discrete convergence is the unit and the chaotic convergence is the null element of the composition:

(3.6) $$\sigma o = o\sigma = o; \quad \sigma\iota = \iota\sigma = \sigma.$$

The composition is not commutative.

EXAMPLE 3.1. Denote by $\boldsymbol{Q}$ the set of rationals in $\boldsymbol{R}$. Let ν be the usual topology of $\boldsymbol{R}$. Define

$$\mathcal{N}_\pi(r) = \begin{cases} \mathcal{N}_\iota(r) & \text{if } r \in \boldsymbol{Q}, \\ \mathcal{N}_o(r) & \text{if } r \notin \boldsymbol{Q}; \end{cases}$$

we have $\pi\nu = o$, while

$$\mathcal{N}_{\nu\pi}(r) = \begin{cases} \mathcal{N}_\nu(r) & \text{if} \quad r \in \boldsymbol{Q}, \\ \mathcal{N}_o(r) & \text{if} \quad r \notin \boldsymbol{Q}. \end{cases}$$

PROPOSITION 3.2. *The composition is isotone and less than the infimum:*

(3.7) $$\pi \leqslant \sigma \text{ and } \varrho \leqslant \theta \Rightarrow \pi\varrho \leqslant \sigma\theta,$$

(3.8) $$\pi\sigma \leqslant \pi \wedge \sigma.$$

Proof. In fact, (3.7) follows from (3.2). To prove (3.8), note that $\pi\sigma \leqslant \pi\iota = \pi$ and $\pi\sigma \leqslant \iota\sigma = \sigma$.

PROPOSITION 3.3. *We have*

(3.9) $$(\sigma_1 \wedge \sigma_2)\pi = \sigma_1\pi \wedge \sigma_2\pi \quad \text{and} \quad \pi(\bigwedge \Sigma) = \bigwedge_{\sigma \in \Sigma} \pi\sigma.$$

A map $\mathcal{M}\colon X \to \varphi X$ is called a *selector* of a convergence σ if, for every $x \in X$, $x \in \operatorname{Lim}^\sigma \mathcal{M}(x)$. We shall denote this by $\mathcal{M}(\cdot) \in (\operatorname{Lim}^\sigma)^{-1}(\cdot)$.

THEOREM 3.4. *Let σ, π be pretopologies. Then*

(3.10) $$\operatorname{Lim}^{\sigma\pi}\mathcal{F} = \bigcup_{\substack{\mathcal{M}(\cdot) \in (\operatorname{Lim}^\sigma)^{-1}(\cdot) \\ \mathcal{M}(\mathcal{G}) \subset \mathcal{F}}} \operatorname{Lim}^\pi \mathcal{G}.$$

Proof. Let $x \in \operatorname{Lim}^{\sigma\pi}\mathcal{F}$, that is, $\mathcal{F} \supset \mathcal{N}_\sigma(\mathcal{N}_\pi(x))$. Thus $\mathcal{N}_\sigma$ is a selector and by setting $\mathcal{M} = \mathcal{N}_\sigma$, $\mathcal{G} = \mathcal{N}_\pi(x)$, we have x in the union. Conversely, if there is a selector $\mathcal{M}$ of σ (that is, such that $\mathcal{M}(x') \supset \mathcal{N}_\sigma(x')$ for each $x' \in X$) and $\mathcal{G}$ such that $\mathcal{G} \supset \mathcal{N}_\pi(x)$ and $\mathcal{M}(\mathcal{G}) \subset \mathcal{F}$, then, by (3.2), $\mathcal{N}_\sigma(\mathcal{N}_\pi(x)) \subset \mathcal{F}$, hence $x \in \operatorname{Lim}^{\sigma\pi}\mathcal{F}$.

Let α be an ordinal number. The αth power of a pretopology π was defined by Hausdorff [10] (in terms of closure operations). Here is an equivalent definition:

(3.11) $$\pi^\alpha = \begin{cases} \pi^{\alpha-1}\pi & \text{if} \quad \alpha-1 \text{ exists,} \\ \bigwedge_{\beta<\alpha} \pi^\beta & \text{otherwise.} \end{cases}$$

Using (3.9), one obtains

(3.12) $$(\pi \wedge \sigma)^2 \leqslant \pi\sigma,$$

which is a companion formula of (3.8).

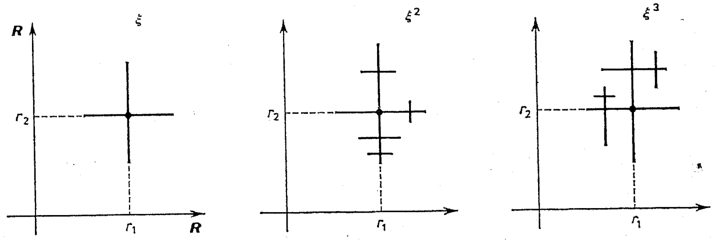
THEOREM 3.5. *Let π, σ be pretopologies, α, β ordinal numbers. Then*

(3.13) $$\pi^{\alpha+\beta} = \pi^\alpha \pi^\beta,$$

(3.14) $$(\pi^\alpha)^\beta = \pi^{\alpha\beta}.$$

4. Examples of pretopologies.

EXAMPLE 4.1 (Féron cross [6]). Let $\iota \times \nu$ be the "vertical" pretopology on $\boldsymbol{R}^2$: $\mathcal{N}_{\iota\times\nu}(r_1, r_2) = \mathcal{N}_\iota(r_1) \times \mathcal{N}_\nu(r_2)$, where ν is, as usual, the natural (usual) topology of $\boldsymbol{R}$. Let $\nu \times \iota$ be the "horizontal" pretopology on $\boldsymbol{R}^2$: $\mathcal{N}_{\nu\times\iota}(r_1, r_2) = \mathcal{N}_\nu(r_1) \times \mathcal{N}_\iota(r_2)$. The pretopology $\xi = (\iota \times \nu) \wedge (\nu \times \iota)$ is called the *Féron cross*. A base for ξ at (r_1, r_2) is formed of crosses $[(r_1-\varepsilon, r_1+\varepsilon) \times \{r_2\}] \cup [\{r_1\} \times (r_2-\varepsilon, r_2+\varepsilon)]$:



Recall that a filter $\mathcal{E}$ is called *elementary* if there exists a sequence $\{x_n\}_n$ such that $\{\{x_k\colon k \geqslant n\}\}_{n\in\boldsymbol{N}}$ constitues a base of $\mathcal{E}$. Denote by εX the collection of all elementary filters on X.

Each filter of the form $\mathcal{M}(\mathcal{E})$, where $\mathcal{M}\colon X \to \varepsilon X$ and $\mathcal{E} \in \varepsilon X$ is called *dielementary* (*2-elementary*). A filter is said to be *n-elementary* if it is of the form $\mathcal{M}(\mathcal{F})$, where $\mathcal{M}\colon X \to \varepsilon X$ and $\mathcal{F}$ is $(n-1)$-elementary; the collection of all such filters is denoted by $\varepsilon^n X$. Every n-elementary filter is $(n+1)$-elementary.

A pretopology π is called *n-elementary at x* if $\mathcal{N}_\pi(x)$ is an intersection of n-elementary filters; 1-elementary pretopologies are called *Fréchet pretopologies* (see [4]). The Féron cross pretopology is a Fréchet pretopology.

THEOREM 4.2. *A pretopology is n-elementary at x if and only if $x \in \operatorname{cl} A$ implies that there is an n-elementary filter $\mathcal{F}$ convergent to x such that $A \in \mathcal{F}$.*

Proof. In view of (2.10), (2.11), we only need prove that if $\{\mathcal{F}_i\}_{i\in I}$ is a family of n-elementary filters such that every filter finer than some $\mathcal{F}_i$ belongs to the family, then $(\bigcap_{i\in I} \mathcal{F}_i)^{\#} = \bigcup_{i\in I} \mathcal{F}_i$. But this follows from the fact that each trace of an n-elementary filter is n-elementary.

COROLLARY 4.3. *The pretopology $\pi_1\pi_2\ldots\pi_n$ is n-elementary if π_k is a Fréchet pretopology for each $k \leqslant n$.*

Consider n-sequences $\{x_{k_1k_2\ldots k_n}\}_{(k_1,k_2,\ldots,k_n)\in\boldsymbol{N}^n}$. Then a filter $\mathcal{F}$ is n-elementary if and only if there is an n-sequence such that the family of sets

(4.1) $$\{x_{k_1k_2\ldots k_n}\colon k_1 \geqslant K_1, k_2 \geqslant K_2(k_1), \ldots, k_n \geqslant K_n(k_1, k_2, \ldots, k_{n-1})\},$$

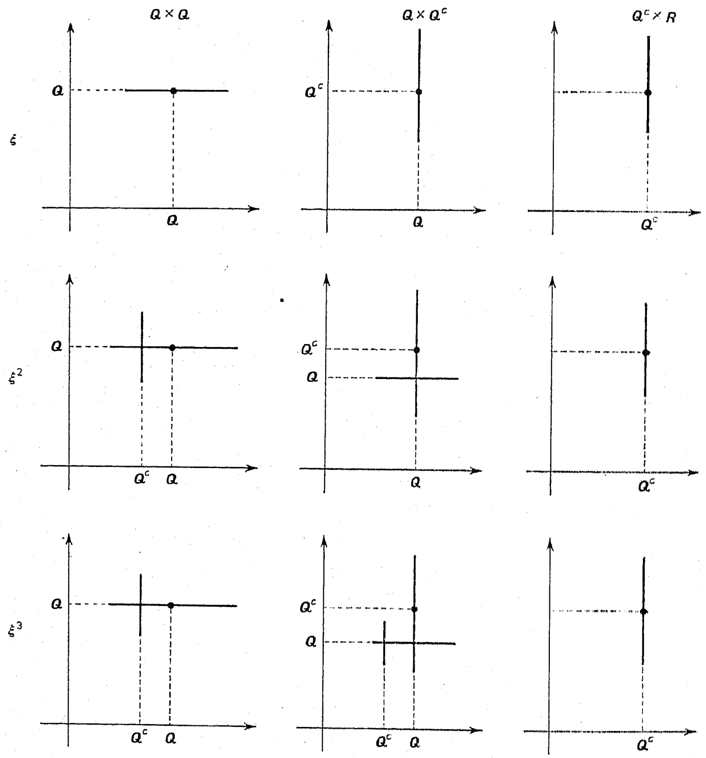
where $K_1 \in \boldsymbol{N}$, $K_2\colon \boldsymbol{N}\to\boldsymbol{N}, \ldots$, is a base of $\mathscr{F}$. We say that the filter F is *generated by the n-sequence* $\{x_{k_1k_2\ldots k_n}\}_{(k_1,k_2,\ldots,k_n)\in \boldsymbol{N}^n}$.

It follows from Theorems 4.2, 3.4 and Corollary 4.3 that for each $n \in \boldsymbol{N}$, the power ξ^{n+1} of the Féron cross is strictly coarser than ξ^n. Indeed, the $(n+1)$-sequence (say, for n even)

$$(4.2)\qquad \{(2^{-k_1}+2^{-k_1k_2k_3}+\ldots+2^{-k_1k_2\ldots k_{n+1}},\ 2^{-k_1k_2}+2^{-k_1k_2k_3k_4}+\ldots \\ \ldots+2^{-k_1k_2\ldots k_n})\}_{(k_1,\ldots,k_{n+1})\in \boldsymbol{N}^{n+1}}$$

generates an $(n+1)$-elementary filter convergent to $(0,0)$ in ξ^{n+1}, but not in ξ^n.

EXAMPLE 4.4 (Cat fur).



By $\boldsymbol{Q}$ we denote the subset of rational numbers of $\boldsymbol{R}$ and let $\boldsymbol{Q}^c = \boldsymbol{R}\setminus\boldsymbol{Q}$. We define the "rational horizontal" pretopology of $\boldsymbol{R}^2$ by

$$(4.3)\qquad \mathscr{N}_\psi(r_1, r_2) = \begin{cases} \mathscr{N}_\nu(r_1)\times\mathscr{N}_\iota(r_2) & \text{if } (r_1, r_2)\in \boldsymbol{Q}^2, \\ \mathscr{N}_\iota(r_1, r_2) & \text{otherwise} \end{cases}$$

and the "irrational vertical" pretopology φ of $\boldsymbol{R}^2$ by

$$(4.4)\qquad \mathscr{N}_\varphi(r_1, r_2) = \begin{cases} \mathscr{N}_\iota(r_1, r_2) & \text{if } (r_1, r_2)\in \boldsymbol{Q}^2, \\ \mathscr{N}_\iota(r_1)\times\mathscr{N}_\nu(r_2) & \text{otherwise.} \end{cases}$$

The infimum $\xi = \psi\wedge\varphi$ is called the *cat fur*.

To describe the consecutive powers of the cat fur we use the above diagram. To every point of a neighborhood in ξ we add a neighborhood of that point in ξ^{n-1}, obtaining a generic neighborhood in ξ^n. Since the composition is associative and we only shall consider the finite powers, we may act differently by adding to each point of a neighborhood in ξ^{n-1} its neighborhood in ξ; there will also result a generic neighborhood in ξ. In the diagram we shall take into consideration only one typical point at each stage. One observes that all the three pretopologies ξ, ξ^2, ξ^3 are distinct, and that $\xi^3 = \xi^\alpha$ for each $\alpha > 3$.

EXAMPLE 4.5 (Domino). Consider $\boldsymbol{R}_+ = [0, \infty)$ with the usual topology ν induced on $\boldsymbol{R}_+$ from $\boldsymbol{R}$. Define the *domino* δ by

$$(4.5)\qquad \mathscr{N}_\delta(0) = \mathscr{N}_\nu(0);\quad \mathscr{N}_\delta(x) = \mathscr{N}_\iota([x/2, 3x/2]) \quad \text{if} \quad x > 0.$$

We have, for $n \in \boldsymbol{N}$,

$$(4.6)\qquad \mathscr{N}_{\delta^n}(0) = \mathscr{N}_\nu(0);\quad \mathscr{N}_{\delta^n}(x) = \mathscr{N}_\iota([(1/2)^n x, (3/2)^n x]) \quad \text{if} \quad x > 0,$$

while for the first infinite ordinal,

$$(4.7)\qquad \mathscr{N}_{\delta^\omega}(0) = \mathscr{N}_\nu(0);\quad \mathscr{N}_{\delta^\omega}(x) = \mathscr{N}_\iota((0, +\infty)) \quad \text{if} \quad x > 0.$$

Finally, for $\alpha \geqslant \omega+1$,

$$(4.8)\qquad \mathscr{N}_{\delta^\alpha}(0) = \{[0, +\infty)\};\quad \mathscr{N}_{\delta^\alpha}(x) = \mathscr{N}_\iota((0, +\infty)) \quad \text{if} \quad x > 0.$$

EXAMPLE 4.6. Define δ_0 on $\boldsymbol{R}$ by: $\mathscr{N}_{\delta_0}(x) = \mathscr{N}_\iota([x-1, x+1])$. Then $\mathscr{N}_{\delta_0^n}(x) = \mathscr{N}_\iota([x-2n+1, x+2n-1])$ for $n\in\boldsymbol{N}$, and $\delta_0^\omega = o$.

5. Topologization. A pretopology τ on X is called a *topology* if for each $x \in X$, and each $Q \in \mathcal{N}_\tau(x)$, there is $V \in \mathcal{N}_\tau(x)$ such that $Q \in \mathcal{N}_\tau(V)$ (Čech [2]). In other words, τ is a topology, whenever for every $Q \in \mathcal{N}_\tau(x)$, $\mathrm{int}^\tau Q \in \mathcal{N}_\tau(x)$. By definition (3.4) we have

THEOREM 5.1. *A pretopology τ is a topology if and only if $\tau = \tau^2$.*

It is known (e.g. [2]) that the above notion of topology amounts to the usual one.

Let π be a pretopology on X. A subset Q of X is called *open* if it is equal to its interior: $Q = \mathrm{int}^\pi Q$.

THEOREM 5.2. *A pretopology τ is a topology if and only if its every neighborhood filter has a base composed of open sets.*

Open sets of each pretopology π satisfy all the classical axioms of open sets of a topological space. By virtue of Theorem 5.2, the neighborhood system generated by the open sets of π determines a topology. This topology is denoted by $\mathfrak{T}\pi$ and called the *topologization* (or the *topology*) of π ([3])). It follows from the definitions that the map $\mathfrak{T}$ is descending and idempotent and that τ is a topology if and only if $\tau \in \mathrm{fix}\,\mathfrak{T}$. Consequently, the topology of π is the finest topology which is less than π. More generally, topologies constitute a coilology in the lattice of all convergences; the corresponding projection is the extension of the topologization to all convergences and is also denoted by $\mathfrak{T}$.

THEOREM 5.3 ([2], [9], [10], [11], [12]). *For each pretopology π there is the least ordinal number $t(\pi)$ such that, for $\alpha \geqslant t(\pi)$,*

(5.1) $$\mathfrak{T}\pi = \pi^\alpha.$$

The ordinal number $t(\pi)$ in the above theorem is called the *topological defect* of π. A pretopology is a topology if and only if the topological defect is one.

As a corollary of Theorem 5.3 one has that for each ordinal β and each pretopology π,

(5.2) $$\mathfrak{T}(\pi^\beta) = \mathfrak{T}\pi.$$

Let $\mathfrak{O}(\pi)$ denote the collection of open sets of a pretopology π. Denote by $\wedge_\mathfrak{T}$ the infimum in the coilology of topologies.

PROPOSITION 5.4. *Let $\{\pi_i\}_{i\in I}$ be a set of pretopologies. The topologization of its infimum is equal to the infimum of the topologizations in the coilology of topologies*

(5.3) $$\mathfrak{T}(\bigwedge_{i\in I} \pi_i) = \bigwedge_{\mathfrak{T}}{}_{i\in I} \mathfrak{T}\pi_i.$$

Proof. Since $\wedge_\mathfrak{T} = \mathfrak{T}\wedge$ in the lattice of topologies ([4]), we have the inequality $\geqslant$ in (5.3). The opposite inequality follows from $\mathfrak{T}(\bigwedge_{i\in I} \pi_i) \leqslant \mathfrak{T}\pi_j$ for each $j \in I$.

In other words,

(5.4) $$\mathfrak{O}(\bigwedge_{i\in I} \pi_i) = \bigcap_{i\in I} \mathfrak{O}(\pi_i).$$

It is interesting that

(5.5) $$\mathfrak{T}(\sigma\pi) = \mathfrak{T}\sigma \wedge_\mathfrak{T} \mathfrak{T}\pi.$$

Indeed, in view of (3.8), $\mathfrak{T}(\sigma\pi) \leqslant \mathfrak{T}(\pi\wedge\sigma)$, while (3.12) and (5.2) yield $\mathfrak{T}(\pi\wedge\sigma) \leqslant \mathfrak{T}(\sigma\pi)$. It is now enough to apply Proposition 5.4.

In general the infimum of a family of topologies (in the lattice of pretopologies) is not a topology.

PROPOSITION 5.5. *Let π and σ be topologies. The infimum $\pi\wedge\sigma$ is a topology if and only if*

(5.6) $$\pi\wedge\sigma = \pi\sigma = \sigma\pi.$$

Proof. We have $(\sigma\wedge\pi)^2 = \sigma\pi\wedge\pi\sigma$, because of (3.7), (3.9) and Theorem 5.1. Again by Theorem 5.1, $\sigma\wedge\pi$ is a topology if and only if $\sigma\wedge\pi = \sigma\pi\wedge\pi\sigma$. By virtue of (3.8), we get (5.6) as a necessary and sufficient condition.

On the other hand, we have

PROPOSITION 5.6. *Let π and σ be topologies. They commute if and only if $\sigma\pi$, $\pi\sigma$ and $\sigma\pi\wedge\pi\sigma$ are topologies.*

Proof. If π and σ commute, then $(\sigma\pi)^2 = \sigma\pi\sigma\pi = \sigma^2\pi^2 = \sigma\pi$ and the condition follows. Conversely, applying Proposition 5.5 to $\sigma\pi$ and $\pi\sigma$ we have that $\sigma\pi\wedge\pi\sigma = \sigma\pi\sigma = \pi\sigma\pi$. On the other hand, $\sigma\pi\wedge\sigma = \sigma\pi$ and $\pi\sigma\wedge\pi = \pi\sigma$. Consequently, $\sigma\pi = \sigma\pi\sigma = \pi\sigma$.

COROLLARY 5.7. *Let π, σ be topologies. If $\pi\wedge\sigma$ is a topology, then*

(5.7) $$\pi\sigma = \sigma\pi.$$

If (5.7) *holds, then the topological defect of $\pi\wedge\sigma$ is less or equal to 2.*(*)

(The second part of Corollary 5.7 follows from Proposition 5.5 and the equality $(\pi\wedge\sigma)^2 = \sigma\pi\wedge\pi\sigma$.)

We shall analyze now some pointwise aspects of topologization. We say that an $x \in X$ is a *topological point of* π whenever

(5.8) $$\mathcal{N}_\pi(x) = \mathcal{N}_{\mathfrak{T}\pi}(x).$$

An x is said to be *hypotopological point of* π if

(5.9) $$\mathcal{N}_\pi(x) = \mathcal{N}_{\pi^2}(x).$$

(*) Added in proof. Let $\mathcal{N}_\sigma(x) = \mathcal{N}_\iota(\{-x, x\})$ for $x \in \mathbf{R}$. Topologies σ and ν commute and the topological defect of $\delta\wedge\nu$ is 2. This example is due to François Laubie of the University of Limoges.

Every topological point is hypotopological. It follows from the definition that if all the points are hypotopological points of π, then π is a topology.

PROPOSITION 5.8. *If the set of hypotopological points is a neighborhood of x, then x is a topological point.*

Proof. Let Q be a neighborhood of x consisting of hypotopological points. Then $\operatorname{int} Q$ is a neighborhood of x (since x is hypotopological) and we show that it is open. Indeed if $x' \in \operatorname{int} Q$, then, equivalently, $Q \in \mathcal{N}(x')$ and by (5.9) $x' \in \operatorname{int}(\operatorname{int} Q)$.

PROPOSITION 5.9. *If the topological defect is finite, then each hypotopological point is topological.*

Proof. It is enough to show that $\mathcal{N}_{\pi^n}(x) = \mathcal{N}_{\pi^{n+1}}(x)$ implies $\mathcal{N}_{\pi^n}(x) = \mathcal{N}_{\pi^{n+2}}(x)$. Indeed, $\mathcal{N}_{\pi^{n+2}}(x) = \mathcal{N}_\pi(\mathcal{N}_{\pi^{n+1}}(x)) = \mathcal{N}_\pi(\mathcal{N}_{\pi^n}(x)) = \mathcal{N}_{\pi^{n+1}}(x) = \mathcal{N}_{\pi^n}(x)$.

Let σ be a pretopology on X. A *hypotopological defect of σ at x* is the least ordinal α for which $\mathcal{N}_{\sigma^{\alpha 2}}(x) = \mathcal{N}_{\sigma^\alpha}(x)$; we denote it by $h(x, \sigma)$. The *topological defect of σ at x* $(t(x, \sigma))$ is the least ordinal α for which $\mathcal{N}_{\sigma^\alpha}(x) = \mathcal{N}_{\mathfrak{T}\sigma}(x)$. Comparing these pointwise notions with that of topological defect one has

$$\sup_{x\in X} h(x, \sigma) \leqslant t(\sigma) = \sup_{x\in X} t(x, \sigma). \tag{5.10}$$

The inequality in (5.10) may be strict.

EXAMPLE 5.10. Let σ be the following pretopology on $\boldsymbol{Z}\cup\{+\infty\}$:

$$\mathcal{N}_\sigma(n) = \begin{cases} \mathcal{N}_\iota(\{n-1, n\}) & \text{if } n \in \boldsymbol{Z}, \\ \bigcup_{k\in \boldsymbol{Z}} \mathcal{N}_\iota(\{s\colon s \geqslant k\} \cup \{+\infty\}) & \text{if } n = +\infty. \end{cases} \tag{5.11}$$

Then we have $t(\sigma) = \omega+1$, while

$$h(n, \sigma) = \begin{cases} 1 & \text{if } n = +\infty, \\ \omega & \text{otherwise,} \end{cases}$$

$$t(n, \sigma) = \begin{cases} \omega+1 & \text{if } n = +\infty, \\ \omega & \text{otherwise.} \end{cases}$$

EXAMPLE 5.11. Take the domino δ (Example 4.5):

$$h(x, \delta) = \begin{cases} 1 & \text{if } x = 0, \\ \omega & \text{if } x > 0, \end{cases} \qquad t(x, \delta) = \begin{cases} \omega+1 & \text{if } x = 0, \\ \omega & \text{if } x = 0. \end{cases}$$

We observe that in the cat fur ζ (Example 4.4), $\boldsymbol{Q}^c\times\boldsymbol{R}$ are topological points, the topological defect of the points in $\boldsymbol{Q}\times\boldsymbol{Q}$ is 2 while of those in $\boldsymbol{Q}\times\boldsymbol{Q}^c$ is 3. Consequently, ζ^3 is the topology of ζ.

The hypotopological defect of all the points of the Féron cross (Example 4.1) is at least ω.

6. Topologically maximal pretopologies. Consider the inverse image $\mathfrak{T}^{-1}(\tau)$ of a topology τ by the topologization. Under what condition does $\mathfrak{T}^{-1}(\tau)$ reduce to $\{\tau\}$?

More generally, we write $\pi \leqslant_{\mathfrak{T}} \theta$ whenever $\pi \leqslant \theta$ and $\mathfrak{T}\pi = \mathfrak{T}\theta$ and say that π is *topologically maximal* whenever $\pi \leqslant_{\mathfrak{T}} \theta$ implies $\pi = \theta$.

Clearly if $\mathfrak{T}^{-1}(\tau)$ reduces to $\{\tau\}$, then τ is topologically maximal.

A pretopology π is called *topologically maximal at* x_0 if for each pretopology θ such that $\pi \leqslant_{\mathfrak{T}} \theta$, $\mathcal{N}_\pi(x_0) = \mathcal{N}_\theta(x_0)$. Obviously, a pretopology is topologically maximal if and only if it is topologically maximal at each point.

THEOREM 6.1. *A pretopology is topologically maximal at x_0 if and only if*

(6.1) *for each set A such that $x_0 \in \operatorname{cl} A^c \cap A$, there is a set Ω such that $\{x_0\} = \operatorname{cl}\Omega\setminus\Omega$ and $x_0 \notin \operatorname{cl}(\Omega\cap A)$.*

Proof. Suppose that the condition does not hold for a pretopology π: there is a set A, such that $x_0 \in A \cap \operatorname{cl}^\pi A^c$ and for every Ω,

$$\{x_0\} = \operatorname{cl}^\pi\Omega\setminus\Omega \Rightarrow x_0 \in \operatorname{cl}^\pi(\Omega\cap A). \tag{6.2}$$

Define the following pretopology σ:

$$\mathcal{N}_\sigma(x_0) = \mathcal{N}_\pi(x_0)\vee A \quad \text{and} \quad \mathcal{N}_\sigma(x) = \mathcal{N}_\pi(x) \quad \text{if} \quad x \neq x_0. \tag{6.3}$$

Clearly σ is finer than π. Since $x_0 \in \operatorname{cl}^\pi A^c$, $A \notin \mathcal{N}_\pi(x_0)$ so that $\mathcal{N}_\sigma(x_0)$ is strictly finer than $\mathcal{N}_\pi(x_0)$. One has, for an arbitrary set Ω,

$$\operatorname{cl}^\sigma\Omega = \operatorname{cl}^\pi(A\cap\Omega)\cup[\operatorname{cl}^\pi(A^c\cap\Omega)\setminus\{x_0\}]. \tag{6.4}$$

In order to prove that $\mathfrak{T}\sigma = \mathfrak{T}\pi$, it is enough to show that if $\operatorname{cl}^\sigma\Omega = \Omega$, then $\operatorname{cl}^\pi\Omega = \Omega$. Suppose that $\operatorname{cl}^\sigma\Omega = \Omega$ and that $\operatorname{cl}^\pi\Omega\setminus\Omega$ is not empty. In view of (6.4), $\operatorname{cl}^\pi\Omega\setminus\Omega = \{x_0\}$, thus by (6.2) and (6.4), $x_0 \in \operatorname{cl}^\sigma\Omega$: a contradiction.

We have proved that every topologically maximal pretopology satisfies (6.1).

Suppose that π is not topologically maximal at x_0; there are θ such that $\mathfrak{T}\pi = \mathfrak{T}\theta$ and $A \in \mathcal{N}_\theta(x_0)\setminus\mathcal{N}_\pi(x_0)$. Consequently $A^c \in \mathcal{N}_\pi(x_0)^\#$ and, on the other hand, $A \in \mathcal{N}_\pi(x_0)^\#$. Define a pretopology σ by (6.3). Clearly, $\theta \geqslant \sigma \geqslant \pi$; hence $\mathfrak{T}\sigma = \mathfrak{T}\pi$.

Let Ω be such that $\{x_0\} = \operatorname{cl}^\pi\Omega\setminus\Omega$. In particular, Ω is not π-closed, hence not σ-closed and since $\operatorname{cl}^\pi\Omega\setminus\operatorname{cl}^\sigma\Omega \subset \{x_0\}$, $x_0 \in \operatorname{cl}^\sigma\Omega$. By virtue of (6.4), $x_0 \in \operatorname{cl}^\pi(\Omega\cap A)$, so that (6.2) is proved.

We recall that a pretopology π is said to be *Fréchet at x_0* if its neighborhood filter at x_0 is an intersection of elementary filters (convergent to x_0); equivalently, if for every set A such that $x_0 \in \mathrm{cl}^\pi A$, there exists a sequence $\{x_n\}_{n\in \boldsymbol{N}}$ of elements of A such that the elementary filter generated by $\{\{x_k\colon k \geqslant n\}\}_{n\in\boldsymbol{N}}$ converges to x_0.

COROLLARY 6.2. *Every pretopology, which is Fréchet at x_0 and such that for each sequence $\{x_n\}_{n\in\boldsymbol{N}}$ convergent to x_0 the set $\{x_n\colon n \in \boldsymbol{N}\}\cup\{x_0\}$ is closed, is maximal at x_0.*

Proof. Let $x_0 \in A \cap \mathrm{cl} A^c$. Then there is a sequence $\{x_n\}_{n\in\boldsymbol{N}}$ convergent to x_0 and such that $\{x_n\colon n\in\boldsymbol{N}\} \subset A^c$. Consequently $\mathrm{cl}(\{x_n\colon n\in\boldsymbol{N}\}\cap \cap A)$ is empty. Since x_0 is the only point of $\mathrm{cl}\{x_n\colon n\in\boldsymbol{N}\}\setminus\{x_n\colon n\in\boldsymbol{N}\}$, we have that condition (6.1) is satisfied.

In particular, it follows that every Fréchet pretopology with the unicity of limits for elementary filters is maximal. Consequently the Féron cross (Example 4.1) and the cat fur (Example 4.4) are maximal.

There are Fréchet pretopologies which are not maximal (for instance, the chaotic pretopology) so that the latter condition of Corollary 6.2 must not be dropped. On the other hand, there are maximal Fréchet pretopologies that do not satisfy that condition.

EXAMPLE 6.3. Let π be the following pretopology on $\{1, 2, \ldots, n\}$: $\mathcal{N}_\pi(k) = \mathcal{N}_\iota\{k, k+1\}$ if $k \neq n$ and $\mathcal{N}_\pi(n) = \mathcal{N}_\iota\{n, 1\}$. This pretopology is Fréchet and maximal but does not fulfil the conditions of Corollary 6.2. Indeed, $\mathfrak{T}\pi = o$ and if $\theta \geqslant \pi$, then there is k such that $\mathcal{N}_\theta(k) = \{k\}$, hence $\{k\}$ is open and $\mathfrak{T}\theta \neq o$.

We give here a criterion for a pretopology to be not only topologically maximal but also "topologically greatest".

THEOREM 6.4. *Let π be a Fréchet pretopology with the unicity of limits of elementary filters. Then π is the finest pretopology in $\mathfrak{T}^{-1}(\mathfrak{T}\pi)$.*

Proof. Let π satisfy the assumptions and let $\mathfrak{T}\theta = \mathfrak{T}\pi$. We shall show that $\theta \leqslant \pi$. Otherwise there is x_0 and $A \in \mathcal{N}_\theta(x_0)$ such that $x_0 \in \mathrm{cl}^\pi A^c$ and since π is a Fréchet pretopology, there is a sequence $\{x_n\}_{n\in\boldsymbol{N}}$ of elements of A^c π-convergent to x_0. By the unicity of limits $[\{x_n\colon n\in\boldsymbol{N}\}\cup\{x_0\}]^c$ is open. The set $A\cup[\{x_n\colon n\in\boldsymbol{N}\}\cup\{x_0\}]^c = \{x_n\colon n \in \boldsymbol{N}\}^c$ is thus open in θ but not in π; a contradiction.

Is there for every pretopology π a maximal pretopology θ such that $\theta \geqslant_{\mathfrak{T}} \pi$? The answer is negative. We give an example.

EXAMPLE 6.5. Let π be a cofinite topology on $\boldsymbol{N}$, that is, the topology which open sets are the empty set and all cofinite sets. One sees that

$$\mathrm{cl}^\pi A = \begin{cases} A & \text{if } A \text{ is finite,} \\ \boldsymbol{N} & \text{otherwise.}\end{cases}$$

If θ is a topologically maximal pretopology such that $\theta \geqslant_{\mathfrak{T}} \pi$ then, by Theorem 6.1, for every subset A of $\boldsymbol{N}$ and every n in $\boldsymbol{N}$

$$n \in \mathrm{cl}^\theta(A^c\setminus\{n\}) \quad \text{implies} \quad n \notin \mathrm{cl}^\theta(A\setminus\{n\}). \tag{6.5}$$

Indeed, the only set Ω for which $\{n\} = \mathrm{cl}^\theta\Omega\setminus\Omega$ is $\boldsymbol{N}\setminus\{n\}$. It is because such Ω is infinite and $\Omega\cup\{n\}$ is closed.

Consequently, for each n the free filter $\mathscr{U}_n = \mathcal{N}_\theta(n)\vee\{n\}^c$ is an ultrafilter. Since the space $\beta\boldsymbol{N}\setminus\boldsymbol{N}$ (with the topology induced by the Stone–Čech compactification of the discrete space $\boldsymbol{N}$) is not separable, there exists an infinite subset B of $\boldsymbol{N}$ such that

$$B \notin \mathscr{U}_n \quad \text{for every} \quad n \in \boldsymbol{N}.$$

This set is closed in θ but not in π, in contradiction with $\mathfrak{T}\theta = \mathfrak{T}\tau$. We have proved that there is no maximal pretopology in $\mathfrak{T}^{-1}\pi$.

A pretopology π is said to be *topologically irreducible* if for every pretopology θ distinct from π and each ordinal α, $\pi \neq \theta^\alpha$.

Clearly, every topologically maximal pretopology is topologically irreducible (in view of (5.2)). Is the converse true?

The answer is negative. Moreover, we shall give an example of a pretopology which is not topologically maximal at any point and which is irreducible.

EXAMPLE 6.6. The pretopology π is defined on $\boldsymbol{R}^2$ as follows: $\pi = (\nu\times\iota)\xi$, where ξ is the Féron cross (Example 4.1), while $\nu\times\iota$ is the product of the usual topology (of $\boldsymbol{R}$) and the discrete topology (of $\boldsymbol{R}$).

One observes that for each $x\in\boldsymbol{R}^2$: $\mathcal{N}_{\xi^2}(x) \subset \mathcal{N}_\pi(x) \subset \mathcal{N}_\xi(x)$ and $\mathcal{N}_{\xi^2}(x) \neq \mathcal{N}_\pi(x) \neq \mathcal{N}_\xi(x)$; so that $\mathfrak{T}\pi = \mathfrak{T}\xi$ and π is not topologically maximal at x for every x.

In view of Theorem 6.4 we conclude that if $\theta \geqslant_{\mathfrak{T}} \pi$, then $\theta \leqslant \xi$, thus $\theta^\alpha \leqslant \xi^\alpha < \pi$ for every $\alpha > 1$. Therefore π is topologically irreducible.

7. Application to elementary topologies. Given a convergence θ and a family of filters $\mathfrak{F}$, one defines the restriction $\theta\vee\mathfrak{F}$ (of θ to $\mathfrak{F}$) by

$$\mathrm{Lim}^{\theta\vee\mathfrak{F}}\mathscr{F} = \begin{cases} \mathrm{lim}^\theta\mathscr{F} & \text{if } \mathscr{F}\in\mathfrak{F}, \\ \emptyset & \text{otherwise.}\end{cases} \tag{7.1}$$

Let εX be the family of all elementary filters on X. Then a pretopology π is Fréchet if and only if

$$\pi = \mathfrak{P}(\pi\vee\varepsilon X). \tag{7.2}$$

A topology τ is called *elementary* if it satisfies

$$\tau = \mathfrak{T}(\tau\vee\varepsilon X). \tag{7.3}$$

Such topologies are traditionally called "sequential" ([1], [7]). Indeed, a topology is elementary if and only if all sequentially closed sets are closed.

It is well known that every Fréchet topology (that is, a topology which is a Fréchet pretopology) is elementary, because $\mathfrak{P}(\tau\vee\varepsilon X) \geqslant \mathfrak{T}(\tau\vee\varepsilon X) \geqslant \tau$ for every topology τ. There is an example of an elementary non Fréchet topology ([5], [7]). In [4], Thm. 15.4, we give a general scheme of costructing such topologies. Here we show how the results on topologically maximal pretopologies may be used to the same end.

THEOREM 7.1. *Let π be a Fréchet pretopology such that π is Hausdorff. If $\mathcal{N}_\pi(x) \neq \mathcal{N}_{\mathfrak{T}\pi}(x)$, then $\mathfrak{T}\pi$ is elementary but not Fréchet at x.*

Proof. If $\mathfrak{T}\pi$ is Fréchet at x, then, by Corollary 6.2, $\mathfrak{T}\pi$ is maximal at x in contradiction with the assumptions.

Since π is a Fréchet pretopology, $\mathfrak{T}\pi = \mathfrak{T}(\pi\vee\varepsilon X)$ and since $\mathfrak{T}(\pi\vee\varepsilon X) \geqslant \mathfrak{T}(\mathfrak{T}\pi\vee\varepsilon X) \geqslant \mathfrak{T}\pi$, we have that $\mathfrak{T}\pi = \mathfrak{T}(\mathfrak{T}\pi\vee\varepsilon X)$.

EXAMPLE 7.2. The Féron topology $\mathfrak{T}\xi$ (see Example 4.1) is elementary but is nowhere Fréchet. Indeed, since $\xi > \nu\times\nu$ (ν the usual topology of $\boldsymbol{R}$), also $\mathfrak{T}\xi \geqslant \nu\times\nu$, hence is Hausdorff. On the other hand, ξ is a Fréchet pretopology and for each x, $\mathcal{N}_\xi(x) \neq \mathcal{N}_{\mathfrak{T}\xi}(x)$.

To prove this fact we may also use [4], II. Thm. 7.7. A sequential convergence θ is said to be *topologically induced* if there is a topology τ such that $\theta = \tau\vee\varepsilon X$.

The Féron cross ξ has the property that $\xi\vee\varepsilon X$ is topologically induced (by [4], II, Corollary 7.5), because $\xi\vee\varepsilon X$ is an Urysohn convergence and ξ is Hausdorff). On the other hand, ξ is Fréchet. The cited theorem claims that then $\mathfrak{T}\xi$ ($\neq \xi$) is elementary but not Fréchet.

EXAMPLE 7.3. The cat fur topology $\mathfrak{T}\xi$ (Example 4.4) is elementary and is not Fréchet at the points at which it is different from the cat fur (pretopology) ξ, namely on $\boldsymbol{Q}\times\boldsymbol{R}$. This may be checked directly or inferred from Theorem 7.1.

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