Consequently,

$$|\langle\varphi_i, D^\alpha\psi\rangle - \langle\varphi_j, D^\alpha\psi\rangle| < \varepsilon \quad \text{for} \quad i, j > i_\varepsilon.$$

Thus the sequence $(\langle\varphi_n, D^\alpha\psi\rangle)$ is convergent for every $\psi \in D_N$. Denoting $f^*(\psi) = \lim_{n\to\infty} \langle\varphi_n, D^\alpha\psi\rangle$, f^* is obviously linear, and continuity of f^* in D_N follows from the Hölder inequality

$$|f^*(\psi)| \leqslant 2L\,\|D^\alpha\psi\|_N \quad \text{for} \quad \psi \in D_N.$$

We have still to show that if $f_m \in D_M^*$, $f_m \xrightarrow{M} 0$, then $f_m \to 0$ in D'_M. Let $f_m = [D^\alpha\varphi_{m,n}]$, $\|\varphi_{m,n}\|_M \leqslant L_m$ and $\|\varphi_{m,n}\|_M \to 0$ as $m\to\infty$ uniformly with respect to n. Let B be a bounded set in D_N, i.e., for every β there is a $C_\beta > 0$ such that $\|D^\beta\psi\|_N \leqslant C_\beta$ for all $\psi \in B$. Taking $\varepsilon_m = \sup_n \|\varphi_{m,n}\|_M$ we then obtain

$$|f_m^*(\psi)| \leqslant 2\,\|\varphi_{m,n}\|_M \|D^\alpha\psi\|_N \leqslant 2\varepsilon_m C_\alpha \to 0 \quad \text{as} \quad m\to\infty$$

for $\psi \in B$, whence $f_m^* \to 0$ in D'_M.

Let us still remark that due to the linearity of the space D'_M, also distributions $f \in \tilde{D}_M$ may be embedded in D'_M.

References

[1] H. Hudzik, A. Kamińska, *Equivalence of the Orlicz and Luxemburg norm in generalized Orlicz spaces* $L_M^\varphi(T)$, Funct. Approximatio 9 (1980), 29–37.

[2] A. Kamińska, R. Płuciennik, *Some theorems on compactness in generalized Orlicz spaces with application of the* Δ_∞*-condition*, ibid. 10 (1980), 135–146.

[3] S. Koshi, T. Shimogaki, *On quasi-modular spaces*, Studia Math. 21 (1961), 15–35.

[4] J. Magdziarz, *On a modular space of infinitely differentiable functions*, Comment. Math. 17 (1973), 159–177.

[5] J. Mikusiński, R. Sikorski, *The elementary theory of distributions* (II), Rozprawy Mat. 25 (1961), 1–47.

[6] J. Musielak, *On some spaces of functions and distributions* (I), *Spaces* D_M *and* D'_M, Studia Math. 21 (1962), 195–202.

[7] — *Modular approximation by a filtered family of linear operators*, Proceed. Confer. held at the Mathematical Institute at Oberwolfach, Black Forest, August 9–16, 1980, Basel–Boston–Stuttgart 1981, 99–110.

[8] J. Musielak, W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), 49–65.

Received September 30, 1982 (1807)

STUDIA MATHEMATICA, T. LXXVII. (1984)

On the range of purely atomic probability measures

by

C. FERENS (Tychy)

Dedicated to Professor Jan Mikusiński
on his 70th birthday

Abstract. An example is given of some purely atomic probability measure $P = (p_n|\ n \in N)$, with a range non-homeomorphic to the Cantor ternary discontinuum, such that $p_{n+1} \leqslant p_n$ for all positive integers n and the inequality $p_n > \sum_{i=n+1}^{\infty} p_i$ holds for infinitely many n.

It is well known that the range of non-atomic probability measures is the unit interval I. On the other hand, in the case of a purely atomic probability measure $P = (p_n|\ n \in N)$ with $p_{n+1} \leqslant p_n$ for each n belonging to the set N of all positive integers, the condition

$$0 < p_n \leqslant \sum_{i=n+1}^{\infty} p_i$$

is necessary and sufficient for the range of P to be the unit interval (e.g., see [1], p. 80). Jim Nymann has proved that if the above inequalities hold for almost all $n \in N$, the range of P is a finite union of some intervals. He has also proved that if

$$p_n > \sum_{i=n+1}^{\infty} p_i$$

for almost all $n \in N$, then the range of P is homeomorphic to the Cantor ternary discontinuum C and asked if the same holds under the weaker assumption that the last inequality is satisfied for infinitely many $n \in N$. The aim of this paper is to construct a counterexample.

Let $P = (p_n|\ n \in N)$ be a purely atomic probability measure with $p_{n+1} \leqslant p_n$ whenever $n \in N$. Let us extend the mapping $f\colon C \to I$ given by the formula

$$f(x) = \tfrac{1}{2} \sum_{n=1}^{\infty} x_n p_n,$$

where $x = (x_1, x_2, \ldots)$ is the triadic expansion of x in which the digit 1 does not occur, to

$$F\colon I \to I$$

by linear interpolation on each of the components of $I \setminus C$. If A_n denotes the set obtained in the nth step of the geometrical construction of C, i.e.,

$$A_n = \bigcup_{d_i=0,2} \Big\langle \sum_{i=1}^{n} \frac{d_i}{3^i},\ \sum_{i=1}^{n} \frac{d_i}{3^i} + \frac{1}{3^n} \Big\rangle,$$

then

$$\operatorname{Rng} P = f(C) = F(C) = F\big(\bigcap A_n\big) = \bigcap F(A_n)$$

since F is continuous. The set $F(A_n)$ can be represented in the form

$$F(A_n) = \bigcup_{k \in K_n} \langle k, k + r_n \rangle,$$

where $K_n = \{k = \sum_{i=1}^{n} e_i p_i \mid e_i = 0, 1\}$ and $r_n = \sum_{i=n+1}^{\infty} p_i$, $r_0 = 1$. Let us put

$$p_{5l-m} = (m+3)\, 2^{l-1}/3^{3l},$$

where $m = 0, 1, 2, 3, 4$; $l = 1, 2, \ldots$, and record the following definition:

An interval $J \subset I$ is ε-approximated by a set $K \subset I$ if $\forall x \in J\, \exists k \in K$: $0 \leqslant x - k \leqslant \varepsilon$.

Let $\varepsilon_n = \frac{1}{2} r_{5n}$ and $a_n = \frac{1}{9} \sum_{i=0}^{n-1} r_{5i}$. We shall show that:

(i) *The interval* $\langle k + \frac{2}{9}\varepsilon_n,\ k + \frac{11}{9}\varepsilon_n \rangle$ *for each* $k \in K_{5n}$ *is* ε_{n+1}-*approximated by* K_{5n+5};

(ii) *The interval* $\langle a_n, \frac{1}{2} \rangle$ *is* ε_n-*approximated by* K_{5n}.

Proof. Since

$$r_{5n} = \tfrac{1}{2} \sum_{m=0}^{4} \sum_{i=n+1}^{\infty} (m+3) \left(\tfrac{2}{27}\right)^i = \left(\tfrac{2}{27}\right)^n,$$

every element k of K_{5n+5} is of the form

(1) $$k = k' + \varepsilon_{n+1} \sum_{m=0}^{4} e_m (m+3)$$

where $k' \in K_{5n}$ and $e_m = 0$ or 1. But

$$\Big\{ \sum_{m=0}^{4} e_m (m+3) \;\Big|\; m = 0, \ldots, 4 \Big\} = \{0, 3, 4, \ldots, 21, 22, 25\},$$

and so (1) implies ε_{n+1}-approximation of the interval

$$\langle k' + 3\varepsilon_{n+1},\ k' + 22\varepsilon_{n+1} \rangle$$

by K_{5n+5}. Now, it suffices to note that

$$3\varepsilon_{n+1} = \tfrac{2}{9}\varepsilon_n < \tfrac{11}{9}\varepsilon_n < 22\varepsilon_{n+1},$$

and assertion (i) is proved.

Similarly, it can be shown that $\langle \frac{1}{9}, \frac{22}{27} \rangle$ is $\frac{1}{27}$-approximated by K_5, i.e., assertion (ii) is true for $n = 1$. Suppose that (ii) holds for some n and let $x \in \langle a_{n+1}, \frac{1}{2} \rangle$. We have

$$x - \tfrac{1}{9} r_{5n} \in \langle a_n, \tfrac{1}{2} \rangle$$

and, by the induction hypothesis, there exists a $k(x) \in K_{5n}$ such tha

$$x \in \langle k(x) + \tfrac{2}{9}\varepsilon_n,\ k(x) + \tfrac{11}{9}\varepsilon_n \rangle.$$

Therefore, the induction assertion follows from (i), which ends the proot of (ii).

The immediate consequence of (ii) is

(2) $$\langle a_n, \tfrac{1}{2} \rangle \subset F(A_{5n}).$$

Letting $n \to \infty$ in (2), we get

$$\langle \tfrac{3}{25}, \tfrac{22}{25} \rangle \subset \operatorname{Rng} P$$

using the symmetry argument. Since

$$p_{5n} = 3 \cdot 2^{n-1}/27^n > 2/27^n = r_{5n}$$

for $n \in N$, our counterexample is furnished.

Reference

[1] A. Rényi, *Probability Theory*, Akademiai Kiadó, Budapest 1970.

Received November 16, 1982 (1835,