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On sequentially M-integrable distributions

by

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To Professor Jan G. Mikusiński on his 70th birthday

Abstract. A sequential definition of M-integrable distributions is given and connections between the set D_M^* of such distributions and the generalized Orlicz space L_M and the space D'_M of distributions are established.

1.1. A space D'_M of distributions generalizing an Orlicz space $L_M(R^q)$ was introduced [6], 1962, and extended to the case of a generalized Orlicz space [4], 1973, applying the functional definition of a distribution. Here we shall show how to define sequentially M-integrable distributions starting with the well-known elementary theory of distributions by J. Mikusiński and R. Sikorski [5], 1961.

Let us recall that a sequence of functions $\varphi_n \in C^\infty(R^q)$, $n = 1, 2, \ldots$, is called *fundamental* if for every q-dimensional parallelepiped $I \subset R^q$ there exist a sequence of functions $\Phi_n \in C^\infty(R^q)$, $n = 1, 2, \ldots$, and a multiindex α such that $D^\alpha \Phi_n(x) = \varphi_n(x)$ in I and the sequence (Φ_n) is uniformly convergent on I.

Two fundamental sequences of functions (φ_n) and (ψ_n) are called *equivalent* if the sequence $\varphi_1, \psi_1, \varphi_2, \psi_2, \ldots$ is fundamental. Equivalence classes of fundamental sequences with regard to the above notion of equivalency are called *distributions*; the distribution f determined by the fundamental sequence (φ_n) is denoted by $f = [\varphi_n]$.

If $f = [\varphi_n]$, then $D^\alpha f = [D^\alpha \varphi_n]$ is called the *derivative of f of order α* (see [5]).

1.2. In the following the sign $\int$ will mean always the Lebesgue integral over the whole space R^q. A φ-function with parameter will mean a real function $M(t, u)$ defined in $R^q \times R^1$ and such that (a) $M(t, u) \geqslant 0$ always and $M(t, u) = 0$ if and only if $u = 0$, (b) $M(t, u)$ is an even, continuous and nondecreasing (for $u \geqslant 0$) function of u for every $t \in R^q$,

(c) $M(t,u)$ is Lebesgue measurable as a function of t for every $u\in R^1$. If, moreover, $M(t,u)$ is a convex function of u for every $t\in R^q$, then M is called a *convex* φ-function with parameter.

The vector space of all Lebesgue measurable functions f on R^q finite a.e., with equality a.e., such that

$$\varrho_M(\lambda f)=\int M\big(t,\lambda f(t)\big)\,dt<\infty \quad \text{for some } \lambda>0,$$

is called the *generalized Orlicz space* generated by the φ-function M with parameter and is denoted by $L_M(R^q)$, or briefly L_M.

Supposing M to be convex, the functional

$$\|f\|_M=\inf\{u>0\colon\ \varrho_M(f/u)\leqslant 1\}$$

is a norm in L_M called the *Luxemburg norm*; L_M is a Banach space with this norm (see e.g. [8]).

We say that a convex φ-function with parameter M is an *N-function* if $u^{-1}M(t,u)\to\infty$ as $u\to\infty$ for every $t\in R^q$.

$$N(t,u)=\sup\{|u|v-M(t,v)\colon\ v>0\}$$

is called the *complementary N-function* to the N-function M. The functional

$$\|f\|_M^0=\sup\Big\{\int f(t)g(t)\,dt\colon\ \varrho_N(g)\leqslant 1\Big\}$$

is then another norm in L_M, called the *Orlicz norm*. If M is an N-function, then there hold the Hölder inequalities

$$\Big|\int f(t)g(t)\,dt\Big|\leqslant\|f\|_M^0\|g\|_N \quad\text{and}\quad \Big|\int f(t)g(t)\,dt\Big|\leqslant\|f\|_M\|g\|_N^0$$

for $f\in L_M$ and $g\in L_N$ and both norms are equivalent in L_M, namely,

$$\|f\|_M\leqslant\|f\|_M^0\leqslant 2\|f\|_M \quad\text{for}\quad f\in L_M \tag{1}$$

(see e.g. [1]).

A φ-function with parameter M satisfies the *condition* (Δ_2) if there exist a constant $A>0$ and a nonnegative Lebesgue integrable function h on R^q such that $M(t,2u)\leqslant AM(t,u)+h(t)$ for all $t\in R^q$ and real u (see [3]). It is easily seen that if M satisfies (Δ_2), then L_M is equal to the space of f such that $\varrho_M(\lambda f)<\infty$ for every $\lambda>0$.

1.3. Definition. Let M be a convex φ-function depending on a parameter. A distribution f will be called *sequentially M-integrable* if there exist a multiindex α and a fundamental sequence (φ_n) of functions $\varphi_n\in C^\infty(R^q)\cap L_M$ such that $f=[D^\alpha\varphi_n]$ and the sequence $(\|\varphi_n\|_M)$ is bounded. The set of all sequentially M-integrable distributions will be denoted by D_M^*. Finite sums of sequentially M-integrable distributions will be called *sequentially M-summable distributions*; the space of all such distributions will be denoted by $\tilde D_M$.

A sequence (f_m) of sequentially M-integrable distributions will be called *sequentially M-convergent to* 0, $f_m\xrightarrow{M}0$, if there exist a multiindex α and fundamental sequences $(\varphi_{m,n})$, $m=1,2,\ldots$, such that $f_m=[D^\alpha\varphi_{m,n}]$, $\varphi_{m,n}\in C^\infty(R^q)\cap L_M$, $\|\varphi_{m,n}\|_M\leqslant L_m$ for $n=1,2,\ldots$ and all m, and $\|\varphi_{m,n}\|_M\to 0$ as $m\to\infty$ uniformly with respect to n. Moreover, if $f_m=[D^\alpha\varphi_{m,n}]$, $\varphi_{m,n}\in C^\infty(R^q)\cap L_M$, $(\|\varphi_{m,n}\|_M)$ bounded for every m separately, $m=0,1,2,\ldots$, and if $f_m-f_0\xrightarrow{M}0$, then we say that $f_m\xrightarrow{M}f_0$.

We are going to answer two problems: (1) Does the set D_M^* contain the Orlicz space L_M? (2) What is the connection between D_M^*, resp. $\tilde D_M$, and the space D_M' of distributions defined as linear continuous functionals over a space D_N (see [6])?

2.1. The following condition (Δ^∞) will be of use: we say that M satisfies (Δ^∞) if there exist constants $k>0$, $u_0>0$ and a locally integrable nonnegative function g on R^q such that $|u|\leqslant kM(t,u)+g(t)$ for all $u\geqslant u_0$ and $t\in R^q$ (see also [2], p. 140). Let us remark that if $M(t,u)=M(u)$ is convex and independent of t, then (Δ^∞) is always satisfied with $g(t)=0$. Obviously, we have

2.2. Proposition. *If M is a φ-function with parameter satisfying (Δ^∞), then every function $f\in L_M$ is locally integrable in R^q.*

In order to answer problem (1) we shall need the notion of M-boundedness of the function M.

2.3. Definition. A convex φ-function M with parameter is called *M-bounded* if there exist numbers $k>0$, $K\geqslant 1$ and a function $h(s,u)\geqslant 0$ in $R^q\times R^q$ with $\int h(s,u)\,du\leqslant K$ such that

$$M(s+u,v)\leqslant M(u,kv)+h(s,u) \quad\text{for all } s,u\in R^q \text{ and } v\in R^1$$

(compare [7], p. 103). Let us remark that if $M(t,u)$ does not depend on t, then it is always M-bounded with $k=1$ and $h(s,u)=0$.

Now let (δ_n) be a δ-sequence in the sense of [5], i.e. $\delta_n\in C_0^\infty(R^q)$ with $\delta_n(x)\geqslant 0$ everywhere, $\delta_n(x)=0$ for $|x|\geqslant\varepsilon_n$, where $0<\varepsilon_n\to 0$, and $\int\delta_n(x)\,dx=1$ for $n=1,2,\ldots$ Then for every locally integrable function f in R^q, $(f*\delta_n)$ is a fundamental sequence defining thus a distribution $\tilde f=U(f)=[f*\delta_n]$ (see [5]).

2.4. Theorem. *Let M be an M-bounded convex φ-function with parameter satisfying the condition (Δ^∞) and let $\varphi_n=f*\delta_n$, $n=1,2,\ldots,$*

where $f \in L_M$ and (δ_n) is a δ-sequence. Then $\varphi_n \in L_M$ and

$$\|\varphi_n\|_M \leqslant 2kK\|f\|_M,$$

where k and K are the constants from 2.3.

Consequently, $\tilde{f} = U(f) = [\varphi_n]$ is a sequentially M-integrable distribution. Moreover, if $f_m \in L_M$, $m = 1, 2, \ldots$, and $\|f_m\|_M \to 0$ as $m \to \infty$, then the corresponding sequence of sequentially M-integrable distributions $(\tilde{f}_m)$ is sequentially M-convergent to 0.

Proof. By Proposition 2.2, f is locally integrable. Applying Jensen's inequality and M-boundedness of M, we obtain for arbitrary $\lambda > 0$

$$\varrho_M\left(\frac{\lambda\varphi_n}{4K^2}\right) \leqslant \frac{1}{2}\varrho_M\left(\frac{k}{2K}\lambda f\right) + \frac{1}{2} \quad \text{for} \quad n = 1, 2, \ldots$$

Supposing $\|f\|_M \leqslant 2K/\lambda k$, we thus get $\varrho_M(\lambda\varphi_n/4K^2) \leqslant 1$, whence $\|\varphi_n\|_M \leqslant 4K^2/\lambda$. Hence $\|\varphi_n\|_M \leqslant 2kK\|f\|_M$. The remaining parts of the theorem follow from this inequality, immediately.

3.1. Let us now recall the definitions of spaces D_N and D'_M from [6] and [4], where M is an N-function with parameter and N is the N-function complementary to M.

Namely, D_N is the space of functions $\psi \in C^\infty(R^q)$ such that the derivatives $D^\alpha\psi \in L_N$ for every multiindex α. If $p_\alpha > 0$ are chosen so that $\sum_\alpha p_\alpha = 1$, then

$$\|\psi\|_{(N)} = \sum_\alpha p_\alpha\|D^\alpha\psi\|_N(1+\|D^\alpha\psi\|_N)^{-1}$$

is an F-norm in D_N. As it is well known, convergence (boundedness) in D_N means convergence (boundedness) with respect to every $\|D^\alpha\psi\|_N$, separately. The space of all linear, continuous functionals over D_N is denoted by D'_M. Convergence to 0 of a sequence of elements $T_n \in D'_M$ is defined as uniform convergence $T_n(\psi) \to 0$ over every set B of ψ's, bounded in D_N.

3.2. Let $f \in D^*_M$, i.e., $f = [D^\alpha\varphi_n]$, where (φ_n) is the fundamental sequence given in Def. 1.3. We are going to associate with f a linear continuous functional f^* over D_N. For this purpose we shall denote

$$\langle\varphi, \psi\rangle = \int \varphi(t)\psi(t)\,dt \quad \text{for} \quad \varphi \in L_M,\ \psi \in L_N,$$

and we shall write

$$K_r = \{x \in R^q\colon x = (x_1, \ldots, x_q),\ |x_i| \leqslant r \text{ for } i = 1, 2, \ldots, q\},$$

$r = 1, 2, \ldots$

3.3. Theorem. *Let M be an N-function with parameter satisfying the condition (Δ_2) and such that $\int_K M(t, \lambda)\,dt \to 0$ as $\lambda \to 0$ for every compact $K \subset R^q$. Let N be complementary to M. If*

$$f = [D^\alpha\varphi_n] \in D^*_M$$

with a fundamental sequence (φ_n) such that

$$\varphi_n \in C^\infty(R^q) \cap L_M, \quad \|\varphi_n\|_M \leqslant L \text{ for } \quad n = 1, 2, \ldots$$

and with a multiindex α, then the sequence $(\langle\varphi_n, D^\alpha\psi\rangle)$ is convergent for every $\psi \in D_N$ and $f^*(\psi) = \lim_{n\to\infty}\langle\varphi_n, D^\alpha\psi\rangle$ defines a functional $f^* = F(f) \in D'_M$.

Moreover, the embedding F is continuous from D^*_M with sequential M-convergence to D'_M.

Proof. First, let us remark that due to condition (Δ_2), $C_0^\infty(R^q)$ is dense in D_N (see [6], where this is shown for M independent of the parameter t; the extension to $M(t, u)$ is obvious). Now let $\psi \in D_N$ and let $\tilde{\psi}_r \in C_0^\infty(R^q)$ be such that $\operatorname{supp}\tilde{\psi}_r \subset K_r$ and $\tilde{\psi}_r \to \psi$ in D_N. Then, taking $f \in D^*_M$ and applying Hölder inequality, inequalities (1) and the boundedness assumption $\|\varphi_n\|_M \leqslant L$ for $n = 1, 2, \ldots$ with some multiindex α, we obtain

$$|\langle\varphi_i, D^\alpha\psi\rangle - \langle\varphi_j, D^\alpha\psi\rangle| \leqslant 4L\|D^\alpha(\psi - \tilde{\psi}_r)\|_N + |\langle\varphi_i - \varphi_j, D^\alpha\tilde{\psi}_r\rangle|.$$

Let $\varepsilon > 0$ be given and let us fix r so that $\|D^\alpha(\psi - \tilde{\psi}_r)\|_N < \varepsilon/8L$. Now there exist a multiindex β and functions $\Phi_n \in C^\infty(R^q)$, $n = 1, 2, \ldots$, such that $D^\beta\Phi_n(x) = \varphi_n(x)$ on K_r and (Φ_n) is uniformly convergent in K_r. Then

$$|\langle\varphi_i - \varphi_j, D^\alpha\tilde{\psi}_r\rangle| \leqslant 2\|(\Phi_i - \Phi_j)\chi_{K_r}\|_M\|D^{\alpha+\beta}\tilde{\psi}_r\|_N,$$

where χ_{K_r} is the characteristic function of the set K_r. Now let us take an arbitrary $\eta > 0$; then there is an index i_0 such that $|\Phi_i(t) - \Phi_j(t)| < \eta$ for $i, j > i_0$. Hence

$$\varrho_M(\lambda(\Phi_i - \Phi_j)\chi_{K_r}) \leqslant \int_{K_r} M(t, \eta)\,dt \quad \text{for} \quad i, j > i_0.$$

This shows that $\varrho_M(\lambda(\Phi_i - \Phi_j)\chi_{K_r}) \to 0$ as $i, j \to \infty$ for every $\lambda > 0$. Consequently,

$$\|(\Phi_i - \Phi_j)\chi_{K_r}\|_M \to 0 \quad \text{as} \quad i, j \to \infty.$$

Hence, there is an i_ε such that

$$|\langle\varphi_i - \varphi_j, D^\alpha\tilde{\psi}_r\rangle| < \tfrac{1}{2}\varepsilon \quad \text{for} \quad i, j > i_\varepsilon.$$

Consequently,

$$|\langle\varphi_i, D^\alpha\psi\rangle - \langle\varphi_j, D^\alpha\psi\rangle| < \varepsilon \quad \text{for} \quad i, j > i_\varepsilon.$$

Thus the sequence $(\langle\varphi_n, D^\alpha\psi\rangle)$ is convergent for every $\psi \in D_N$. Denoting $f^*(\psi) = \lim_{n\to\infty} \langle\varphi_n, D^\alpha\psi\rangle$, f^* is obviously linear, and continuity of f^* in D_N follows from the Hölder inequality

$$|f^*(\psi)| \leqslant 2L\|D^\alpha\psi\|_N \quad \text{for} \quad \psi \in D_N.$$

We have still to show that if $f_m \in D_M^*$, $f_m \xrightarrow{M} 0$, then $f_m \to 0$ in D'_M. Let $f_m = [D^\alpha\varphi_{m,n}]$, $\|\varphi_{m,n}\|_M \leqslant L_m$ and $\|\varphi_{m,n}\|_M \to 0$ as $m\to\infty$ uniformly with respect to n. Let B be a bounded set in D_N, i.e., for every β there is a $C_\beta > 0$ such that $\|D^\beta\psi\|_N \leqslant C_\beta$ for all $\psi \in B$. Taking $\varepsilon_m = \sup_n \|\varphi_{m,n}\|_M$ we then obtain

$$|f_m^*(\psi)| \leqslant 2\|\varphi_{m,n}\|_M \|D^\alpha\psi\|_N \leqslant 2\varepsilon_m C_\alpha \to 0 \quad \text{as} \quad m\to\infty$$

for $\psi \in B$, whence $f_m^* \to 0$ in D'_M.

Let us still remark that due to the linearity of the space D'_M, also distributions $f \in \tilde{D}_M$ may be embedded in D'_M.

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On the range of purely atomic probability measures

by

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Abstract. An example is given of some purely atomic probability measure $P = (p_n | n \in N)$, with a range non-homeomorphic to the Cantor ternary discontinuum, such that $p_{n+1} \leqslant p_n$ for all positive integers n and the inequality $p_n > \sum_{i=n+1}^{\infty} p_i$ holds for infinitely many n.

It is well known that the range of non-atomic probability measures is the unit interval I. On the other hand, in the case of a purely atomic probability measure $P = (p_n | n \in N)$ with $p_{n+1} \leqslant p_n$ for each n belonging to the set N of all positive integers, the condition

$$0 < p_n \leqslant \sum_{i=n+1}^{\infty} p_i$$

is necessary and sufficient for the range of P to be the unit interval (e.g., see [1], p. 80). Jim Nymann has proved that if the above inequalities hold for almost all $n \in N$, the range of P is a finite union of some intervals. He has also proved that if

$$p_n > \sum_{i=n+1}^{\infty} p_i$$

for almost all $n \in N$, then the range of P is homeomorphic to the Cantor ternary discontinuum C and asked if the same holds under the weaker assumption that the last inequality is satisfied for infinitely many $n \in N$. The aim of this paper is to construct a counterexample.

Let $P = (p_n | n \in N)$ be a purely atomic probability measure with $p_{n+1} \leqslant p_n$ whenever $n \in N$. Let us extend the mapping $f\colon C \to I$ given by the formula

$$f(x) = \tfrac{1}{2}\sum_{n=1}^{\infty} x_n p_n,$$