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L^p-behaviour of the integral means of analytic functions

by

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Abstract. Various results on L^p-behaviour of power series with positive coefficients are extended to Lipschitz spaces. For example, we have a characterization (decomposition) of these spaces, which enables us to describe an isomorphism of a Lipschitz space onto a solid sequence space and to establish new connections between some classical inequalities concerning Hardy spaces.

1. Introduction. In [15] we have considered some theorems on L^p-behaviour of power series with positive coefficients and their applications to H^p spaces. In this paper we continue the investigation in this direction. First we introduce some notations and then we list some known results from this area.

Throughout the paper let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the open unit disc. Unless specified otherwise, the letters p, q, r, α denote numbers satisfying $0 < p, q \leqslant \infty$, $0 < r < 1$, $0 < \alpha < \infty$. The letter φ always denotes a non-negative increasing function defined on $(0,1]$ for which

(1.1) $$\varphi(tr) \leqslant Ct^{\alpha}\varphi(r), \quad 0 < t < 1,$$

and

(1.2) $$\varphi(tr) \geqslant C^{-1}t^{\beta}\varphi(r), \quad 0 < t < 1,$$

where C and β are positive real numbers. Note that $\beta \geqslant \alpha > 0$.

We use the usual notations for the integral means of f:

$$M_p^p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt, \quad p < \infty,$$

$$M_\infty(r,f) = \sup_t |f(re^{it})|,$$

and we write

$$\|f\|_p = \sup_r M_p(r, f).$$

For our purposes it is convenient to introduce the class $X = H(p, q, \varphi)$ of functions f for which $F \in L^q(0, 1)$, where

$$F(r) = (1-r)^{-1/q}\varphi(1-r)M_p(r, f)$$

and $L^q(0, 1)$ is the usual Lebesgue space. The norm in X is given by

$$\|f\|_X = \|F\|_{L^q}.$$

If $\varphi(r) = r^a$, we write $H(p, q, a)$ instead of $H(p, q, \varphi)$. An account of the properties of $H(p, q, a)$ may be found in [5].

Since $M_2^2(r, f) = \sum |a_n|^2 r^{2n}$, various results on L^p-behaviour of power series with positive coefficients may be expressed in terms of the spaces $H(2, q, a)$. We begin with a result of Hardy and Littlewood [7], Theorem 3.

THEOREM (HL I). *Let* $q \leqslant 2$. *Then* $f \in H(2, q, a)$ *implies*

(1.3) $$\{(n+1)^{-a+1/2-1/q} a_n\}_{n=0}^{\infty} \in l^q.$$

If $q \geqslant 2$, *then* (1.3) *implies* $f \in H(2, q, a)$.

Askey and Boas [1], Theorem 2, have proved the following stronger result.

THEOREM (AB). *The function* f *is in* $H(2, q, a)$ *if and only if*

$$\{(n+1)^{-a-1/q}\|s_n\|_2\}_{n=0}^{\infty} \in l^q,$$

where

$$s_n(z) = s_n f(z) = \sum_{k=0}^{n} a_k z^k.$$

In [15] another characterization of $H(2, q, a)$ is given.

THEOREM (MP). *Let* $I_0 = \{0\}$, $I_n = \{k: 2^{n-1} \leqslant k < 2^n\}$ $(n = 1, 2, \ldots)$ *and*

$$\Delta_n(z) = \Delta_n f(z) = \sum_{k\in I_n} a_k z^k.$$

Then $f \in H(2, q, a)$ *if and only if*

$$\{2^{-na}\|\Delta_n\|_2\}_{n=0}^{\infty} \in l^q.$$

The first implication in Theorem (HL I) is an immediate consequence of the inequality

$$\Big(\sum_{n=0}^{\infty} |b_n| r^n\Big)^{\beta} \geqslant (1-r)^{1-\beta} \sum_{n=0}^{\infty} |b_n|^{\beta} r^n, \quad 0 < \beta \leqslant 1,$$

and may be used (cf. [6]) to prove the following result of Hardy and Littlewood [4], Theorem 6.2.

THEOREM (HL II). *If* $f \in H^p$, $p \leqslant 2$, *then*

$$\sum (n+1)^{p-2}|a_n|^p < \infty.$$

Indeed, this theorem follows from Theorem (HL I) and the inclusion $H^p \subset H(2, p, 1/p - 1/2)$, $p < 2$, which is a special case of another theorem of Hardy and Littlewood [4], Theorem 5.11.

THEOREM (HL III). *Let* $p < q \leqslant \infty$, $s \geqslant p$. *Then*

$$H^p \subset H(q, s, 1/p - 1/q).$$

It is the aim of this paper to extend Theorems (AB) and (MP). The main results are Theorems 2.1 and 2.2. Using some elementary inequalities, we prove that

(1.4) $$\{\varphi(2^{-n})\|\Delta_n\|_p\}_{n=0}^{\infty} \in l^q$$

is a sufficient condition for f to belong to $H(p, q, \varphi)$ (see Theorem 2.1 (a) below). Applying the Riesz projection theorem, we prove that condition (1.4) is also necessary in the case $1 < p < \infty$ (Theorem 2.1 (b)). The counter-examples given in Section 4 show that this equivalence does not hold for the extreme values of p ($p = 1$ or $p = \infty$). However, Theorem 2.2 gives a characterization of $H(p, q, \varphi)$ for $1 \leqslant p \leqslant \infty$ in terms of the (C.1) means of $\sum a_k z^k$. From Theorem 2.2 we derive a generalization of Theorem (AB) (Theorem 2.3). An easy consequence of Theorem 2.1 (a) is Theorem (HL III) (see Corollary 2.1).

We briefly discuss some applications of Theorem 2.1 to multiplier problems. (A more complete discussion will appear in [16].) Using only Theorem 2.1(a) and Khintchine's inequality, we show that the smallest solid superspace containing $H(p, q, \varphi)$, for $2 \leqslant p < \infty$, is $H(2, q, \varphi)$. A profound result of Kisliakov [12] and Theorem 2.1(a) show that this is also valid for $p = \infty$.

Theorem 2.1(b) and a classical result on interpolating polynomials enable us to describe an isomorphism of $H(p, q, \varphi)$, $1 < p < \infty$, onto a solid sequence space (Theorem 2.6). As a corollary we have (for $1 < p < \infty$) Lindenstrauss–Pełczyński's result [13] that $H(p, p, 1/p)$ is isomorphic to l^p.

In Section 3 we give further consequences of the main results. Theorem 3.1 generalizes Theorem 2.1 to the fractional derivatives of f and as a special case includes a stronger version of a theorem of Sledd [18], Theorem 3.2. Section 4 is devoted to the proofs of Theorems 2.1 and 2.2.

2. Main results. For a positive function ψ defined on $(0, 1]$ we denote by $H\Delta(p, q, \psi)$ the space of those functions f for which

$$\lambda = \{\psi(2^{-n})\|\Delta_n\|_p\}_{n=0}^{\infty} \in l^q.$$

We define the norm in $Y = H\Delta(p, q, \psi)$ by

$$\|f\|_Y = \|\lambda\|_{l^q}.$$

In the case $\psi(r) = r^\beta$ for a real number β we write $H\Delta(p, q, \beta) = H\Delta(p, q, \psi)$.

THEOREM 2.1. (a) $H\Delta(p, q, \varphi) \subset H(p, q, \varphi)$.

(b) $H(p, q, \varphi) = H\Delta(p, q, \varphi), 1 < p < \infty$.

(c) $H(1, q, \varphi) \subset H\Delta(p, q, \varphi),\ p < 1$.

(d) *If $p \leqslant 1$ or $p = \infty$, then inclusion* (a) *is proper.*

An inspection of the proof, which is postponed to Section 4, shows that all the inclusion mappings in (a), (b), (c) are continuous. This means, for example, that for $X = H(p, q, \varphi)$, $Y = H\Delta(p, q, \varphi)$ and $1 < p < \infty$ we have

$$C^{-1}\|f\|_X \leqslant \|f\|_Y \leqslant C\|f\|_X.$$

Here and elsewhere the letter C denotes a positive constant which depends only on $p, q, \alpha, \beta, \varphi, \psi$ and need not be the same on each occurrence.

COROLLARY 2.1. *Theorem* (HL III).

Proof. In the usual way (cf. [4], p. 87) the theorem reduces to the case $p = 2$. Let $f \in H^2$, $2 < q \leqslant \infty$, $s \geqslant 2$ and $\alpha = 1/2 - 1/q$. Then

$$\|\Delta_n\|_q \leqslant \|\Delta_n\|_\infty^{1-2/q}\|\Delta_n\|_2^{2/q} \leqslant (2^{n/2}\|\Delta_n\|_2)^{1-2/q}\|\Delta_n\|_2^{2/q} = 2^{n\alpha}\|\Delta_n\|_2.$$

Hence

$$\sum_{n=0}^{\infty} 2^{-ns\alpha}\|\Delta_n\|_q^s \leqslant \sum_{n=0}^{\infty}\|\Delta_n\|_2^s < \infty.$$

Thus $f \in H\Delta(q, s, \alpha)$ and, by Theorem 2.1 (a), $f \in H(q, s, \alpha)$.

In Section 4 we shall prove the following characterization of $H(p, q, \varphi)$. (See also Theorem 5.1.)

THEOREM 2.2. *Let $1 \leqslant p \leqslant \infty$. Then $f \in H(p, q, \varphi)$ if and only if*

$$\{\varphi(1/(n+1))(n+1)^{-1/q}\|\sigma_n\|_p\}_{n=0}^{\infty} \in l^q, \tag{2.1}$$

where

$$\sigma_n(z) = \sigma_n f(z) = \sum_{k=0}^{n}(1 - k/(n+1))a_k z^k.$$

As an application of Theorem 2.2 we prove the following generalization of Theorem (AB).

THEOREM 2.3. *Let $1 < p < \infty$. Then $f \in H(p, q, \varphi)$ if and only if*

$$\{\varphi(1/(n+1))(n+1)^{-1/q}\|s_n\|_p\}_{n=0}^{\infty} \in l^q. \tag{2.2}$$

Proof. The implication (2.2) $\Rightarrow f \in H(p, q, \varphi)$ follows from Theorem 2.2 and the inequality $\|\sigma_n\|_p \leqslant \|s_n\|_p$, $p \geqslant 1$, [19], Ch. IV, p. 145.

Let $f \in H(p, q, \varphi)$, $1 < p < \infty$. By the Riesz projection theorem [19], Ch. VII, Theorem 6.4, $\|\sigma_{2n}\|_p \geqslant C\|s_n\sigma_{2n}\|_p$. Hence

$$\begin{aligned}\|\sigma_{2n}\|_p &\geqslant C\left\|s_n - \frac{z}{2n+1}s_n'\right\|_p \\ &\geqslant C\left(\|s_n\|_p - \frac{1}{2n+1}\|s_n'\|_p\right) \\ &\geqslant C\left(\|s_n\|_p - \frac{n}{2n+1}\|s_n\|_p\right) \geqslant C\|s_n\|_p,\end{aligned}$$

where we have used Bernstein's inequality [19], Ch. X, Theorem 3.15, in the form $\|s_n'\|_p \leqslant n\|s_n\|_p$, $p \geqslant 1$. Now the desired result follows from Theorem 2.2 and inequality (1.2).

A further consequence of Theorem 2.1 is

THEOREM 2.4. *If $f \in H(2, q, \varphi)$, then for almost every choice of signs $\{\varepsilon_n\}$, the function $g(z) = \sum \varepsilon_n a_n z^n$ belongs to $H(p, q, \varphi)$ for all $p < \infty$.*

Proof. Let $f \in H(2, q, \varphi)$. We have to prove that $\operatorname{mes}(T) = 1$, where

$$T = \{t\colon f_t \in H(p, q, \varphi) \text{ for all } p < \infty\},$$

$$f_t(z) = \sum a_n R_n(t) z^n$$

and R_n are the Rademacher functions on $[0, 1]$. To prove this we use the inequality

$$\int_0^1 \|\Delta_n f\|_p\, dt \leqslant C(p)\|\Delta_n\|_2, \quad p < \infty, \tag{2.3}$$

which is an immediate consequence of Khintchine's inequality. We assume that $C(p) > 1$. Put

$$T_{x,p} = \{t\colon \|\Delta_n f_t\|_p \leqslant C(p)^{x+1}\|\Delta_n\|_2\}, \qquad x > 0.$$

Since $f \in H(2, q, \varphi) = H\Delta(2, q, \varphi)$, it follows from the definition of $T_{x,p}$ and Theorem 2.1 (a) that

(2.4) $$T \supset \bigcap_{p<\infty} T_{x,p} \quad \text{for all } x > 0.$$

On the other hand, using inequality (2.3), it may easily be seen that

$$\operatorname{mes}(T_{x,p}) \geqslant 1 - C(p)^{-x}, \qquad x > 0.$$

Applying this to relation (2.4), we conclude that

$$\operatorname{mes}(T) \geqslant 1 - \sum_{m=1}^{\infty} C(m)^{-x} \quad \text{for all } x > 0$$

and consequently $\operatorname{mes}(T) \geqslant 1$. This concludes the proof.

A sequence space A is called *solid* if $\{a_n\} \in A$ and $|b_n| \leqslant |a_n|$ for all $n \geqslant 0$ imply $\{b_n\} \in A$. Regarding $H(p, q, \varphi)$ as being a sequence space, we have

COROLLARY 2.2. *The smallest solid space containing $H(p, q, \varphi)$ for $2 \leqslant p < \infty$ is $H(2, q, \varphi)$.*

In fact, a much stronger result is valid.

THEOREM 2.5. *The smallest solid space containing $H(\infty, q, \varphi)$ is $H(2, q, \varphi)$. Moreover, if $f \in H(2, q, \varphi)$, then there is a sequence $\{b_n\}$ such that $\sum b_n z^n \in H(\infty, q, \varphi)$ and $|b_n| \geqslant |a_n|$ for all $n \geqslant 0$.*

This is an easy consequence of Theorem 2.1 and the following profound result of Kisliakov [12].

THEOREM (K). *For any sequence $\{a_k\}_{k=m}^{n}$ $(0 \leqslant m \leqslant n)$ there is a polynomial $h(z) = \sum_{k=m}^{n} b_k z^k$ satisfying $|b_k| \geqslant |a_k|$ for $m \leqslant k \leqslant n$ and*

$$\|h\|_\infty \leqslant C\Big(\sum_{k=m}^{n} |a_k|^2\Big)^{1/2}.$$

We finish this section by showing that the space $H(p, q, \varphi)$ for $1 < p < \infty$ is isomorphic to the space $l(p, q)$ of those sequences $\{b_n\}_{n=0}^{\infty}$ for which

$$\Big\{\Big(\sum_{k\in I_n} |b_k|^p\Big)^{1/p}\Big\}_{n=0}^{\infty} \in l^q.$$

The norm in $l(p, q)$ is defined in an obvious way (cf. [11]). We point out that $l(p, p) = l^p$. We shall generalize the case $1 < p < \infty$ of the following result of Lindenstrauss and Pełczyński [13].

THEOREM (LPe). *The space $H(p, p, 1/p)$, $1 \leqslant p \leqslant \infty$, is isomorphic to l^p.*

Although our method does not work in the cases $p = 1$ and $p = \infty$ we can explicitly describe an isomorphism of $H(p, q, \varphi)$ onto $l(p, q)$. We need

LEMMA 2.1 ([19], Ch. X, Theorem 7.10). *Let $e_n = \exp(2\pi i/2^{n-1})$, $n \geqslant 0$ and $1 < p < \infty$. Then*

$$C^{-1}\|\Delta_n\|_p \leqslant 2^{-n}\Big(\sum_{k\in I_n} |\Delta_n(e_n^k)|^p\Big)^{1/p} \leqslant C\|\Delta_n\|_p.$$

THEOREM 2.6. *Let $1 < p < \infty$ and $U(f) = \{b_n\}_{n=0}^{\infty}$, where*

$$b_k = \varphi(2^{-n})2^{-n/p}\Delta_n(e_n^k), \qquad k \in I_n.$$

Then U is an isomorphism of $H(p, q, \varphi)$ onto $l(p, q)$.

Proof. Let $X = H(p, q, \varphi)$ and $Z = l(p, q)$. Theorem 2.1(b) and Lemma 2.1 show that $U(X) \subset Z$ and, moreover,

$$C^{-1}\|f\|_X \leqslant \|U(f)\|_Z \leqslant C\|f\|_X.$$

It remains to be shown that $Z \subset U(X)$. Let $\{c_n\}_{n=0}^{\infty} \in Z$. Since

$$\operatorname{card}\{e_n^k\colon k \in I_n\} = 2^{n-1}, \qquad n \geqslant 1,$$

there is a sequence $\{h_n\}_{n=1}^{\infty}$ of polynomials such that h_n is of degree $2^{n-1}-1$ and $h_n(e_n^k) = c_k$ for $k \in I_n$, $n \geqslant 1$. Then $\{c_n\} = U(g)$, where

$$g(z) = c_0 + \sum_{n=1}^{\infty} \varphi(2^{-n})^{-1} 2^{n/p} z^{2^{n-1}} h_n(z).$$

This completes the proof.

3. Some applications to H^p spaces. Theorem 2.1 enables us to establish a connection between some classical inequalities concerning H^p spaces. The following result is due to Hardy and Littlewood and may be found in [8].

THEOREM (HL IV). *Let $p \leqslant 2 \leqslant q < \infty$ and $f(0) = 0$. Then*

$$\int_0^1 (1-r) M_p^2(r, f')\,dr \leqslant C\|f\|_p^2,$$

$$\|f\|_q^2 \leqslant C\int_0^1 (1-r) M_q^2(r, f')\,dr.$$

A consequence of the well-known theorem of Littlewood and Paley [19], Ch. XV, p. 233, is

THEOREM (LP). *Let* $1 < p \leqslant 2 \leqslant q < \infty$. *Then*

$$\sum_{n=0}^{\infty} \|\Delta_n\|_p^2 \leqslant C\|f\|_p^2 \quad \text{and} \quad \|f\|_q^2 \leqslant C\sum_{n=0}^{\infty} \|\Delta_n\|_q^2.$$

It has been shown by Sledd [18], Theorem 3.2, that Theorem (LP) may be sharpened in the following way.

THEOREM (S). *Let* $1 < p \leqslant 2 \leqslant q < \infty$. *Then*

$$\sum_{n=1}^{\infty} \|\Delta_n\|_p^2 \leqslant C\int_0^1 (1-r)M_p^2(r, f')\,dr \tag{3.1}$$

and

$$\int_0^1 (1-r)M_q^2(r, f')\,dr \leqslant C\sum_{n=1}^{\infty} \|\Delta_n\|_q^2. \tag{3.2}$$

We shall prove that the conditions $p \leqslant 2$ and $q \geqslant 2$ are superfluous. More precisely, we have

PROPOSITION 3.1. *The inequalities* (3.1) *and* (3.2) *are valid for* $1 < p < \infty$ *and* $0 < q \leqslant \infty$, *respectively. If* $p < 1$, *then*

$$\sum_{n=1}^{\infty} \|\Delta_n\|_p^2 \leqslant C\int_0^1 (1-r)M_1^2(r, f')\,dr.$$

Combining this with Theorem (HL IV) we obtain the inequality

$$\sum_{n=0}^{\infty} \|\Delta_n\|_p^2 \leqslant C\|f\|_1^2, \quad p < 1.$$

Proposition 3.1 is a consequence of the following more general fact. Here $f^{[\beta]}$ is the fractional derivative of f, i.e.

$$f^{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\beta+1)}{n!} a_n z^n, \quad \beta \geqslant 0.$$

THEOREM 3.1. *Let* $X = H(p, q, \varphi)$, $Y = H\Delta(p, q, \psi)$ $Z = H(1, q, \varphi)$, *where* $\psi(r) = \varphi(r)r^{-\beta}$. *Then* $\|f^{[\beta]}\|_X \leqslant C\|f\|_Y$ *for all* p; $\|f\|_Y \leqslant C\|f^{[\beta]}\|_X$ *for* $1 < p < \infty$; $\|f\|_Y \leqslant C\|f^{[\beta]}\|_Z$ *for* $p < 1$.

For the proof we need the following lemma.

LEMMA 3.1. *Let* $h(z) = \sum_{k=m}^{n} a_k z^k$, $0 \leqslant m \leqslant n$. *Then*

$$\|h\|_p r^n \leqslant M_p(r, h) \leqslant \|h\|_p r^m.$$

Proof. It is easily seen that

$$M_p(r, h) = M_p(1/r, g)r^n,$$

where

$$g(z) = \sum_{k=m}^{n} \bar{a}_k z^{n-k}.$$

Since g is a polynomial and $1/r > 1$, we have

$$M_p(1/r, g) \geqslant M_p(1, g) = \|h\|_p.$$

This proves the left-hand side inequality. The rest of the proof is similar.

Proof of Theorem 3.1. Let

$$K_n(\beta) = \sup_r (1-r)^{\beta+1} M_p(r, \Delta_n^{[\beta]}), \quad n \geqslant 0, \beta \geqslant 0.$$

Then, by a result of Hardy and Littlewood [9],

$$C^{-1}K_n(0) \leqslant K_n(\beta) \leqslant CK_n(0). \tag{3.3}$$

On the other hand, using Lemma 3.1, it is easily proved that

$$C^{-1}K_n(\beta) \leqslant 2^{-n(\beta+1)}\|\Delta_n^{[\beta]}\|_p \leqslant CK_n(\beta).$$

Thus

$$C^{-1}2^{n\beta}\|\Delta_n\|_p \leqslant \|\Delta_n^{[\beta]}\|_p \leqslant C2^{n\beta}\|\Delta_n\|_p. \tag{3.4}$$

Now the desired result follows from Theorem 2.1.

We remark that, for $p \geqslant 1$, inequality (3.4) is of an elementary character and can easily be proved by standard arguments (without appealing to (3.3)). Namely, if $p \geqslant 1$ and $\beta > 0$, we apply Minkowski's inequality (in continuous form) to the relations

$$\Gamma(\beta)\Delta_n(e^{it}) = \int_0^1 (1-r)^{\beta-1}\Delta_n^{[\beta]}(re^{it})\,dr,$$

$$2\pi\Delta_n^{[\beta]}(re^{it}) = \Gamma(\beta+1)\int_0^{2\pi} \Delta_n(e^{i(t+s)})(1-re^{-is})^{-\beta-1}\,ds$$

and obtain

$$\|\Delta_n\|_p \leqslant C\int_0^1 (1-r)^{\beta-1} M_p(r, \Delta_n^{[\beta]})\,dr,$$

$$M_p(r, \Delta_n^{[\beta]}) \leqslant C(1-r)^{-\beta}\|\Delta_n\|_p.$$

These estimates together with Lemma 3.1 give inequality (3.4).

Combining Theorems 2.1 (b) and 2.3 with Theorem (HL III), we obtain

THEOREM 3.2. *Let $p < \infty$ and $q > \max\{1, p\}$. Then*

$$\sum_{n=0}^{\infty} 2^{n(p/q-1)}\|\Delta_n\|_q^p \leqslant C\|f\|_p^p \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} (n+1)^{p/q-2}\|s_n\|_q^p \leqslant C\|f\|_p^p. \tag{3.6}$$

In the case $p \geqslant 2$ inequality (3.5) is weaker than the inequality

$$\sum_{n=0}^{\infty} \|\Delta_n\|_p^p \leqslant C\|f\|_p^p, \quad 2 \leqslant p < \infty.$$

This is a known result of Littlewood and Paley [19], Ch. XV, Theorem 4.22, and can be derived from the inequality

$$\int_0^1 (1-r)^{p-1} M_p^p(r, f')\,dr \leqslant C\|f\|_p^p, \quad 2 \leqslant p < \infty,$$

by use of Theorem 3.1. The last inequality is also a result of Littlewood and Paley [19], Ch. XIV, Theorem 3.24.

It may easily be proved that the dual of $H\Delta(p, q, \beta)$, for $1 < p, q < \infty$, is $H\Delta(p', q', -\beta)$, where p' and q' are the conjugate indices of p and q. The pairing is given by

$$(f, g) = \sum_{n=0}^{\infty} a_n b_n.$$

Using this duality and inequality (3.5), one can prove that

$$\|f\|_p^p \leqslant C\sum_{n=0}^{\infty} 2^{n(p/q-1)}\|\Delta_n\|_q^p, \quad 1 < q < p < \infty. \tag{3.7}$$

An immediate consequence of (3.7) and Theorem 3.1 is the following result of Flett [6], Theorem 1.

COROLLARY 3.1. *Let $1 < q < p < \infty$. Then*

$$\|f\|_p^p \leqslant C\int_0^1 (1-r)^{-p/q+p} M_q^p(r, f^{[1]})\,dr.$$

Now from Theorem 2.3 and Corollary 3.1 it follows that

$$\|f\|_p^p \leqslant C\sum_{n=0}^{\infty} (n+1)^{p/q-p-2}\|s_n^{[1]}\|_q^p, \quad 1 < q < p < \infty. \tag{3.8}$$

From these various inequalities one can deduce some extensions of Theorem (HL II). For example, taking $q = 2$ in (3.6) and (3.8), we obtain the following result of Holland and Twomey [10].

COROLLARY 3.2. *Let $A_n = \sum_{k=0}^{n} (k+1)|a_k|^2$. Then*

$$\sum_{n=0}^{\infty} (n+1)^{-2} A_n^{p/2} \leqslant C\|f\|_p^p \quad \textit{if} \quad p \leqslant 2,$$

and

$$\|f\|_p^p \leqslant C\sum_{n=0}^{\infty} (n+1)^{-2} A_n^{p/2} \quad \textit{if} \quad 2 \leqslant p < \infty.$$

Finally, we remark that a number of inequalities, including the (C.1) means, can be proved if we use Theorem 2.2 and Hardy–Littlewood's and Littlewood–Paley's results. An example is the inequality

$$\sum_{n=0}^{\infty} (n+1)^{p-2}\|\sigma_n\|_1^p \leqslant C\|f\|_p^p, \quad p < 1,$$

which is a consequence of Theorem (HL III) and Theorem 2.2.

4. Proofs of the main results. The proof of Theorem 2.1 is based on L^q-behaviour of the functions

$$F_1(r) = (1-r)^{-1/q}\varphi(1-r)\sup\{\lambda_n r^{2^n}:\ n \geqslant 0\}$$

and

$$F_2(r) = (1-r)^{-1/q}\varphi(1-r)\sum_{n=0}^{\infty} \lambda_n r^{2^n},$$

where $\{\lambda_n\}$ is a sequence of non-negative real numbers.

PROPOSITION 4.1. *Let $F = F_1$ or $F = F_2$. Then*

$$C^{-1}\|F\|_{L^q} \leqslant \|\{\varphi(2^{-n})\lambda_n\}\|_{l^q} \leqslant \|F\|_{l^q}.$$

For the proof we need some lemmas.

LEMMA 4.1. *Let* $\psi(r) = \varphi(r)^q r^{-\varepsilon}$, *where* $q < \infty$, $q\alpha - \varepsilon > -1$ *and* α *satisfies condition* (1.1). *Then*

$$C^{-1}x^{-1}\psi(1/x) \leqslant \int_0^1 \psi(1-r)r^{x-1}dr \leqslant Cx^{-1}\psi(1/x), \qquad x \geqslant 1.$$

Proof. We have

$$I(x) := \int_0^1 \psi(1-r)r^{x-1}dr = x^{-1}\int_0^x \psi(t/x)(1-t/x)^{x-1}dt.$$

Since φ satisfies the conditions (1.1) and (1.2) we have

$$\varphi(t/x) \leqslant C(t^\alpha + t^\beta)\varphi(1/x), \qquad 0 < t < x \qquad (\beta \geqslant \alpha),$$

and consequently

$$I(x) \leqslant Cx^{-1}\psi(1/x)\int_0^x (t^\alpha + t^\beta)^q t^{-\varepsilon}(1-t/x)^{x-1}dt$$
$$\leqslant Cx^{-1}\psi(1/x).$$

This proves the right-hand side inequality. The left-hand side inequality is easy and does not depend on the conditions (1.1) and (1.2).

LEMMA 4.2. *Let* $\psi(r) = \varphi(r)r^{-\varepsilon}$, $\varepsilon \leqslant \alpha$ *and* α *satisfies* (1.1). *Then*

$$C^{-1}\psi(1/x) \leqslant \sup_r \psi(1-r)r^{x-1} \leqslant C\psi(1/x), \qquad x \geqslant 1.$$

The proof is similar to that of Lemma 4.1.

Besides these lemmas we shall use the familiar estimate

(4.1) $$\sum_{n=0}^{\infty} 2^{n\beta} r^{2^n} \leqslant Cr(1-r)^{-\beta}, \qquad \beta > 0.$$

Proof of the proposition. We shall consider only the case $q < \infty$. In the case $q = \infty$ the proof is similar and is based on Lemma 4.2.

Let $q < \infty$. Then

$$F(r)^q \geqslant F_1(r)^q \geqslant (1-r)^{-1}\varphi(1-r)^q \lambda_k^q r^{2^k q} \qquad \text{for all } k$$

and, by (4.1),

$$F(r)^q \geqslant C^{-1}\varphi(1-r)^q \sum_{n=0}^{\infty} 2^n r^{2^n} \lambda_k r^{2^k q}.$$

Hence

(4.2) $$F(r)^q \geqslant C^{-1}\varphi(1-r)^q \sum_{n=0}^{\infty} 2^n \lambda_n^q r^{2^n(1+q)}.$$

On the other hand, from Lemma 4.1 and hypothesis (1.2) it follows that

$$\int_0^1 \varphi(1-r)^q r^{2^n(q+1)}dr \geqslant C^{-1}2^{-n}\varphi(2^{-n}).$$

Combining this with (4.2), we obtain the right-hand side inequality in Proposition 4.1.

To prove the left-hand side inequality, let

$$\eta_n = 2^{n\delta} r^{2^{n-1}}, \qquad \theta_n = 2^{-n\delta}\lambda_n r^{2^{n-1}},$$

where $\delta = \alpha/2$ and α satisfies (1.1). Then

$$\Big(\sum_{n=0}^{\infty} \lambda_n r^{2^n}\Big)^q = \Big(\sum_{n=0}^{\infty} \eta_n\theta_n\Big)^q$$
$$\leqslant \Big(\sum_{n=0}^{\infty} \eta_n\Big)^q \sum_{n=0}^{\infty} \theta_n^q$$
$$\leqslant C(1-r)^{-q\delta} \sum_{n=0}^{\infty} 2^{-nq\delta}\lambda_n^q r^{2^{n-1}q},$$

where we have used inequality (4.1). Hence

$$F(r)^q \leqslant F_2(r)^q \leqslant C\psi(1-r)\sum_{n=0}^{\infty} 2^{-nq\delta}\lambda_n^q r^{2^{n-1}q},$$

where $\psi(r) = \varphi(r)^q r^{-q\delta-1}$. Now the desired result follows from Lemma 4.1.

The assertions (a), (b) and (c) of Theorem 2.1 are a simple consequence of Proposition 4.1 and the following three lemmas.

LEMMA 4.3. *Let* $s = \min\{p, 1\}$. *Then*

$$M_p^s(r, f) \leqslant |a_0|^s + \sum_{n=0}^{\infty} \|\Delta_{n+1}\|_p^s r^{2^n s}.$$

Proof. By the triangle inequality and Lemma 3.1,

$$M_p^s(r, f) \leqslant |a_0|^s + \sum_{n=0}^{\infty} M_p^s(r, \Delta_{n+1})$$
$$\leqslant |a_0|^s + \sum_{n=0}^{\infty} \|\Delta_{n+1}\|_p^s r^{2^n s}.$$

LEMMA 4.4. *Let* $1 < p < \infty$. *Then*

$$\|\Delta_n\|_p r^{2^n} \leqslant CM_p(r, f), \qquad n \geqslant 0.$$

Proof. By the Riesz projection theorem and Lemma 3.1, we have

$$\|\Delta_n\|_p r^{2^n} \leqslant M_p(r, \Delta_n) \leqslant CM_p(r, f).$$

LEMMA 4.5. *If $p < 1$, then*

$$\|\Delta_n\|_p r^{2^n} \leqslant CM_1(r, f).$$

Proof. This follows from the inequality

$$M_p(r, \Delta_n) \leqslant CM_1(r, f), \quad p < 1,$$

[19], Ch. VII, Theorem 6.8.

Proof of Theorem 2.1 (d). We shall consider only the case $p \leqslant 1$. In the case $p = \infty$ the assertion is proved in a similar way by use of [3], Lemma 1.14.

Using the inequality $M_1(r, \Delta_n) \leqslant C(1-r)^{1-1/p} \|\Delta_n\|_p$, $p \leqslant 1$, [4], Theorem 5.9, and Lemma 3.1, it may easily be seen that

$$C\|\Delta_n\|_p \geqslant 2^{n(1/p-1)} \|\Delta_n\|_1, \quad p \leqslant 1.$$

Hence

$$H\Delta(p, q, \varphi) \subset H\Delta(p, \infty, \varphi) \subset H\Delta(1, \infty, \psi), \quad p \leqslant 1,$$

where $\psi(r) = \varphi(r) r^{1/p-1}$. Thus Theorem 2.1 (d) is a consequence of the following stronger result.

PROPOSITION 4.2. *Let $p \leqslant 1$ and $\psi(r) = \varphi(r) r^{1/p-1}$. Then*

$$H(p, q, \varphi) \not\subset H\Delta(1, \infty, \psi).$$

Proof. Let $X = H(p, q, \varphi)$ and $Y = H\Delta(1, \infty, \psi)$ and suppose that $X \subset Y$. Then, by the closed graph theorem (X and Y are complete), there is a positive constant C such that

$$\psi(2^{-n}) \|\Delta_n f\|_1 \leqslant C \|f\|_X, \quad n \geqslant 0, f \in X. \tag{4.3}$$

To obtain a contradiction we use a generalization of an example due to F. Riesz [2], p. 599.

Let

$$f(z) = z^{2^n} (1-z)^{-2/p} (1-z^{2^n})^{2/p} = z^{2^n} (1-z^{2^n})^{2/p} \sum_{k=0}^{\infty} c_k z^k,$$

where

$$c_k = \frac{\Gamma(k+2/p)}{k!\,\Gamma(2/p)} \geqslant C^{-1}(k+1)^{2/p-1}.$$

Hence

$$\Delta_{n+1} f(z) = z^{2^n} \Big(\sum_{k=0}^{2^n-1} c_k z^k\Big), \quad n \geqslant 0.$$

Using the inequality

$$C \Big\| \sum_{k=0}^{m} c_k z^k \Big\|_1 \geqslant \Big| \sum_{k=0}^{m} (k+1)^{-1} c_{m-k+1} \Big|,$$

[2], p. 476, we obtain

$$C \|\Delta_{n+1} f\|_1 \geqslant (n+1) 2^{n(2/p-1)}.$$

This is a contradiction of (4.3) because, as is easily verified,

$$\|f\|_X \leqslant C 2^{n/p} \varphi(2^{-n}).$$

We pass now to the proof of Theorem 2.2. The following lemmas will be needed.

LEMMA 4.6. *Let $1 \leqslant p \leqslant \infty$ and $k = 0, 1, \ldots$ Then*

$$\|\sigma_k\|_p r^k \leqslant M_p(r, f) \leqslant (1-r)^2 \sum_{n=0}^{\infty} \|\sigma_n\|_p (n+1) r^n.$$

Proof. The first inequality follows from the inequality $M_p(r, f) \geqslant M_p(r, \sigma_n)$ and Lemma 3.1; the second follows from the formula

$$f(re^{it}) = (1-r)^2 \sum_{n=0}^{\infty} \sigma_n(e^{it})(n+1) r^n.$$

LEMMA 4.7. *Let $1 \leqslant p \leqslant \infty$, $0 \leqslant k < n$. Then*

$$(n-k+1)\|\sigma_k\|_p \leqslant (n+1)\|\sigma_n\|_p.$$

Proof. We have

$$\|\sigma_n\|_p \geqslant \|\sigma_k \sigma_n\|_p = \Big\| \sigma_k - \frac{z}{n+1} \sigma_k' \Big\|_p \geqslant \|\sigma_k\|_p - \frac{1}{n+1} \|\sigma_k'\|_p \geqslant \|\sigma_k\|_p - \frac{k}{n+1} \|\sigma_k\|_p,$$

where Bernstein's inequality has been used.

LEMMA 4.8. *Let $F(r) = (1-r)^{4-1/q} \varphi(1-r) \sum_{n=0}^{\infty} x_n r^n$, where $\{x_n\}$ is a monotone sequence of non-negative real numbers. Then F belongs to $L^q(0, 1)$*

if and only if

$$\left\{\varphi\left(\frac{1}{n+1}\right)(n+1)^{-3-1/q}x_n\right\}_{n=0}^{\infty}\in l^q.$$

This is obtained from Proposition 4.1 by a Cauchy condensation test type argument.

Proof of Theorem 2.2. Consider first the case $q<\infty$. Let $f\in H(p,q,\varphi)$, $1\leqslant p\leqslant\infty$. Then

$$(1-r)^{-1}\varphi(1-r)^qM_p^q(r,f)=\varphi(1-r)^q\sum_{n=0}^{\infty}M_p^q(r,f)r^n$$
$$\geqslant\varphi(1-r)^q\sum_{n=0}^{\infty}\|\sigma_n\|_p^q r^{n(q+1)},$$

by Lemma 4.6. Now integration yields

$$\infty>\int_0^1(1-r)^{-1}\varphi(1-r)^qM_p^q(r,f)\,dr\geqslant C^{-1}\sum_{n=0}^{\infty}\varphi(1/(n+1))^q(n+1)^{-1}\|\sigma_n\|_p^q,$$

where Lemma 4.1 and condition (1.2) have been used.

Conversely, suppose that (2.1) holds. Let

$$x_n=\sum_{k=}^{n}(k+1)(n-k+1)\|\sigma_k\|_p.$$

Then

$$\sum_{n=0}^{\infty}\|\sigma_n\|_p(n+1)r^n=(1-r)^2\sum_{n=0}^{\infty}x_nr^n. \tag{4.4}$$

On the other hand, using Lemma 4.7, we see that

$$x_n\leqslant C(n+1)^3\|\sigma_n\|_p$$

and therefore

$$\sum_{n=0}^{\infty}\varphi(1/(n+1))^q(n+1)^{-3q-1}x_n^q<\infty.$$

Now we use Lemma 4.8, equality (4.4) and the right-hand side inequality in Lemma 4.6 to conclude that $f\in H(p,q,\varphi)$.

Finally, suppose that $q=\infty$. The implication $f\in H(\infty,p,\varphi)\Rightarrow(2.1)$ is a direct consequence of Lemmas 4.2 and 4.6. To prove the converse, observe that, by Lemma 4.6,

$$M_p(r^2,f)\leqslant 4(1-r)^2\sum_{n=0}^{\infty}\|\sigma_n\|_p(n+1)r^{2n}.$$

Hence

$$M_p(r^2,f)\leqslant 4\sup_n\|\sigma_n\|_pr^n,\quad p\geqslant 1, \tag{4.5}$$

because

$$(1-r)^2\sum_{n=0}^{\infty}(n+1)r^n=1.$$

Now the result follows from (4.5), by Lemma 4.2.

5. Remarks. Let ψ be a positive increasing function defined on $(0,1]$ and let $\varepsilon\geqslant 0$. Then there is a positive function $K(b)$, $b>0$, such that

$$K(b)x^{-\varepsilon}\psi(b/x)\leqslant\int_0^1\psi(1-r)(1-r)^{\varepsilon-1}r^{x-1}\,dr,\quad x\geqslant b, \tag{5.1}$$

and

$$K(b)\psi(b/x)\leqslant\sup_r\psi(1-r)r^{x-1},\quad x\geqslant b. \tag{5.2}$$

On the other hand, condition (1.2) is equivalent to

$$\varphi(2^{-n})\leqslant C\varphi(2^{-n-1}),\quad n\geqslant 0. \tag{5.3}$$

Using these estimates, one can prove that condition (1.2) is necessary for the validity of Theorem 2.1 (a). However, Theorem 2.1 (c) is valid for any increasing function φ. This may be seen from Lemma 4.5 and the proof of Proposition 4.1 if the inequalities (5.1) and (5.2) are used.

We also remark that if $q\geqslant 1$, the proof of Theorem 2.2 can be simplified by using the inequality

$$M_p^q(r,f)\leqslant(1-r)^2\sum_{n=0}^{\infty}\|\sigma_n\|_p^q(n+1)r^n,\quad 1\leqslant q<\infty \tag{5.4}$$

which is an immediate consequence of Lemma 4.6 and Jensen's inequality. In fact, we can do somewhat more. Using (4.5), (5.4) and the fact that the results of Lemmas 4.1 and 4.2 remain true for $\alpha=0$, we obtain the following partial generalization of Theorem 2.2.

THEOREM 5.1. *Let $1\leqslant q\leqslant\infty$ and let φ satisfies* (5.3). *Then f belongs to $H(p,q,\varphi)$ if and only if condition* (2.1) *is satisfied.*

The special case $p = q = \infty$ is obtained by Bennett, Stegenga and Timoney [3], Theorem 1.4.

EXAMPLE. The function

$$\varphi(r) = (1+|\log r|)^{-1}$$

satisfies (1.2) (or, equivalently, (5.3)) but not (1.1). Theorem 5.1 shows that, if $1 < q < \infty$, then $f \in H(p, q, \varphi)$ $(p \geqslant 1)$ if and only if (2.1) holds. Let us observe that the space $H(p, q, \varphi)$ is infinite-dimensional because

$$\int_0^1 (1-r)^{-1}\varphi(1-r)^q dr < \infty, \quad 1 < q < \infty.$$

Consider the function

$$g(z) = \sum_{n=0}^{\infty} z^{2^n}.$$

It is easily verified that

$$M_\infty(r, g) \geqslant C^{-1}\varphi(r)^{-1}$$

and, consequently, $g \notin H(\infty, 2, \varphi)$. On the other hand, $g \in HA(\infty, 2, \varphi)$. This shows that condition (2.1) in Theorem 5.1 cannot be replaced by (1.4).

In conclusion we mention a problem concerning Theorems (LPe) and 2.6.

PROBLEM. If φ satisfies the conditions (1.1) and (1.2), is the space $H(1, q, \varphi)$ isomorphic to $l(1, q)$?

We remark that some properties of $H(1, q, a)$ and $l(1, q)$ are similar. For example, if $1 < q < \infty$, then these spaces are reflexive. The reflexivity of $H(1, q, a)$ follows from a very general result of Muramatu [17], Theorem 1.4. In particular, Muramatu's theorem asserts that the dual of $H(p, q, a)$ is isomorphic to $H(p', q', a)$, where $p \in \{1, \infty\}$ and p', q' are the conjugate indices: $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. See also [5].

After completing this paper we knew that if $\varphi(r) = r^a$ and $q \geqslant 1$, then Theorem 2.1 (b) can be derived from a result of Lizorkin [14], Theorem 3.

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