On some distributional multiplicative products

by

SUSANA ELENA TRIONE (Buenos Aires)

Abstract. In this note we extend the one-dimensional multiplicative products due to B. Fisher (cf. [4], [5], [6] and [7]) and A. González Domínguez and R. Searle (cf. [16]) to certain kinds of multidimensional distributions called causal and anticausal distributions (1.1).

It mainly deals with a method which gives sense to many heterodox multiplicative distributional products which appear often in applications, especially in the quantum theory of fields.

We evaluate some multiplicative products, such as \( P_r^{*} \cdot \delta (r)(P) \), \( r = 0, 1, 2, \ldots ; \)
\( P_r^{*} \cdot \delta (r-1)(P) \cdot \delta (r-1)(P), P_r^{*} \cdot \delta (r-1)(P) \cdot P_r^{*} \cdot \delta (r-1)(P), \) (with \( r \) fixed and \( P \) finite part), where \( P_r^{*} \) are given by formulas (1.4), (1.6) and (1.7), respectively.

1. Definitions. We begin with some definitions. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Consider a non-degenerate quadratic form in \( n \) variables of the form

\[
P = P(x) = x_1^2 + \ldots + x_n^2 - x_{n+1}^2 - \ldots - x_{2n}^2,
\]

where \( n = p + q \). The distributions \( (P \pm \delta)^j \) are defined by

\[
(P \pm \delta)^j = \lim_{s \to 0} \left( P \pm is|s|^{1/j} \right)
\]

where \( s > 0, |s|^j = x_1^2 + \ldots + x_n^2, j \in \mathbb{C} \). The distributions \( (P \pm \delta)^j \) are analytic in \( s \) everywhere except at \( s = -n \), \( s = 0, 1, \ldots, \) where they have simple poles ([8], p. 275).

The distributions \( (P \pm \delta)^j \) are called, respectively, causal and anticausal distributions.

Furthermore, we can write ([8], formulas (2) and (2'), p. 276),

\[
(P \pm \delta)^j = P_+^j + e^{i\pi \delta} P_-^j,
\]

where

\[
P_+^j = \begin{cases} P^j & \text{for } P \geq 0, \\ 0 & \text{for } P < 0, \end{cases} \quad \text{and} \quad P_-^j = \begin{cases} (-P)^j & \text{for } P \geq 0, \\ 0 & \text{for } P < 0. \end{cases}
\]
The distributions $P_+^1$ and $P_+^2$ have two sets of singularities, namely, $\lambda = -1, -2, \ldots, -k, \ldots,$ and $\lambda = -n/2, -n/2 - 1, \ldots, -n/2 - k, \ldots$ If $\lambda = r = 0, 1, 2, \ldots,$ it follows that

$$P_{+1}^r = P_{+2}^r + e^{+i\pi r} P_{-2}^r = P_{+1}^r.$$ 

We shall define (cf. [8], p. 211, formula (7) and (8))

$$\langle \delta^r(P), \varphi \rangle = \int_{R^n} \delta(P) \varphi(x) \, dx = \int_{R^n} \mathcal{V}(0, u_1, \ldots, u_n) \, du_1 \ldots du_n,$$

where

$$\mathcal{V} = \varphi_1(u) D(u).$$

and

$$\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(u_1, \ldots, u_n).$$

We write $P = u_1$ and choose the remaining $u_i$ coordinates (with $i = 2, 3, \ldots, n$) arbitrarily except that the Jacobian of the $u_i$ with respect to the $u_1$ which we shall denote by $D(u_1)$, does not vanish.

Similarly we put

$$\langle \delta^{+r}(P), \varphi \rangle = \int_{R^n} \delta^{+r}(P) \varphi(x) \, dx$$

$$= (-1)^r \int_{R^n} \mathcal{V}^0(0, u_1, \ldots, u_n) \, du_1 \ldots du_n.$$

Finally, we observe that the following formulae are valid:

$$\text{sgn}(P) \mathcal{P}_+^1 = P_{+}^1 - P_{-}^1,$$

and

$$\mathcal{P}^1 = P_{+}^1 + P_{-}^1.$$

2. Introduction. In this paper we generalize the formulae due to A. González Domínguez and R. Scarfello (cf. [10])

$$\frac{1}{\omega} \delta(x) = -\frac{1}{\alpha} \delta'(x),$$

and several formulae due to B. Fisher (cf. [4], [5], [6] and [7]); some of these formulae read

$$\delta(x) = -\frac{1}{(n-1)\omega} \delta(x), \quad \lambda = 0, \pm 1, \pm 2, \ldots;$$

$$\delta^{(r)}(x) = \frac{(-1)^r}{r!} \delta^{(r)}(x), \quad r = 0, 1, 2, \ldots;$$

$$\delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{(2r-1)!} \delta^{(2r-1)}(x), \ldots$$

In this formula $\delta^{(r)}(x)$ is the $r$th derivative of the one-dimensional $\delta$-measure and $\delta^{(r)}$, $\lambda \in \mathbb{C}$, are the distributions defined by the formulae

$$\delta^{(r)} = \begin{cases} \omega^r & \text{for } x > 0, \\ 0 & \text{for } x \leq 0, \end{cases}$$

These one-dimensional formulae will be extended to $n$-dimensional distributions called causal and anticausal distributions (cf. (1.9)).

To evaluate our results we need the following formulae:

$$(P \pm i0)^{+r} = (P \pm i0)^{+r},$$

where $\lambda, \mu \in \mathbb{C}$ and $\lambda, \mu$ and $\lambda + \mu$ are different from $-n/2 - n_0$, $n_0 = 0, 1, 2, \ldots$ and

$$\mathcal{P}^1 = P_{+}^1 \mathcal{P}^{-1} + \frac{(-1)^r}{(r-1)!} \mathcal{P}^{-1} \mathcal{P}^{+r}(P),$$

for $r = 1, 2, \ldots, n \neq n, n+1, \ldots, n+n_0$, $n_0 = 0, 1, 2, \ldots$ (Pf means the finite part).

Formula (2.1) was proved in [12], Theorem 2, formula (i, 3.17), p. 23.

Formula (2.2) appears in [3], p. 577, formula (4.9).

In order to give a sense to our products we shall make use of a special change of variable. We begin by defining the multiplicative product of two distributions "a la Mikusiński" (13), p. 242, § 12.4, formula (1.1). If $S$ and $T$ are distributions, we shall define their multiplicative product by the Mikusiński formula:

$$(S \cdot T)_{n=0} \lim \{S \ast g_n(x) \cdot T \ast g_n(x)\}$$

if the limit exists for every mollifier $g_n(x).$ The symbol $\ast$ denotes, as usual, convolution. By a mollifier we mean a sequence $g_n(x) = ng(nx)$, where the function $g(x)$ has the properties

$$g(x) \geq 0,$$

$$g \in C_0^\infty,$$

$$\int g(x) \, dx = 1,$$

$$g(x) = \delta(\mathcal{I}x),$$

$$\text{supp} g(x) = [-1, 1],$$

$$g(x) \text{ is increasing for } -1 < x < 0 \text{ and decreasing for } 0 < x < 1.$$
critical points; \( \varphi_{\omega_{\alpha}} \) denotes the distribution defined on \( R^n \) by the formula (called the Leray formula, cf. [11], p. 102).

\[
(2.4) \quad \int_{\mathbb{R}^n} \varphi_{\omega_{\alpha}} f(x) \, dx = \int_{-\infty}^{\infty} \varphi_{\alpha} \int_{\partial B_{\alpha}} f(x) \omega_{\alpha}(x, \, dx);
\]

here \( \omega_{\alpha} \) is an \((n-1)\)-dimensional form \( \omega \) defined as follows:

\[
d\omega = d\omega_{\alpha} \wedge d\omega_{\alpha} \wedge \ldots \wedge d\omega_{\alpha};
\]

the manifold \( U(x) = \partial \) has the orientation such that \( \omega_{\alpha}(x, \, dx) > 0 \).

We remark that the one-dimensional products we have considered when applying Leray’s formula are due to A. Braédimas ([2]) and B. Fisher ([4]-[7]).

3. First we shall prove the following formulæ:

\[
(3.1) \quad P^{1+} \cdot P^{-1-1} = -\frac{i}{2} \pi \cos \theta z \cdot \delta(P)
\]

and

\[
(3.2) \quad P^{1+} \cdot P^{-1-1} - P^{-1-1} = P^{-1-1},
\]

where \( \lambda \) and \(-\lambda - 1\) \( \neq -n/2 - h, n \neq 2, h = 0, 1, \ldots \) and \( \lambda \) and \(-\lambda - 1 \neq -1, -2, \ldots, -k, k = 1, 2, \ldots \)

Taking into account formula (1.3), we have \( \lambda \in \mathbb{C} \)

\[
(3.3) \quad (P+i\theta)^{-1} = P^{1+} + e^{i\theta} P^{1-},
\]

and

\[
(3.4) \quad (P+i\theta)^{-1} = P^{-1+1} + e^{i\theta} P^{-1-1}.
\]

By multiplying the left-hand members of (3.3) and (3.4), one verifies that

\[
(3.5) \quad (P+i\theta)^{-1}(P+i\theta)^{-1} = (P+i\theta)^{-2},
\]

The last formula is valid when \( \lambda \) and \(-\lambda - 1\) are complex numbers different from \(-n/2 - h, h = 0, 1, 2, \ldots \), and \( n \neq 2 \) (cf. (2.3)).

We also have from (2.2), for \( n \neq 2 

\[
(3.6) \quad (P+i\theta)^{-2} = \frac{1}{P} - i\pi \delta(P).
\]

From (3.3) and (3.6) we get

\[
(3.7) \quad (P+i\theta) \cdot (P+i\theta)^{-1} = \frac{1}{P} - i\pi \delta(P),
\]

\( \lambda \) and \(-\lambda - 1\) different from \(-n/2 - h, h = 0, 1, \ldots, n \neq 2 \).

On the other hand, by multiplying the right-hand members of (3.3) and (3.4), we obtain the formula

\[
(3.8) \quad \{P^{1+} + e^{i\theta} P^{1-}\} \cdot \{P^{-1+1} + e^{i\theta} P^{-1-1}\}
\]

\[
= P^{1+} \cdot P^{-1+1} - P^{1-1} \cdot P^{-1-1} + (2i\sin \theta) P^{1+} \cdot P^{-1-1},
\]

valid when \( \lambda \) and \(-\lambda - 1\) are complex numbers different from \(-n/2 - k, k = 0, 1, 2, \ldots \)

\( \lambda \) and \(-\lambda - 1\) are complex numbers different from \(-1, -2, \ldots, -k, k = 1, 2, \ldots \)

We remark that the heterodox multiplicative products which appear in (3.8) and the products that we evaluate in what follows are perfectly justified by the Leray formula (2.4) and also by Theorems 8, 9, 10, 11 and 12 of [9].

From (3.7) and (3.8) we get

\[
(3.9) \quad \frac{1}{P} - i\pi \delta(P) = P^{1+} \cdot P^{-1+1} - P^{1-1} \cdot P^{-1-1} + (2i\sin \theta) P^{1+} \cdot P^{-1-1},
\]

\( \lambda \) and \(-\lambda - 1 \neq -n/2 - k, n \neq 2, k = 0, 1, 2, \ldots \) and \( \lambda \) and \(-\lambda - 1 \neq -1, -2, \ldots, -k, k = 1, 2, \ldots \)

It is immediately seen by equating the real and the imaginary parts of both members of (3.9) that

\[
(3.10) \quad P^{1+} \cdot P^{-1+1} - P^{1-1} \cdot P^{-1-1} = \frac{1}{P},
\]

\( \lambda \) and \(-\lambda - 1 \neq -n/2 - k, n \neq 2, k = 0, 1, 2, \ldots \) and \( \lambda \) and \(-\lambda - 1 \neq -1, -2, \ldots, -k, k = 1, 2, \ldots \) and

\[
(3.11) \quad P^{1+} \cdot P^{-1+1} - \frac{i}{2} \pi \cos \theta \lambda \delta(P),
\]

where \( \lambda \) and \(-\lambda - 1 \neq -n/2 - k, n \neq 2, k = 0, 1, \ldots \) and \( \lambda \) and \(-\lambda - 1 \neq -1, -2, \ldots, -k, k = 1, 2, \ldots \) as we wish to prove.

Formulae (3.10) and (3.11) are generalisations of the one-dimensional distributional multiplicative products due to B. Fisher (cf. [7], p. 317, formula (1.1) and [4], formula (1), p. 296, respectively).

\[
\alpha^{1+} \cdot \alpha^{-1-1} = \alpha^{1+} \cdot \alpha^{-1-1} = \alpha^{-1},
\]

and

\[
\alpha^{1+} \cdot \alpha^{1-1} = \frac{1}{\alpha} \pi \cos \pi \alpha,
\]

where \( \lambda \neq 0, \pm 1, \pm 2, \ldots \).
4. In this section we shall prove the following:

\[ P^r_\pm \cdot \delta^{(r)}(P) = \frac{1}{r!} (-1)^r r! \cdot \delta(P), \]

where \( r = 0, 1, 2, \ldots; r \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, \ldots \)

We begin by considering formula (2.2):

\[ (P+\theta)^{-\theta+1} = \frac{1}{P^{\theta+1}} + \frac{(-1)^{\theta+1}}{\theta!} \delta^{(\theta)}(P), \]

for \( \theta = 0, 1, 2, \ldots; \theta \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, 2, \ldots \)

Taking into account (3.7), with \( \lambda = r \), we obtain

\[ (P+\theta)^{r-1}(P+\theta)^{-\theta+1} = (P+\theta)^{-\theta+1} = \frac{1}{P} - i\pi \delta(P), \]

where \( r = 0, 1, 2, \ldots; r - 1 \neq n/2 - h, n \neq 2, h = -1, 0, 1, 2, \ldots \)

From the right-hand member of (3.3), where \( \lambda = r \), and from (4.2) we obtain, taking into account formula (1.5),

\[ (P_\pm^r + e^{ir\theta} P_\pm^r) \cdot (P^{\theta-1}) = \frac{(-1)^{\theta+1}}{\theta!} \delta^{(\theta)}(P), \]

for \( r = 0, 1, 2, \ldots; r \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, 2, \ldots \) We get, by taking the imaginary parts of the right-hand members of (4.3) and (4.4),

\[ P^r_\pm \cdot \delta^{(r)}(P) = (-1)^r r! \delta(P), \]

for \( r = 0, 1, \ldots; r \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, \ldots \) We know that

\[ P^r = P^r_+ + (-1)^r P^r_-, \]

for a non-negative integer. From (4.3) and (4.6), we get

\[ P^r_\pm \cdot \delta^{(r)}(P) = \frac{1}{r!} (-1)^r r! \delta(P), \]

for \( r = 0, 1, \ldots; r \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, \ldots \),

analogously we get

\[ P^r_\pm \cdot \delta^{(r)}(P) = \frac{1}{r!} (-1)^r r! \delta(P), \]

for \( r = 0, 1, \ldots; r \neq n/2 - 1, n/2, \ldots, n/2 + h, h = -1, 0, 1, \ldots \)

Formulae (4.7) and (4.8) are extensions of the one-dimensional formulae due to B. Fisher ([4], formula (2), pp. 297 and 298):

\[ \psi \cdot \delta^{(r)} = \frac{1}{r!} (-1)^r r! \delta, \]

where \( r = 0, 1, 2, \ldots \),

and

\[ \psi \cdot \delta^{(r)} = \frac{1}{r!} (-1)^r r! \delta, \]

where \( r = 1, 2, \ldots \)

5. Here we shall establish the formula

\[ \frac{1}{F^r} \cdot \delta_{\psi}^{(r)}(P) = \frac{1}{2} (-1)^r \left( \frac{r-1}{(2r-1)!} \right) \delta^{(2r-1)}(P), \]

where \( r = 0, 1, \ldots; 2r \neq n/2, n/2+1, \ldots, r \neq n/2, n/2+1, \ldots \)

Formula (5.1) is a generalization of the formula \( \frac{1}{x} \cdot \delta(x) = -\frac{i}{2} \delta'(x) \) due to A. González Domínguez and R. Scarfelli ([10]) and is identical with formula (5.1), p. 17, [9]. From (2.2) we get

\[ (P+\theta)^{-2r} = \frac{1}{P^{2r}} + \frac{(-1)^{2r-1}}{(2r-1)!} i\pi \delta^{(2r-1)}(P), \]

for \( r = 1, 2, \ldots; 2r \neq n/2, n/2+1, \ldots \) Also we have (2.1)

\[ (P+\theta)^{-2r} = [(P+\theta)^{-1}]^2 = (P+\theta)^{-r}(P+\theta)^{2r} = \frac{1}{F^r} \prod_{(r-1)!}^{(2r-1)!} \frac{1}{\delta^{(r-1)}(P), \]

for \( r = 1, 2, \ldots \) and \( 2r \neq n/2 - h, h = 0, 1, \ldots \)

By comparing the real and imaginary parts of the right-hand members of (5.2) and (5.3), we get

\[ \frac{1}{F^r} \cdot \delta^{(r)}(P) = \frac{1}{r!} \left( \frac{(-1)^r}{(2r-1)!} \right) \delta^{(2r-1)}(P), \]

and

\[ \frac{1}{F^r} \cdot \delta^{(r)}(P) = \frac{1}{2r} \left( \frac{(-1)^r}{(2r-1)!} \right) \delta^{(2r-1)}(P), \]

for \( r = 1, 2, \ldots; 2r \neq n/2, n/2+1, \ldots, r \neq n/2, n/2+1, \ldots \)

2 – Studia Mathematica 213.
6. Formula (3.3) for \( \lambda = -r-1/2, r = 0, 1, \ldots \), reads

\[
(P \pm i)^{-r-1/2} = P_+^{-r-1/2} + e^{i \pi (r+1)} P_-^{-r-1/2}.
\]

Therefore, we obtain

\[
(P\pm i)^{-r-1/2} = \left[ P_+^{-r-1/2} \right]^r + e^{i \pi (r+1)} P_-^{-r-1/2} + \left[ e^{i \pi (r+1)} P_-^{-r-1/2} \right]^r + 2 P_+^{-r-1/2} e^{i \pi (r+1)} P_-^{-r-1/2},
\]

for \( r = 0, 1, 2, \ldots \); \( -r-1/2 \neq -n/2 - k, k = 0, 1, \ldots \); \( -2r-1 \neq -n/2 - k \). The left-hand member of (6.2) is (2.1) and (2.2).

\[
(P\pm i)^{-r-1/2} = (P \pm i)^{-r-1/2}, \quad (P \pm i)^{-r-1/2} = (P \pm i)^{-r-1/2},
\]

(6.3)

\[
(P \pm i)^{-r-1/2} = P_+^{-r-1/2} \quad (P \pm i)^{-r-1/2} = P_+^{-r-1/2}, \quad (P \pm i)^{-r-1/2} = P_+^{-r-1/2},
\]

where \( -r-1/2 \neq -n/2 - k, -2r-1 \neq -n/2 - k \) and \( r = 0, 1, 2, \ldots \). By taking the imaginary parts of the right-hand members of (6.2) and (6.3), we obtain

\[
P_+^{-r-1/2} P_-^{-r-1/2} = \frac{(-1)^r}{2(2r)!} \pi \delta^{(2r)}(P),
\]

for \( -r-1/2 \neq -n/2 - k, -2r-1 \neq -n/2 - k \); \( k = 0, 1, \ldots \); \( r = 0, 1, 2, \ldots \). Formula (6.4) is a generalization of the formula

\[
\delta^{(r-1)}(P) = \frac{(-1)^r}{2(2r)!} \pi \delta^{(2r)}(P),
\]

where \( r = 0, 1, \ldots \), due to Fisher ([5], p. 120).

7. We shall extend the formula

\[
\langle \text{sgn}\, P|s| \delta^{(2r)}(s) \rangle = 0,
\]

for \( \lambda > 2r-1 \) and \( r = 0, 1, 2, \ldots \), which appears in [7], p. 318, formula (3.1).

From (6.4) we get

\[
\delta^{(2r)}(P) = \frac{(-1)^r}{2(2r)!} \pi P_+^{-r-1/2} P_-^{-r+1/2},
\]

for \( -r-1/2 \neq -n/2 - k, -2r-1 \neq -n/2 - k \); \( k = 0, 1, \ldots \); \( r = 0, 1, \ldots \). The following formula is valid (\( \lambda \in C \))

\[
\langle \text{sgn}\, P|P_+^\lambda - P_-^\lambda \rangle = P_+^\lambda - P_-^\lambda.
\]

(7.3)

By multiplying the left and the right-hand members of the preceding formula, we get

\[
\langle \text{sgn}\, P|P_+^\lambda \cdot \delta^{(2r)}(P) \rangle = (-1)^r 2(2r)! \pi P_+^{-r-1/2} P_-^{-r+1/2} \cdot P_+^\lambda - P_-^\lambda = 0,
\]

that is to say,

\[
\langle \text{sgn}\, P|P_+^\lambda \cdot \delta^{(2r)}(P) \rangle = 0,
\]

for \( -r-1/2 \neq -n/2 - k, k = 0, 1, \ldots \); \( r = 0, 1, \ldots \). Formula (3.1) is an n-dimensional extension of

\[
\langle \omega|\delta^{(2r+1)}(\omega) = 0,
\]

where \( \omega > 2r \) and \( r = 0, 1, \ldots \), due to Fisher ([7], p. 318, formula (3.1)).

From (6.4) we have, for \( -r-1 \neq -n/2 - k, k = 0, 1, \ldots \); \( -2r-2 \neq -n/2 - k \); \( r = 0, 1, 2, \ldots \),

(8.3)

\[
\delta^{(2r+1)}(P) = (-1)^r e^{i \pi} 2(2r+1)! \pi P_+^{-r-1/2} P_-^{-r+1/2}.
\]

We also know that (\( \lambda \in C \))

\[
\langle \text{sgn}\, P|P_+^\lambda - P_-^\lambda \rangle = P_+^\lambda - P_-^\lambda.
\]

(8.3)

Multiplying the two last equalities, we obtain

\[
\langle \text{sgn}\, P|P_+^\lambda \cdot \delta^{(2r+1)}(P) \rangle = 0,
\]

where \( \lambda \in C \); \( -r-1 \neq -n/2 - k, k = 0, 1, \ldots \); \( -2r-2 \neq -n/2 - k \); \( -2r-2 \neq -r-1 \); \( r = 0, 1, \ldots \).

9. Taking into account formula (7.2), we have

\[
\langle \text{sgn}\, P|P_+^\lambda \cdot P_+^{r+1} = P_+^\lambda - P_-^{r+1} = P_+^{r+1} - P_-^{r+1},
\]

(9.1)

The last formula

\[
\langle \text{sgn}\, P|P_+^\lambda \cdot P_+^{r+1} = P_+^{r+1} - P_-^{r+1},
\]

(9.2)
where \( r = 0, 1, \ldots \), \( i = C \) and \( 2r + \lambda + 1 \neq -1, \lambda \neq -k \) and \( \neq n/2 - k \), \( k = 0, 1, \ldots \), generalizes the equality

\[(|8^n_x| - 2^n_{r+1}) = 2^{n_{r+1}},\]

for \( i > -2r - 2, \lambda \neq -2r - 2 \) and \( r = 0, 1, \ldots \), ([(7), p. 319, formula (4.2)].

10. We shall obtain the distributional multiplicative product

\[(P + \delta)^{2r} = (P + \delta)^{r+1},\]

which generalizes formula (4.1), p. 319 of [7]. First we register the formula

\[(P + \delta)^{2r} = (P + \delta)^{r+1},\]

for \( r = 0, 1, \ldots \). Multiplying the left-hand members of (10.2) and (10.3) we obtain

\[(P + \delta)^{2r} = (P + \delta)^{r+1},\]

for \( \lambda \neq -n/2 - k, k = 0, 1, \ldots \) and \( r = 0, 1, \ldots \). By multiplying the right-hand members of (10.2) and (10.3) we obtain

\[(P + \delta)^{2r} = (P + \delta)^{r+1},\]

for \( \lambda \neq -n/2 - k, k = 0, 1, \ldots \). From (10.4) and (10.5) we conclude

\[(P + \delta)^{2r} = (P + \delta)^{r+1},\]

where \( r = 0, 1, \ldots \; \lambda \neq -n/2 - k, k = 0, 1, \ldots \)

11. In this last paragraph we shall evaluate the product \((P \pm i)^{2r+1} \times (P \pm i)^{2r+1}\). This formula is a particular case of our Theorem 2, formula (1, 3; 17), p. 23 of (12), or, equivalently, formula (2.1). We have

\[(P \pm i)^{2r} = (P \pm i)^{r+1},\]

where \( i \) and \( -1 \) are different from \(-n/2 - k, k = 0, 1, \ldots \). We also have (2.2)

\[(P + \delta)^{2r} = \frac{1}{P} \mp \delta(P),\]

for \( n \neq 2 \). Formulas (11.1) and (11.2) are generalizations of the formulae

\[(\lambda \neq 0, \pm \delta)^{2r+1} = (\lambda \neq 0, \pm \delta)^{r+1} = (\lambda \neq 0, \pm \delta)^{r+1},\]

where \( \lambda \neq 0, \pm 1, \pm 2, \ldots \); due to B. Fischer ([7], p. 324, formula (6.1)).

References


[5] — The product of distributions \( \alpha \neq i^{1/3} \) and \( \alpha \neq r^{1/2} \), Proc. Cambridge Philos. Soc. 71 (1972), 122–130.

[6] — The product of the distributions \( \alpha \neq r \) and \( \delta(r–1)(r) \), ibid. 72 (1972), 201–204.


[10] A. González Domínguez and E. Scalfiello, Nota sobre la fórmula \( \text{sp} \lambda \delta \delta \lambda \text{sp} \), Rev. Un. Mat. Argentina 17 (1969), 33–68.


Received June 3, 1983 (1707)