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On some distributional multiplicative products

by

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Abstract. In this note we extend the one-dimensional multiplicative products due to B. Fisher (cf. [4], [5], [6] and [7]) and A. González Domínguez and R. Scarfiello (cf. [10]) to certain kinds of n-dimensional distributions called causal and anticausal distributions ((1.2)).

It mainly deals with a method which gives sense to many heterodox multiplicative distributional products which appear often in applications, especially in the quantum-theory of fields.

We evaluate some multiplicative products, such as $P_+^r \cdot \delta^{(r)}(P)$, $r = 0, 1, 2, \ldots$; $P_+^{-r-1/2} \cdot P_-^{-r-1/2}$, $\{(\operatorname{sgn} P)|P|^\lambda\} \cdot P^{2r+1}$, $\mathrm{Pf}(1/P^r) \cdot \delta^{(r-1)}(P)$, ($\mathrm{Pf} \overset{\text{def}}{=}$ Finite Part), where P_+, P_-, $\delta^{(r)}(P)$, $(\operatorname{sgn} P)|P|^\lambda$ are given by formulae (1.4), (1.7) and (1.8), respectively.

1. Definitions. We begin with some definitions. Let $x = (x_1, x_2, \ldots, x_n)$ be a point of the n-dimensional Euclidean space R^n. Consider a non-degenerate quadratic form in n variables of the form

(1.1)
$$P = P(x) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2,$$

where $n = p+q$. The distributions $(P \pm i0)^\lambda$ are defined by

(1.2)
$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \{P \pm i\varepsilon |x|^2\}^\lambda,$$

where $\varepsilon > 0$, $|x|^2 = x_1^2 + \ldots + x_n^2$, $\lambda \in \boldsymbol{C}$. The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = -n/2-k$, $k = 0, 1, \ldots$, where they have simple poles ([8], p. 275).

The distributions $(P \pm i0)^\lambda$ are called, respectively, *causal and anticausal distributions.*

Furthermore, we can write ([8], formulas (2) and (2′), p. 276),

(1.3)
$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm i\pi\lambda} P_-^\lambda,$$

where

(1.4)
$$P_+^\lambda = \begin{cases} P^\lambda & \text{for } P \geq 0, \\ 0 & \text{for } P \leq 0, \end{cases} \quad \text{and} \quad P_-^\lambda = \begin{cases} 0 & \text{for } P \geq 0, \\ (-P)^\lambda & \text{for } P \leq 0. \end{cases}$$

The distributions P_+^λ and P_-^λ have two sets of singularities, namely, $\lambda = -1, -2, \ldots, -k, \ldots$, and $\lambda = -n/2, -n/2-1, \ldots, -n/2-k, \ldots$ If $\lambda = r = 0, 1, 2, \ldots$, it follows that

$$(1.5)\qquad (P \pm i0)^r = P_+^r + e^{\pm i\pi r} P_-^r = P^r.$$

We shall define ([8], p. 211, formulae (7) and (8))

$$(1.6)\qquad \langle \delta(P), \varphi \rangle = \int_{P=0} \delta(P)\varphi(x)\,dx = \int_{P=0} \Psi(0, u_2, \ldots, u_n)\,du_2 \ldots du_n,$$

where

$$\Psi = \varphi_1(u) D\left(\tfrac{x}{u}\right)$$

and

$$\varphi_1(u_1, u_2, \ldots, u_n) = \varphi(x_1, \ldots, x_n).$$

We write $P = u_2$ and choose the remaining u_i coordinates (with $i = 2, 3, \ldots, n$) arbitrarily except that the Jacobian of the x_i with respect to the u_i, which we shall denote by $D\left(\frac{x}{u}\right)$, does not vanish.

Similarly we put

$$(1.7)\qquad \langle \delta^{(k)}(P), \varphi \rangle = \int_{P=0} \delta^{(k)}(P)\varphi(x)\,dx = (-1)^k \int_{P=0} \Psi_{u_1}^{(k)}(0, u_2, \ldots, u_n)\,du_2 \ldots du_n.$$

Finally, we observe that the following formulae are valid:

$$(1.8)\qquad (\operatorname{sgn} P)|P|^\lambda = P_+^\lambda - P_-^\lambda,$$

and

$$(1.9)\qquad |P|^\lambda = P_+^\lambda + P_-^\lambda.$$

2. Introduction. In this paper we generalize the formulae due to A. González Domínguez and R. Scarfiello (cf. [10])

$$\frac{1}{x} \cdot \delta(x) = -\frac{1}{2} \cdot \delta'(x),$$

and several formulae due to B. Fisher (cf. [4], [5], [6] and [7]); some of these formulae read

$$x_+^\lambda \cdot x_-^{-1-\lambda} = -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\,\delta(x), \qquad \lambda \neq 0, \pm 1, \pm 2, \ldots;$$

$$x_+^r \cdot \delta^{(r)}(x) = \tfrac{1}{2}(-1)^r r!\,\delta(x), \qquad r = 0, 1, 2, \ldots;$$

$$x^{-r} \cdot \delta^{(r-1)}(x) = \frac{(-1)^r (r-1)!}{2(2r-1)!}\,\delta^{(2r-1)}(x), \ldots$$

In this formula $\delta^{(r)}(x)$ is the rth derivative of the one-dimensional δ-measure and x_+^λ, x_-^λ, $\lambda \in \boldsymbol{C}$, are the distributions defined by the formulae

$$x_+^\lambda = \begin{cases} x^\lambda & \text{for } x > 0, \\ 0 & \text{for } x \leqslant 0, \end{cases} \quad \text{and} \quad x_-^\lambda = \begin{cases} 0 & \text{for } x > 0, \\ |x|^\lambda & \text{for } x \leqslant 0. \end{cases}$$

These one-dimensional formulae will be extended to n-dimensional distributions called causal and anticausal distributions (cf. (1.2)).

To evaluate our results we need the following formulae:

$$(2.1)\qquad (P \pm i0)^\lambda \cdot (P \pm i0)^\mu = (P \pm i0)^{\lambda+\mu},$$

where $\lambda, \mu \in \boldsymbol{C}$, and λ, μ and $\lambda+\mu$ are different from $-n/2-k$, $k = 0, 1, 2, \ldots$; and

$$(2.2)\qquad (P \pm i0)^{-r} = \operatorname{Pf}\frac{1}{P^r} \mp \frac{(-1)^r}{(r-1)!}\pi i \cdot \delta^{(r-1)}(P),$$

for $r = 1, 2, \ldots, r \neq n/2, n/2+1, \ldots, n/2+k$, $k = 0, 1, \ldots$ (Pf means the finite part).

Formula (2.1) was proved in [12], Theorem 2, formula (i, 3; 17), p. 23.

Formula (2.2) appears in [3], p. 577, formula (4.9).

In order to give a sense to our products we shall make use of a special change of variable. We begin by defining the multiplicative product of two distributions "a la Mikusiński" ([1], p. 242, § 12.4, formula (1)). If S and T are distributions, we shall define their multiplicative product by the Mikusiński formula:

$$(2.3)\qquad S \cdot T = \lim_{n\to\infty} \{S * g_n(x)\} \cdot \{T * g_n(x)\}$$

if the limit exists for every mollifier $g_n(x)$. The symbol $*$ denotes, as usual, convolution. By a *mollifier* we mean a sequence $g_n(x) = ng(nx)$, where the function $g(x)$ has the properties

(1) $g(x) \geqslant 0$,
(2) $g \in C_0^\infty$,
(3) $\int_{-\infty}^{\infty} g(x)\,dx = 1$,
(4) $g(x) = g(-x)$,
(5) $\operatorname{supp} g(x) = [-1, 1]$,
(6) $g(x)$ is increasing for $-1 \leqslant x \leqslant 0$ and decreasing for $0 \leqslant x \leqslant 1$.

Let φ_s be a distribution of one variable s and let $u(x) \in C^\infty(R^n)$ be such that the $(n-1)$-dimensional manifold $u(x_1, x_2, \ldots, x_n) = 0$ has no critical points; $\varphi_{u(x)}$ denotes the distribution defined on R^n by the formula (called the *Leray formula*, cf. [11], p. 102).

$$(2.4)\qquad \int_{R^n} \varphi_{u(x)} f(x)\,dx_1 \ldots dx_n = \int_{-\infty}^{\infty} \varphi_s\,ds \int_{u(x)=s} f(x)\,w_u(x,\,dx);$$

here w_u is an $(n-1)$-dimensional form u defined as follows:

$$du \wedge dw = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n;$$

the manifold $u(x) = s$ has the orientation such that $w_u(x, dx) > 0$.

We remark that the one-dimensional products we have considered when applying Leray's formula are due to A. Brédimas ([2]) and B. Fisher ([4]–[7]).

3. First we shall prove the following formulae:

$$(3.1)\qquad P_+^{\lambda}\cdot P_-^{-1-\lambda} = -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\cdot\delta(P)$$

and

$$(3.2)\qquad P_+^{\lambda}\cdot P_+^{-1-\lambda} - P_-^{\lambda}\cdot P_-^{-1-\lambda} = P^{-1},$$

where λ and $-\lambda-1 \neq -n/2-k$, $n \neq 2$, $k = 0, 1, \ldots$ and λ and $-\lambda-1 \neq -1, -2, \ldots, -k$, $k = 1, 2, \ldots$

Taking into account formula (1.3), we have ($\lambda \in \boldsymbol{C}$)

$$(3.3)\qquad (P+i0)^{\lambda} = P_+^{\lambda} + e^{i\pi\lambda} P_-^{\lambda},$$

and

$$(3.4)\qquad (P+i0)^{-\lambda-1} = P_+^{-\lambda-1} + e^{i\pi(-\lambda-1)} P_-^{-\lambda-1}.$$

By multiplying the left-hand members of (3.3) and (3.4), one verifies that

$$(3.5)\qquad (P+i0)^{\lambda}\cdot(P+i0)^{-\lambda-1} = (P+i0)^{-1}.$$

The last formula is valid when λ and $-\lambda-1$ are complex numbers different from $-n/2-k$, $k = 0, 1, 2, \ldots$, and $n \neq 2$ (cf. (2.1)).

We also have from (2.2), for $n \neq 2$,

$$(3.6)\qquad (P+i0)^{-1} = \frac{1}{P} - i\pi\,\delta(P).$$

From (3.5) and (3.6) we get

$$(3.7)\qquad (P+i0)\cdot(P+i0)^{-\lambda-1} = \frac{1}{P} - i\pi\,\delta(P),$$

λ and $-\lambda-1$ different from $-n/2-k$, $k = 0, 1, \ldots$, $n \neq 2$.

On the other hand, by multiplying the right-hand members of (3.3) and (3.4), we obtain the formula

$$(3.8)\qquad \{P_+^{\lambda} + e^{i\pi\lambda} P_-^{\lambda}\}\cdot\{P_+^{-\lambda-1} + e^{i\pi(-\lambda-1)} P_-^{-\lambda-1}\} = P_+^{\lambda}\cdot P_+^{-\lambda-1} - P_-^{\lambda}\cdot P_-^{-\lambda-1} + \{2i\sin\pi\lambda\} P_-^{\lambda}\cdot P_+^{-\lambda-1},$$

valid when λ and $-\lambda-1$ are complex numbers different from $-n/2-k$ and different from $-1, -2, \ldots, -k$, $k = 1, 2, \ldots$

We remark that the heterodox multiplicative products which appear in (3.8) and the other products that we evaluate in what follows are perfectly justified by the Leray formula (2.4) and also by Theorems 8, 9, 10, 11 and 12 of [9].

From (3.7) and (3.8) we get

$$(3.9)\qquad \frac{1}{P} - i\pi\delta(P) = P_+^{\lambda}\cdot P_+^{-\lambda-1} - P_-^{\lambda}\cdot P_-^{-\lambda-1} + \{2i\sin\pi\lambda\} P_+^{-\lambda-1}\cdot P_-^{\lambda},$$

λ and $-\lambda-1 \neq -n/2-k$, $n \neq 2$, $k = 0, 1, 2, \ldots$, λ and $-\lambda-1 \neq -1, -2, \ldots, -k$, $k = 1, 2, \ldots$

It is immediately seen by equalizing the real and the imaginary parts of both members of (3.9) that

$$(3.10)\qquad P_+^{\lambda}\cdot P_+^{-\lambda-1} - P_-^{\lambda}\cdot P_-^{-\lambda-1} = \frac{1}{P},$$

λ and $-\lambda-1 \neq -n/2-k$, $n \neq 2$, $k = 0, 1, \ldots$; λ and $-\lambda-1 \neq -1, -2, \ldots, -k$, $k = 1, 2, \ldots$, and

$$(3.11)\qquad P_+^{-\lambda-1}\cdot P_-^{\lambda} = -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\,\delta(P),$$

where λ and $-\lambda-1 \neq n/2-k$, $n \neq 2$, $k = 0, 1, \ldots$, λ and $-\lambda-1 \neq -1, -2, \ldots, -k$, $k = 1, 2, \ldots$, as we wish to prove.

Formulae (3.10) and (3.11) are generalizations of the one-dimensional distributional multiplicative products due to B. Fisher (cf. [7], p. 317, formula (1.1) and [4], formula (1), p. 296, respectively):

$$x_+^{\lambda}\cdot x_+^{-\lambda-1} - x_-^{\lambda}\cdot x_-^{-\lambda-1} = x^{-1},$$

and

$$x_+^{\lambda}\cdot x_-^{-\lambda-1} = -\tfrac{1}{2}\pi\operatorname{cosec}\pi\lambda,$$

where $\lambda \neq 0, \pm 1, \pm 2, \ldots$

4. In this section we shall prove the following formula:

$$P_+^r \cdot \delta^{(r)}(P) = \tfrac{1}{2}(-1)^r \cdot r! \cdot \delta(P), \tag{4.1}$$

where $r = 0, 1, \ldots, n \neq 2, r \neq n/2-1, n/2, \ldots, n/2+k, k = -1, 0, 1, \ldots$ We begin by considering formula (2.2):

$$(P+i0)^{-(r+1)} = \frac{1}{P^{r+1}} + \frac{(-1)^{r+1}}{r!}\pi i\, \delta^{(r)}(P), \tag{4.2}$$

for $r = 0, 1, 2, \ldots; r \neq n/2-1, n/2, \ldots, n/2+k; k = -1, 0, 1, 2, \ldots$ Taking into account (3.7), with $\lambda = r$, we obtain

$$(P+i0)^r (P+i0)^{-(r+1)} = (P+i0)^{-1} = \frac{1}{P} - i\pi\, \delta(P), \tag{4.3}$$

where $r = 0, 1, 2, \ldots; -r-1 \neq -n/2-k, n \neq 2, k = 0, 1, 2, \ldots$

From the right-hand member of (3.3), where $\lambda = r$, and from (4.2) we obtain, taking into account formula (1.5),

$$\{P_+^r + e^{i\pi r} P_-^r\} \cdot \{P^{-(r+1)} + \frac{(-1)^{r+1}}{r!}\pi i\, \delta^{(r)}(P)\} \tag{4.4}$$

$$= P^r \cdot P^{-(r+1)} + \frac{(-1)^{r+1}}{r!}\pi i\, \delta^{(r)}(P) \cdot P^r,$$

for $r = 0, 1, 2, \ldots, r \neq n/2-1, n/2, \ldots, n/2+k, k = -1, 0, 1, 2, \ldots$ We get, by taking the imaginary parts of the right-hand members of (4.3) and (4.4),

$$P^r \cdot \delta^{(r)}(P) = (-1)^r r!\, \delta(P), \tag{4.5}$$

for $r = 0, 1, \ldots; r \neq n/2-1, n/2, \ldots, n/2+k, k = -1, 0, 1, \ldots$ We know that

$$P^r = P_+^r + (-1)^r P_-^r \tag{4.6}$$

for r a non-negative integer. From (4.5) and (4.6), we get

$$P_+^r \cdot \delta^{(r)}(P) = \tfrac{1}{2}(-1)^r r!\, \delta(P), \tag{4.7}$$

for $r = 0, 1, \ldots, n \neq 2, r \neq n/2-1, n/2, \ldots, n/2+k, k = -1, 0, 1, \ldots$, analogously we get

$$P_-^r \cdot \delta^{(r)}(P) = \tfrac{1}{2} r!\, \delta(P), \tag{4.8}$$

for $r = 0, 1, \ldots, n \neq 2, r \neq n/2-1, n/2, \ldots, n/2+k, k = -1, 0, 1, \ldots$

Formulae (4.7) and (4.8) are extensions of the one-dimensional formulae due to B. Fisher ([4], formula (2), pp. 297 and 298):

$$x_+^r \cdot \delta^{(r)} = \tfrac{1}{2}(-1)^r r!\, \delta,$$

where $r = 0, 1, 2, \ldots$, and

$$x_-^r \cdot \delta^{(r)} = \tfrac{1}{2} r!\, \delta(x),$$

where $r = 1, 2, \ldots$

5. Here we shall establish the formula

$$\mathrm{Pf}\frac{1}{P^r} \cdot \delta^{(r-1)}(P) = \frac{1}{2}(-1)^r \frac{(r-1)!}{(2r-1)!}\delta^{(2r-1)}(P), \tag{5.1}$$

where $r = 0, 1, \ldots; 2r \neq n/2, n/2+1, \ldots, r \neq n/2, n/2+1, \ldots$

Formula (5.1) is a generalization of the formula $\frac{1}{x} \cdot \delta(x) = -\frac{1}{2}\delta'(x)$ due to A. González Domínguez and R. Scarfiello ([10]) and is identical with formula (8.1), p. 17, [9]. From (2.2) we get

$$(P+i0)^{-2r} = \mathrm{Pf}\, P^{-2r} + \frac{(-1)^{2r-1}}{(2r-1)!} i\pi\, \delta^{(2r-1)}(P), \tag{5.2}$$

$r = 1, 2, \ldots; 2r \neq n/2, n/2+1, \ldots$ Also we have ((2.1))

$$(P \pm i0)^{-2r} = [(P \pm i0)^{-r}]^2 = (P \pm i0)^{-r} \cdot (P \pm i0)^{-r} \tag{5.3}$$

$$= \left\{\mathrm{Pf}\frac{1}{P^r}\right\}^r - \frac{\pi^2}{[(r-1)!]^2}\{\delta^{(r-1)}(P)\}^2 \mp$$

$$\mp 2\left\{\mathrm{Pf}\frac{1}{P^r}\right\} \cdot \left\{\frac{(-1)^r}{(r-1)!} i\pi \delta^{(r-1)}(P)\right\},$$

where $r = 1, \ldots, -r$ and $-2r \neq n/2-k, k = 0, 1, \ldots$

By comparing the real and imaginary parts of the right-hand members of (5.2) and (5.3), we get

$$\mathrm{Pf}\, P^{-2r} = \left\{\mathrm{Pf}\frac{1}{P^r}\right\}^2 - \frac{\pi^2}{[(r-1)!]^2}\{\delta^{(r-1)}(P)\}^2, \tag{5.4}$$

and

$$\mathrm{Pf}\frac{1}{P^r} \cdot \delta^{(r-1)}(P) = \frac{1}{2}(-1)^r \frac{(r-1)!}{(2r-1)!}\delta^{(2r-1)}(P), \tag{5.5}$$

for $r = 1, 2, \ldots; 2r \neq n/2, n/2+1, \ldots; r \neq n/2, n/2+1, \ldots$

6. Formula (3.3) for $\lambda = -r-1/2$, $r = 0, 1, \ldots$, reads

$$(6.1) \qquad (P\pm i0)^{-r-1/2} = P_+^{-r-1/2} + e^{\pm i\pi(-r-1/2)} P_-^{-r-1/2}.$$

Therefore, we obtain

$$(6.2) \qquad [(P\pm i0)^{-r-1/2}]^2 = [P_+^{-r-1/2}]^2 + [e^{\pm i\pi(-r-1/2)} P_-^{-r-1/2}]^2 + \\ +2P_+^{-r-1/2} e^{\pm i\pi(-r-1/2)} P_-^{-r-1/2},$$

for $r = 0, 1, 2, \ldots$; $-r-1/2 \neq -n/2-k$, $k = 0, 1, \ldots$; $-2r-1 \neq -n/2-k$. The left-hand member of (6.2) is ((2.1) and (2.2)),

$$(6.3) \qquad [(P\pm i0)^{-r-1/2}]^2 = (P\pm i0)^{-r-1/2}(P\pm i0)^{-r-1/2} = (P\pm i0)^{-(2r+1)} \\ = P^{-(2r+1)} \mp \frac{i\pi(-1)^{2r}}{(2r)!}\delta^{(2r)}(P),$$

where $-r-1/2 \neq -n/2-k$, $-2r-1 \neq -n/2-k$; $k = 0, 1, \ldots$; and $r = 0, 1, 2, \ldots$ By taking the imaginary parts of the right-hand members of (6.2) and (6.3), we obtain

$$(6.4) \qquad P_+^{-r-1/2}\cdot P_-^{-r-1/2} = \frac{(-1)^r}{2(2r)!}\pi\delta^{(2r)}(P)$$

for $-r-1/2 \neq -n/2-k$; $-2r-1 \neq -n/2-k$; $k = 0, 1, \ldots$ and $r = 0, 1, 2, \ldots$ Formula (6.4) is a generalization of the formula

$$x_+^{-r-1/2}\cdot x_-^{-r-1/2} = \frac{(-1)^r}{2(2r)!}\pi\delta^{(2r)}(x),$$

where $r = 0, 1, \ldots$, due to Fisher ([5], p. 125).

7. We shall extend the formula

$$\{\operatorname{sgn} x\,|x|^\lambda\}\cdot\delta^{(2r)}(x) = 0,$$

for $\lambda > 2r-1$ and $r = 0, 1, 2, \ldots$ which appears in [7], p. 318, formula (3.1).

From (6.4) we get

$$(7.1) \qquad \delta^{(2r)}(P) = (-1)^r 2(2r)!\,\pi P_+^{-(r+1/2)}\cdot P_-^{-(r+1/2)},$$

for $-r-1/2 \neq -n/2-k$; $-2r-1 \neq -n/2-k$, $k = 0, 1, \ldots$; and $r = 0, 1, \ldots$ The following formula is valid ($\lambda \in C$)

$$(7.2) \qquad (\operatorname{sgn} P)|P|^\lambda = P_+^\lambda - P_-^\lambda.$$

By multiplying the left and the right-hand members of the preceding formula, we get

$$(7.3) \qquad \{(\operatorname{sgn} P)|P|^\lambda\}\cdot\delta^{(2r)}(P) = (-1)^r 2(2r)!\,\pi\{P_+^{-r-1/2}P_-^{-r-1/2}\}\cdot\{P_+^\lambda - P_-^\lambda\} = 0,$$

that is to say,

$$(7.4) \qquad \{(\operatorname{sgn} P)|P|^\lambda\}\cdot\delta^{(2r)}(P) = 0,$$

for $-r-1/2$ and $-2r-1 \neq -n/2-k$, $k = 0, 1, \ldots$; and $r = 0, 1, \ldots$; $\lambda \in C$, $-r-1/2+\lambda$ and $-2r-1+\lambda \neq -1, -2, \ldots$, and λ, $-2r-1+\lambda \neq -n/2-k$.

8. We shall obtain the following formula:

$$(8.1) \qquad |P|^\lambda\cdot\delta^{(2r+1)}(P) = 0,$$

where $\lambda \in C$, $-r-1 \neq -n/2-k$, $k = 0, 1, \ldots$, $-2r-2 \neq -n/2-k$; and λ, $-r+1+\lambda \neq -k$, and $\neq -n/2-k$, $-2-2r+\lambda \neq -1$, $r = 1, 2, \ldots$ Formula (8.1) is an n-dimensional extension of

$$|x|^\lambda\cdot\delta^{(2r+1)}(x) = 0,$$

where $\lambda > 2r$ and $r = 0, 1, \ldots$, due to B. Fisher ([7], p. 318, formula (3.2)).

From (6.4) we have, for $-r-1 \neq -n/2-k$, $k = 0, 1, \ldots$; $-2r-2 \neq -n/2-k$; $r = 1, 2, \ldots$,

$$(8.2) \qquad \delta^{(2r+1)}(P) = (-1)^{r+1/2} 2(2r+1)!\,\pi P_+^{-r-1}\cdot P_-^{-r-1}.$$

We also know that ($\lambda \in C$)

$$(8.3) \qquad |P|^\lambda = P_+^\lambda + P_-^\lambda.$$

Multiplying the two last equalities, we obtain

$$(8.4) \qquad |P|^\lambda\cdot\delta^{(2r+1)}(P) = 0,$$

where $\lambda \in C$; $-r-1 \neq -n/2-k$, $k = 0, 1, \ldots$; $-2r-2 \neq -n/2-k$; $-2r-2+\lambda \neq -1$, $r = 0, 1, \ldots$

9. Taking into account formula (7.2), we have

$$(9.1) \qquad \{(\operatorname{sgn} P)|P|^\lambda\}\cdot P_+^{2r+1} = \{P_+^\lambda - P_-^\lambda\}\cdot P_+^{2r+1} = P^{\lambda+2r+1}.$$

The last formula

$$(9.2) \qquad \{(\operatorname{sgn} P)|P|^\lambda\}\cdot P_+^{2r+1} = P_+^{\lambda+2r+1},$$

where $r = 0, 1, \ldots, \lambda \in C$ and $2r+\lambda+1 \neq -1$, $\lambda \neq -k$ and $\neq n/2-k$, $k = 0, 1, \ldots$, generalizes the equality

$$\{(\operatorname{sgn} P)\,|x|^{\lambda}\}\cdot x_{+}^{2r+1} = x_{+}^{\lambda+2r+1},$$

for $\lambda > -2r-3$, $\lambda \neq -2r-2$ and $r = 0, 1, \ldots$, ([7], p. 319, formula (4.2)).

10. We shall obtain the distributional multiplicative product $|P|^{\lambda}\cdot P_{+}^{2r}$, which generalizes formula (4.1), p. 319 of [7]. First we register the formula

(10.1) $$(P+i0)^{2r}\cdot(P+i0)^{\lambda} = (P+i0)^{2r+\lambda},$$

where λ and $2r+\lambda$ are complex numbers different from $-n/2-k$, r and $k = 0, 1, \ldots$ From (1.3) and (1.9) we have

(10.2) $$(P+i0)^{\lambda} = (|P|^{\lambda}-P_{-}^{\lambda})+e^{i\pi\lambda}P_{-}^{\lambda},$$

for $\lambda \in C$ and

(10.3) $$(P+i0)^{2r} = P_{+}^{2r}+e^{i2\pi r}P_{-}^{2r},$$

for $r = 0, 1, \ldots$ Multiplying the left-hand members of (10.2) and (10.3) we obtain

10.4) $$(P+i0)^{\lambda}\cdot(P+i0)^{2r} = (P+i0)^{\lambda+2r} = P_{+}^{\lambda+2r}+e^{i\pi(\lambda+2r)}P_{-}^{\lambda+2r},$$

for $\lambda \neq -n/2-k$, $k = 0, 1, \ldots$ and $r = 0, 1, \ldots$ By multiplying the right-hand members of (10.2) and (10.3) we obtain

(10.5) $$[(|P|^{\lambda}-P_{-}^{\lambda})+e^{i\pi\lambda}P_{-}^{\lambda}]\cdot[P_{+}^{2r}+e^{i2\pi r}P_{-}^{2r}] = |P|^{\lambda}\cdot P_{+}^{2r}+e^{i2\pi r}|P|^{\lambda}\cdot P_{-}^{2r}-(e^{i2\pi r}-e^{i\pi(\lambda+2r)}P_{-}^{\lambda+2r}),$$

for $\lambda \in C$, $r = 0, 1, \ldots$, $\lambda \neq -1, -2, \ldots, -k$; $\lambda \neq -n/2-k$ and $k = 0, 1, 2, \ldots$ From (10.4) and (10.5) we conclude

(10.6) $$|P|^{\lambda}\cdot P_{+}^{2r} = P_{+}^{2r+\lambda},$$

where $r = 0, 1, \ldots$; $\lambda \neq -1, -2, \ldots, -k$; $\lambda \neq -n/2-k$, $k = 0, 1, \ldots$

11. In this last paragraph we shall evaluate the product $(P\pm i0)^{\lambda}\times$ $\times(P\pm i0)^{-\lambda-1}$. This formula is a particular case of our Theorem 2, formula (I, 3; 17), p. 23 of [12], or, equivalently, formula (2.1). We have

(11.1) $$(P\pm i0)^{\lambda}\cdot(P\pm i0)^{-\lambda-1} = (P\pm i0)^{-1},$$

where λ and $-\lambda-1$ are different from $-n/2-k$, $n \neq 2$, $k = 0, 1, \ldots$ We also have ((2.2))

(11.2) $$(P+i0)^{-1} = \frac{1}{P}\mp i\pi\,\delta(P),$$

$n \neq 2$. Formulae (11.1) and (11.2) are generalizations of the formulae

$$(x\pm i0)^{\lambda}\cdot(x\pm i0)^{-\lambda-1} = (x\pm i0)^{-1} = x^{-1}\mp i\pi\delta(x)$$

where $\lambda \neq 0, \pm 1, \pm 2, \ldots$; due to B. Fisher ([7], p. 324, formula (6.1)).

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