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A theorem on matrices and its applications in functional analysis

by

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Abstract. We establish a theorem on infinite matrices of real numbers which can be regarded as an abstract "sliding-hump" type result. To illustrate the usefullness of the theorem, we derive several well-known results in functional analysis and measure theory. In particular, we prove the Orlicz–Pettis Theorem and a result of Diestel and Faires on strongly additive vector measures.

Recently there have been several presentations in the literature of abstract "sliding-hump" type results with applications to various topics in functional analysis and measure theory. For example, in the recent book of Diestel and Uhl ([7]), Rosenthal's Lemma ([11]) is used to establish many of the major results of vector measure theory. In the same spirit the Antosik–Mikusiński Diagonal Theorem has been employed to establish many of the classical results of functional analysis and measure theory by sliding-hump type methods (cf. [1], [9], [13]). In this note we present a very simple theorem on infinite matrices of real numbers and show that it too can be viewed as an abstract "sliding-hump" type result. In particular, we indicate how this result can be employed to derive several major theorems of functional analysis and measure theory. Our result is of a much more elementary character than Rosenthal's Lemma since it only concerns matrices of real numbers whereas the Rosenthal Lemma deals with measures and its proof in [7] requires non-trivial methods. There is an elementary proof of the Rosenthal Lemma given in [2] which was motivated by suggestions of Pap. Our result is also of a somewhat simpler nature than the Antosik–Mikusiński Diagonal Theorem in that it deals with matrices of real numbers whereas the diagonal theorem deals with matrices having values in a normed group.

We begin by stating a lemma on matrices of real numbers which will be used in the proof of the abstract sliding-hump theorem.

LEMMA 1. *Let $x_{ij} \in \boldsymbol{R}$ for $i, j \in \boldsymbol{N}$. Then there is a subsequence $\{p_i\}$ such that the diagonal submatrix $x_{p_i p_j} = y_{ij}$ has the property that all elements* *on the diagonal have the same sign, all of the elements over the diagonal have the same sign and all of the elements below the diagonal have the same sign.*

Proof. First we show that there exists a diagonal submatrix of $\{x_{ij}\}$ such that all of the elements below the diagonal have the same sign. In fact, there exists an increasing sequence of positive integers $\{m_{1i}\}$ such that $m_{11}=1$ and the $x_{m_{1i}m_{11}}$ for $i\geqslant 2$ have the same sign. Again there exists a subsequence $\{m_{2i}\}$ of $\{m_{1i}\}$ such that $m_{21}=m_{11}$, $m_{22}=m_{12}$ and the $x_{m_{2i}m_{22}}$ for $i\geqslant 3$ have the same sign. By induction, we select a sequence of sequences such that the jth sequence $\{m_{ji}\}_{i=1}^{\infty}$ is a subsequence of the $(j-1)$st sequence, $m_{jk}=m_{j-1,k}$ for $k=1,\ldots,j$ and the numbers $x_{m_{jk}m_{jj}}$ for $k\geqslant j+1$ have the same sign. Note that the diagonal sequence m_{ii} is a subsequence of each $\{m_{ji}\}_{i=1}^{\infty}$ and the numbers $x_{m_{ii}m_{jj}}$ for $i\geqslant 1+j$ have the same sign. In other words, the columns of the matrix $\{x_{m_{ii}m_{jj}}\}$ starting from the diagonal are nonnegative or nonpositive. Evidently there exists a subsequence $\{m_i\}$ of $\{m_{ii}\}$ such that numbers below the diagonal of the matrix $\{x_{m_im_j}\}$ are of the same sign.

Applying the result above to the transpose matrix $y_{ij}=x_{m_jm_i}$, there is a subsequence $\{k_i\}$ of $\{m_i\}$ such that the elements over the diagonal of $\{x_{k_ik_j}\}$ are all of the same sign. Finally, we take a subsequence $\{p_i\}$ of $\{k_i\}$ such that the elements $x_{p_ip_i}$ have the same sign. The matrix $\{x_{p_ip_j}\}$ then satisfies the required conditions.

We now establish the theorem concerning matrices of real numbers which will serve as our abstract "sliding-hump" result.

THEOREM 2. *Let $\{x_{ij}\}$ be a matrix of real numbers such that each subsequence $\{m_i\}$ has a subsequence $\{n_i\}$ with $\{\sum_{j=1}^{\infty}x_{n_in_j}\}_{i=1}^{\infty}$ bounded. Then for each $\varepsilon>0$ there is a subsequence $\{p_i\}$ of positive integers such that $\sum_{j\neq i}|x_{p_ip_j}|<\varepsilon$ for all $i\in\mathbf{N}$.*

Proof. First we prove the theorem under additional assumptions. Assume x_{ij} also satisfies:

(1) the diagonal $\{x_{ii}\}$ is bounded,

(2) $x_{ij}=0$ if $i<j$, i.e., all elements over the diagonal are zero,

(3) $x_{ij}\geqslant 0$ if $i>j$, i.e., all elements below the diagonal are nonnegative.

By passing to a submatrix if necessary, we may assume that there exists an $M>0$ such that $|\sum_{j=1}^{\infty}x_{ij}|<M$ for all i. By (2) and (3), we have $\sum_{j=1}^{i-1}x_{ij}<|x_{ii}|+M$ so by (1) there is an $M_1>0$ such that $\sum_{j=1}^{i-1}x_{ij}<M_1$ for all i. Set $\varepsilon_j=\varepsilon/2^{j+1}$ for $j\in\mathbf{N}$. We assert that there exist an index j_1 and a subsequence $\{m_{1i}\}$ such that $m_{11}=j_1$ and $x_{m_{1i}m_{11}}<\varepsilon_1$ for $i>1$. For if this is not the case, then for each j there exists an index k_j such that $x_{ij}>\varepsilon_1$ for each $i>k_j$. If $i\geqslant\max\{k_1,\ldots,k_j\}$, then $\sum_{j=1}^{i-1}x_{ij}\geqslant j\varepsilon_1$ so that if j is such that $M_1<j\varepsilon_1$, then $\sum_{j=1}^{i-1}x_{ij}>M_1$ which gives the desired contradiction.

Consider the matrix $\{x_{m_{1i}m_{1j}}\}$. By the same argument as above, there exist an index j_2 and a subsequence $\{m_{2i}\}$ of $\{m_{1i}\}$ such that $m_{21}=m_{11}$, $m_{22}=m_{1j_2}$ and $x_{m_{2i}m_{22}}<\varepsilon_2$ for $i>2$. By induction there exists a sequence of sequences such that $\{m_{k+1,i}\}_{i=1}^{\infty}$ is a subsequence of $\{m_{ki}\}_{i=1}^{\infty}$, $m_{k+1,i}=m_{k,i}$ for $i\leqslant k+1$ and $x_{m_{ki}m_{kk}}<\varepsilon_k$ for $i>k$. Set $p_i=m_{ii}$. Then we have $x_{p_ip_j}<\varepsilon_j$ for $i>j$. Hence $\sum_{j=1}^{i-1}x_{p_ip_j}<\sum_{j=1}^{i-1}\varepsilon/2^{j+1}<\varepsilon/2$ for all i. This establishes the result under the additional hypothesis above.

We now prove the general result under the additional hypothesis (1), i.e., we assume the diagonal is bounded. By using Lemma 1 and passing to a submatrix if necessary, we may assume that the numbers $\{x_{ij}\colon i>j\}$ have the same sign. Set $q_1=1$. Since the rows of the matrix converge to 0, there exists a $q_2>q_1$ such that $|x_{q_1q_2}|<\varepsilon/2^2$. Then there exists a $q_3>q_2$ such that $|x_{q_1q_3}|<\varepsilon/2^3$ and $|x_{q_2q_3}|<\varepsilon/2^3$. By induction, there is a subsequence $\{q_j\}$ such that $|x_{q_iq_j}|<\varepsilon/2^j$ for $i<j$. This implies that $\sum_{j\geqslant i+1}|x_{q_iq_j}|<\varepsilon/2^i$ for all i. By hypothesis there is a subsequence $\{p_i\}$ of $\{q_i\}$ such that $\{\sum_{j=1}^{\infty}x_{p_ip_j}\}$ is bounded.

Consider the matrix $\{y_{ij}\}$ defined by $y_{ij}=|x_{p_ip_j}|$ if $i\geqslant j$ and $y_{ij}=0$ if $i<j$. Clearly $\{y_{ij}\}$ satisfies conditions (1), (2), and (3). From the inequality

$$\sum_{j=1}^{\infty}y_{ij}=\Big|\sum_{j=1}^{i-1}x_{p_ip_j}\Big|+|x_{p_ip_i}|$$
$$\leqslant\Big|\sum_{j=1}^{\infty}x_{p_ip_j}\Big|+\Big|\sum_{j=i+1}^{\infty}x_{p_ip_j}\Big|+2|x_{p_ip_i}|,$$

it follows that $\{y_{ij}\}$ also satisfies the hypothesis of the theorem. By what has been proven above there is a subsequence $\{r_i\}$ of positive integers such that $\sum_{j=1}^{i-1}y_{r_ir_j}<\varepsilon/2$ for all i. Setting $s_i=p_{r_i}$, we have

$$\sum_{j\neq i}^{\infty}|x_{s_is_j}|\leqslant\sum_{j=1}^{i-1}y_{r_ir_j}+\sum_{j=i+1}^{\infty}|x_{s_is_j}|<\varepsilon/2+\varepsilon/2=\varepsilon.$$

Thus, the theorem is established under the additional assumption (1).

We claim that a matrix satisfying the hypothesis of the theorem has a bounded diagonal. If this is not the case, we may assume by passing to a submatrix if necessary that $x_{ii}\to\infty$. Consider the matrix $z_{ij}=|x_{ii}|^{-1}x_{ij}$. Then $\{z_{ij}\}$ has a bounded diagonal and satisfies the hypothesis of the theorem since $\sum_{j=1}^{\infty} z_{n_in_j}=|x_{n_in_i}|^{-1}\sum_{j=1}^{\infty} x_{n_in_j}$. By the hypothesis of the theorem and what has been proven above, there is a subsequence $\{p_i\}$ of positive integers such that $\{\sum_{j=1}^{\infty} x_{p_ip_j}\}$ is bounded and $\sum_{j\neq i}|z_{p_ip_j}|<1/2$. Then

$$1=|z_{p_ip_i}|\leqslant|x_{p_ip_i}|^{-1}\Big|\sum_{j=1}^{\infty}x_{p_ip_j}\Big|+\sum_{j\neq i}|z_{p_ip_j}|$$

and the right-hand side of this inequality is less than 1 for i large since $x_{p_ip_i}\to\infty$. This contradiction establishes the general form of the theorem.

The conclusion of Theorem 2 is very similar to the conclusion of Rosenthal's Lemma for σ-fields as applied to measures which are generated by series ([7], I. 4. 1). That is, if $\{x_{ij}\}$ is such that the series $\sum_j x_{ij}$ is absolutely convergent for each i and P is the power set of N, then $\mu_i(E)=\sum_{j\in E}x_{ij}$ defines a countably additive measure on P. The conclusion of Rosenthal's Lemma for this sequence of measures is that there is a subsequence $\{p_i\}$ such that $\sum_{j\neq i}|\mu_{p_i}|(p_j)=\sum_{j\neq i}|x_{p_ip_j}|<\varepsilon$ for each i; exactly the conclusion of Theorem 2. Note, however, a matrix $\{x_{ij}\}$ can satisfy the hypothesis of Theorem 2 and be such that $\sum_j x_{ij}$ is not convergent. Indeed, the matrix $x_{ij}=1/j$ satisfies the hypothesis of Theorem 2 but not the conditions above.

Even the field version of the Rosenthal Lemma for finitely additive measures does not seem to directly yield Theorem 2. Let $\{x_{ij}\}$ be a real matrix. If Σ is the field of subsets of N consisting of subsets which are either finite or have finite complements, then $\mu_i(E)=\sum_{j\in E}x_{ij}$ for E finite and $\mu_i(E)=-\sum_{j\notin E}x_{ij}$ for E infinite defines a finitely additive measure Σ. However, to apply Rosenthal's Lemma to this sequence of measures requires that the measures be uniformly bounded, and if $\{x_{ij}\}$ is the matrix $x_{ij}=1/j$, each of the μ_i is unbounded, but Theorem 2 is applicable to the matrix $\{x_{ij}\}$.

From these observations it is clear that Theorem 2 may be applicable to situations where the Rosenthal Lemma is not applicable. The applications which follow in the sequel will clearly indicate that Theorem 2 can be regarded as an abstract "sliding-hump" result in much the same spirit as the Rosenthal Lemma.

Note that Theorem 2 cannot be generalized to matrices with values in an infinite dimensional normed space. For example, if e_n is the nth unit vector in c_0 and if $x_{ij}=e_j$ when $i\geqslant j$ and $x_{ij}=0$ if $i<j$, then x_{ij} satisfies the hypothesis of the lemma, but the conclusion clearly does not hold with $\varepsilon=1$.

As a first application of Theorem 2 we derive a result of Diestel and Faires on strongly additive vector measures ([6], [7], I.4.2). Actually, using Theorem 2, we derive a more general result which contains the Diestel–Faires Theorem as well as a result of Rosenthal ([12]) as special cases. For this we require two preliminary lemmas.

First we need a result of Drewnowski on strongly additive measures ([8]). Recall that if Σ is a σ-algebra and X is a normed space, an additive set function $\mu\colon\Sigma\to X$ is strongly additive if whenever $\{E_j\}$ is a disjoint sequence from Σ, then $\mu(E_j)\to 0$. Concerning such measures, Drewnowski has proved

LEMMA 3 (Drewnowski). *Let Σ be a σ-algebra of subsets of S and let $\mu_m\colon\Sigma\to X$ be strongly additive. If $\{E_k\}$ is a disjoint sequence from Σ, then there is a subsequence $\{E_{m_k}\}$ such that each μ_m is countably additive on the σ-algebra generated by the $\{E_{m_k}\}$.*

Drewnowski does not explicitly state this result in [8] but the proof follows from his lemma and the argument on page 728 of [8] (see also Proposition 2). Diestel and Uhl state the result for a single measure ([7], I.6).

We also require the following technical lemma on infinite matrices of real numbers.

LEMMA 4. *Let $z_{ij}\in \boldsymbol{R}$ be such that there exists $\delta>0$ with*

$$\sup_i\sum_{j=1}^{\infty}|z_{ij}|<\infty,\ |z_{ii}|\geqslant\delta\quad\textit{and}\quad\sum_{j\neq i}|z_{ij}|\leqslant\delta/2\ \textit{for each}\ i.$$

Then

$$\sup_i\Big|\sum_{j=1}^{\infty}t_jz_{ij}\Big|\geqslant\|\zeta\|\,\delta/2\quad\textit{for}\quad\zeta=\{t_j\}\in l^{\infty},$$

where $\|\zeta\|=\sup_j|t_j|$.

Proof. For each i,

$$\Big|\sum_{j=1}^{\infty}t_jz_{ij}\Big|\geqslant|t_iz_{ii}|-\Big|\sum_{j\neq i}t_jz_{ij}\Big|$$
$$\geqslant|t_i|\,\delta-\|\zeta\|\sum_{j\neq i}^{\infty}|z_{ij}|\geqslant|t_i|\,\delta-\|\zeta\|\,\delta/2.$$

Taking the sup over i implies $\sup_i|\sum_{j=1}^{\infty}t_jz_{ij}|\geqslant\|\zeta\|\,\delta/2$.

The operator interpretation of Lemma 4 is the following: the matrix $\{z_{ij}\}$ induces a linear operator $U\colon l^\infty\to l^\infty$ by $U\zeta = \{\sum_{j=1}^{\infty} t_j z_{ij}\}$. The conditions of Lemma 4 insure that U is continuous and $\|U\zeta\| \geqslant \|\zeta\|\delta/2$ so that U has a bounded inverse.

We now prove our first main theorem. Throughout the remainder of the paper X will denote a Banach space with dual X'. The duality between X and X' will be denoted by $\langle\ ,\ \rangle$. $\boldsymbol{N}$ will denote the positive integers and for any set S, $P(S)$ will denote the power set of S. If $J\subseteq \boldsymbol{N}$, $l^\infty(J)$ will denote the subspace of l^∞ which consists of those sequences which vanish outside of J.

THEOREM 5. *Let $\mu\colon P(\boldsymbol{N})\to X$ be bounded and finitely additive. If $\{\mu(j)\}$ does not converge to 0, then there is a subsequence $\{m_k\}$ such that for any subsequence $\{p_k\}$ of $\{m_k\}$, $T\zeta = \int_J \zeta\, d\mu$ defines a topological isomorphism $T\colon l^\infty(J)\to X$, where $J = \{p_k\colon k\in \boldsymbol{N}\}$.*

Proof. There are a subsequence $\{l_k\}$ and a $\delta>0$ such that $\|\mu(l_k)\|\geqslant\delta$. For notational convenience assume $l_k = k$. Pick $x_k'\in X'$ such that $\|x_k'\| = 1$ and $\langle x_k', \mu(k)\rangle = \|\mu(k)\|$. Now for each m, $x_m'\mu\colon P(\boldsymbol{N})\to \boldsymbol{R}$ defined by $x_m'\mu(E) = \langle x_m', \mu(E)\rangle$ defines a bounded finitely additive measure. By Drewnowski's Lemma 3, there is a subsequence $\{m_k\}$ such that each $x_m'\mu$ is countably additive on the σ-algebra Σ generated by $\{m_k\}$. Put $x_{ij} = x_{m_i}'\mu(m_j)$. Then x_{ij} satisfies the conditions of Theorem 2 since μ is bounded and each $x_m'\mu$ is countably additive on Σ. Thus, there is a subsequence $\{n_k\}$ of $\{m_k\}$ such that $\sum_{j\neq 1}^{\infty} |x_{n_i}'\mu(n_j)| < \delta/2$ for each i. If $\{p_k\}$ is any subsequence of $\{n_k\}$ and $J = \{p_k\colon k\in \boldsymbol{N}\}$, define $T\colon l^\infty(J)\to X$ by $T\zeta = \int_J \zeta\, d\mu$. By the countable additivity of $x_{p_k}'\mu$ on Σ,

$$\|T\zeta\|\geqslant |\langle x_{p_k}', T\zeta\rangle| = \Big|\int_J \zeta\, dx_{p_k}'\mu\Big| = \Big|\sum_{j=1}^{\infty} t_{p_j} x_{p_k}'\mu(p_j)\Big|$$

for $\zeta = \{t_j\}\in l^\infty$. If $z_{ij} = x_{p_i}'\mu(p_j)$, then Lemma 4 implies $\|T\zeta\|\geqslant\|\zeta\|\delta/2$.

As an immediate consequence of Theorem 5 we obtain the following result of Diestel and Faires ([7], I.4.2, [6]).

COROLLARY 6 (Diestel–Faires). *Let Σ be a σ-algebra of subsets of S and $v\colon \Sigma\to X$ be bounded, finitely additive but not strongly additive. Then X contains a subspace (topologically) isomorphic to l^∞.*

Proof. If v is not strongly additive, there are a disjoint sequence $\{E_j\}\subseteq\Sigma$ and a $\delta>0$ such that $\|v(E_j)\|\geqslant\delta$. Define $\mu\colon P(\boldsymbol{N})\to X$ by $\mu(A) = v(\bigcup_{j\in A} E_j)$. Then μ satisfies the conditions of Theorem 5 and the result is immediate.

Similarly, we obtain as a corollary of Theorem 5 the following result of Rosenthal ([12]).

COROLLARY 7 (Rosenthal). *Let $T\colon l^\infty\to X$ be bounded and linear. If there is an infinite $I\subseteq \boldsymbol{N}$ such that T restricted to $c_0(I)$ is an isomorphism, then there is an infinite $J\subseteq I$ such that T restricted to $l^\infty(J)$ is an isomorphism.*

Proof. Define $\mu\colon P(\boldsymbol{N})\to X$ by $\mu(A) = T(C_A)$, where C_A denotes the characteristic function of A. Then μ is bounded, finitely additive and $T\zeta = \int_{\boldsymbol{N}} \zeta\, d\mu$. By hypothesis there is a $\delta>0$ such that $\|\mu(i)\| = \|TC_{\{i\}}\|\geqslant\delta$ for $i\in I$. Thus, Theorem 5 gives the result.

Rosenthal actually obtains a more general result than Corollary 7 in Proposition 1.2 of [12]. For the countable case of his result which is given in Corollary 7 our methods are much simpler.

In [13], Swartz used the Antosik–Mikusiński Diagonal Theorem to derive several well-known results of Bessaga and Pełczyński, Diestel, and Pełczyński. These results were obtained as corollaries of Lemma 3 of [13]. We now indicate how Lemma 3 of [13] can be obtained from Theorem 2. A series $\sum x_j$ in X is said to be *weakly unconditionally convergent* (w.u.c.) if $\sum |\langle x', x_m\rangle| < \infty$ for each $x'\in X'$ ([3]). Recall that if $\sum x_m$ is w.u. c. and X is a B-space, then for each $\{t_j\}\in c_0$ the series $\sum t_j x_j$ converges, and the map $T\colon \{t_j\}\to\sum t_j x_j$ defines a bounded linear operator from c_0 into X ([3], [13]). Also, we have $\sup\{\sum|\langle x', x_m\rangle|\colon \|x'\|\leqslant 1\}<\infty$ for a w.u.c. series $\sum x_m$ ([3], [13]).

THEOREM 8. *Suppose that X is a Banach space which contains a w.u.c. series $\sum x_m$ with $\|x_m\|\geqslant\delta>0$ for each m. Then there is a subsequence $\{m_k\}$ such that for any subsequence $\{p_k\}$ of $\{m_k\}$, $T\zeta = \sum_{j=1}^{\infty} t_j x_{p_j}$, $\zeta = \{t_j\}\in c_0$, defines a topological isomorphism T of c_0 into X.*

Proof. Pick $x_m'\in X'$ such that $\|x_m'\| = 1$ and $\langle x_m', x_m\rangle = \|x_m\|$. If $x_{ij} = \langle x_i, x_j\rangle$, then x_{ij} satisfies the condition of Theorem 2. Hence, there is a subsequence $\{m_k\}$ with $\sum_{j\neq i}^{\infty} |\langle x_{m_i}', x_{m_j}\rangle| < \delta/2$ for each i. Let $\{p_k\}$ be any subsequence of $\{m_k\}$. If T is defined as above, then by Lemma 4 $\|T\zeta\|\geqslant \sup|\langle x_p', T\zeta\rangle|\geqslant\|\zeta\|\delta/2$ for $\zeta\in c_0$.

Theorem 8 can now be used to derive a classic result of Bessaga and Pełczyński on w.u.c. series ([3], [7], I.4.5), a result of Diestel on strongly additive vector measures ([5], [7], I.4.2), and a result of Pełczyński on unconditionally converging operators ([10]). We refer the reader to [13] for details.

Finally, we show that Theorem 2 can be used to derive one of the most important and interesting theorems of functional analysis, namely, the Orlicz–Pettis Teorem ([7], I.4.4).

THEOREM 9 (Orlicz–Pettis). *If the series $\sum x_m$ in X is weak subseries convergent, then it is norm subseries convergent.*

Proof. It suffices to show $\|x_m\| \to 0$, for once this is established the proof may be completed in the usual fashion (cf. [4], IV.1.1). Let $\varepsilon > 0$.

By replacing X by the closed subspace generated by the $\{x_m\}$ we may assume that X is separable. Pick $x'_m \in X'$, $\|x'_m\| = 1$, such that $\langle x'_m, x_m\rangle = \|x_m\|$. By the Banach–Alaoglu Theorem there exist a subsequence $\{x'_{m_k}\}$ and $x' \in X'$ such that $x'_{m_k} \to x'$ weak*. If $x_{ij} = \langle x'_{m_i} - x', x_{m_j}\rangle$, then for any $A \subseteq N$ $|\sum_{j\in A} x_{ij}| \leqslant 2\|\sum_{j\in A} x_{m_j}\|$, where $\sum_{j\in A} x_{m_j}$ is the weak sum of the corresponding subseries. Thus, $\{x_{ij}\}$ satisfies the condition of Theorem 2. Let $\{m_k\}$ be the subsequence of Theorem 2. Then

$$
\begin{aligned}
(4)\qquad \|x_{m_k}\| &\leqslant |\langle x'_{m_k} - x', x_{m_k}\rangle| + |\langle x', x_{m_k}\rangle| \\
&\leqslant \sum_{j\neq k}^{\infty} |\langle x'_{m_k} - x', x_{m_j}\rangle| + \Big|\sum_{j=1}^{\infty} \langle x'_{m_k} - x', x_{m_j}\rangle\Big| + |\langle x', x_{m_k}\rangle| \\
&< \varepsilon + \Big|\langle x'_{m_k} - x', \sum_{j=1}^{\infty} x_{m_j}\rangle\Big| + |\langle x', x_{m_k}\rangle|.
\end{aligned}
$$

Now the second term on the right-hand side of (4) goes to 0 since $x'_{m_k} - x' \to 0$ weak*, and the third term on the right-hand side of (4) goes to 0 since $x_m \to 0$ weakly. Thus, (4) shows that $\{x_m\}$ has a subsequence which goes to 0, and since the argument can be applied to any subsequence, this implies that $x_m \to 0$.

In conclusion, if one examines the development of vector measure theory as given in [7], I.4, it can be concluded that Theorem 2 can be considered as a reasonable, elementary substitute for the Rosenthal Lemma.

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