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On operator-valued analytic functions with positive real part whose logarithm belongs to a C_p class

by

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and

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Dedicated to Professor Jan Mikusiński

Abstract. The main result in this paper (Theorem 2) is a representation formula of the type $F(z) = \exp(A+B(z))$ for operator valued functions $F(z)$, where: A is a Hermitian operator and

$$B(z) = (i/2\pi)\int_{-\pi}^{\pi}(e^{it}+z)(e^{it}-z)^{-1}G(t)\,dt$$

for G a suitable operator-valued function. The conditions of F for the existence of this representation are the following: (i) $F(z)$ is analytic on $|z|<1$ with values in the bounded invertible operators on a Hilbert space; (ii) $\operatorname{Re}F(z)\geqslant 0$; (iii) $F(0)-I\in C_p$, the von Neumann–Schatten class of order $1\leqslant p<+\infty$; (iv) $\operatorname{Im}F(z)\in C_p$, and (v)

$$\int \|\operatorname{Im}\log F(re^{it})\|_p\,dt \leqslant C$$

for all $0<r<1$ and some C.

Furthermore, the G appearing in the expression for $B(z)$ satisfies: $G(t)$ is Hermitian and $\|G(t)\|\leqslant \pi/2$ for almost all t.

Other related results are also proven.

Introduction and notation. We begin by quoting some definitions and recording the notation used in this article. $\boldsymbol{C}$ designates the field of complex numbers and $D = \{z\in \boldsymbol{C}\colon |z|<1\}$ the unit disc. H denotes a separable complex Hilbert space. With (ξ,η) we designate the scalar product of two vectors ξ and η, with $\|\xi\|$ the norm of ξ. $L(H)$ designates the Banach space of all bounded linear operators on H, $\|A\|$ designates the norm of A: $\|A\| = \sup\{\|A\xi\|\colon \|\xi\| = 1\}$. I and 0 designate, respectively, the unit and the zero operator of H. $\operatorname{Re}A = \frac{1}{2}(A+A^*)$ and $\operatorname{Im}A = 1/2i(A-A^*)$ (where A^* designates the adjoint of A) designate, respectively, the real and the imaginary parts of A.

In this paper we are concerned with the classes $C_p(H)$ of linear operators in H which were introduced by von Neumann and Schatten [8].

DEFINITION. (See [9], p. 75.) When $1 \leqslant p < \infty$, $C_p(H)$ is the set of all operators A in $L(H)$ which satisfy the following condition: for each orthonormal basis $\{\varphi_k\}$ in H,

$$\sum_{k=1}^{\infty} |(A\varphi_k, \varphi_k)|^p < \infty.$$

We shall put by definition $C_\infty(H) = L(H)$. It is easy to see that $C_p(H)$ is a linear subspace of $L(H)$ and that $C_p(H) \subset C_q(H)$ if $1 \leqslant p \leqslant q \leqslant \infty$.

The operators belonging to $C_1(H)$ are called *trace class* operators or, equivalently, *nuclear* operators. If $A \in C_1(H)$, then the *trace* of A, denoted by $\operatorname{tr} A$, is defined by

$$\operatorname{tr} A = \sum_{k=1}^{\infty} (A\varphi_k, \varphi_k),$$

where $\{\varphi_k\}$ is an orthonormal basis in H. $\operatorname{tr} A$ depends only on A (not on the choice of the orthonormal basis) (see [9], Lemma 2.2.1, p. 82).

For $1 \leqslant p < \infty$, $C_p(H)$ is a Banach space with norm

$$\|A\|_p = (\operatorname{tr} |A|^p)^{1/p},$$

where $|A| = (A^*A)^{1/2}$, (see [9], Def. 2.3.2, p. 86 and Th. 2.3.8, p. 93). If $p = \infty$ and $A \in C_\infty(H) = L(H)$, we write $\|A\|_\infty = \|A\|$.

The following relations are valid:

(a) If $A \in C_p(H)$ and $B \in L(H)$, then AB and BA belong to $C_p(H)$ and

$$\|AB\|_p \leqslant \|A\|_p \|B\|, \qquad \|BA\|_p \leqslant \|B\| \; \|A\|_p;$$

(b) If $A \in C_p(H)$, then for every natural integer n, $A^n \in C_p(H)$ and $\|A^n\|_p \leqslant \|A\|_p^n$.

In this note (as well as in our paper [1]) a key role is played by Definition (1) below of the logarithm of an operator $A \in L(H)$ with positive real part. For the interesting properties of this definition, cf. [1], pp. 85–88. The main result of this note is Theorem 2. Its thesis (formula (13)) gives an exponential representation of an operator-valued function $F\colon D \to L(H)$, analytic and of positive real part, whose logarithm belongs to a C_p class. The appearance of these C_p classes is one novelty of Theorem 2, whose thesis is formally identical to formula (3.6), p. 91 of [1]. Another novelty is condition (v). The new conditions (iii), (iv) and (v) entail that the operator-valued function $G(t)$ which appears in the right-hand side of (13) is Bochner integrable. This integrability, which is essential for the proof of Theorems 4, 5 and 6, is not shared by the analogous function $G(t)$ which appears in the right-hand side of formula (3.6) of [1]. The proof of Theorem 2 is essentially based on Lemma 1 and on Theorem 1, which are, respectively, operator-valued versions of classical theorems of Plessner and of Herglotz.

The case $p = 1$ of Theorem 2 (and of its particular case Theorem 3) is especially interesting. This is connected with the fact that the C_1 class coincides with the class of nuclear or trace-class operators. This has a consequence that, when $p = 1$, a determinant can be defined for the operator-valued function $F(z)$ which appears in the left-hand member of (13) (and of (21)). This determinant can be expressed as an infinite product (formulae (28) and (29)). In the particular case in which the Hilbert space H is finite-dimensional, (28) and (29) reduce to known formulae for the determinant of matrices analytical and of positive real part (respectively positive-real or impedance matrices) in D.

We remark, finally, that in an article to appear elsewhere we shall establish similar formulae when the unit disc D is replaced by the right-half plane. These transplanted formulae are relevant for applications to the theory of Hilbert ports.

The logarithm of an operator with positive real part. If $A \in L(H)$, $\operatorname{Re} A \geqslant 0$ and A has a bounded inverse, we have defined in [1] the logarithm of A, denoted by $\log A$, by the formula

$$\log A = \int_0^\infty [(t+1)^{-1}\cdot I - (A + t\cdot I)^{-1}]\,dt, \tag{1}$$

where the integral is a Bochner integral. We have proved in [1] that under the stated conditions $\log A \in L(H)$, $\exp(\log A) = A$ and

$$\|\operatorname{Im} \log A\| \leqslant \pi/2. \tag{2}$$

LEMMA 1. *Let $A \in L(H)$ be an operator such that $\operatorname{Re} A \geqslant 0$, and A has a bounded inverse. Then $A - I \in C_p(H)$ $(1 \leqslant p < \infty)$ if and only if $\log A \in C_p(H)$.*

Proof. If $A - I \in C_p(H)$, then

$$(t+1)^{-1}I - (A+tI)^{-1} = (t+1)^{-1}(A+tI)^{-1}(A-I) \in C_p(H);$$

and

$$\|\log A\|_p \leqslant \|A - I\|_p \int_0^\infty (t+1)^{-1} \|(A+tI)^{-1}\|\,dt < \infty.$$

Conversely, if $B = \log A \in C_p(H)$, $A = \exp B$ and

$$A - I = B + \frac{B^2}{2!} + \ldots + \frac{B^n}{n!} + \ldots$$

Hence

$$\|A-I\|_p \leqslant \sum_{n=1}^{\infty} \frac{\|B\|_p^n}{n!} < \exp(\|B\|_p) < \infty.$$

A Herglotz's Formula for operator-valued functions.

Lemma 2. *Let $F\colon D \to L(H)$ be an analytic operator-valued function such that $1 \leqslant p \leqslant \infty$ and*

(i) $F(0) \in C_p(H)$,

(ii) $V(z) = \operatorname{Im} F(z) \in C_p(H)$ *for each* $z \in D$,

(iii) $\frac{1}{\pi} \int_{-\pi}^{\pi} \|V(re^{it})\|_p \, dt \leqslant C < \infty \quad (0 < r < 1)$.

Then, for $z \in D$,

$$F(z) = \sum_{n=0}^{\infty} z^n A^n \tag{3}$$

with $A_n \in C_p(H)$ $(n = 0, 1, 2, \ldots)$; and

$$\|A_n\|_p \leqslant C \quad (n > 0). \tag{4}$$

Moreover, (the convergence being in the norm of $C_p(H)$)

$$\lim_{r \to 1} \frac{i}{\pi r^{|n|}} \int_{-\pi}^{\pi} V(re^{it}) e^{-int} dt = \begin{cases} A_n & \text{if} \quad n > 0, \\ 2iV(0) & \text{if} \quad n = 0, \\ -A_{-n}^* & \text{if} \quad n < 0. \end{cases} \tag{5}$$

Proof. That the function F admits the Taylor expansion (3), which converges in the norm of $C_p(H)$, follows from the theory of vector-valued analytic functions (see [7], Chap. III, § 2). From (3) we obtain

$$F(re^{it}) = \sum_{n=0}^{\infty} r^n e^{int} A_n \quad (0 \leqslant r < 1),$$

$$F(re^{it})^* = \sum_{n=0}^{\infty} r^n e^{-int} A_n^* \quad (0 \leqslant r < 1).$$

From these formulae it is easy to see that, for $n > 0$,

$$A_n = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} F(re^{it}) e^{-int} dt \quad (0 < r < 1),$$

$$0 = \frac{1}{2\pi r^n} \int_{-\pi}^{\pi} F(re^{it})^* e^{-int} dt.$$

We get, therefore

$$A_n = \frac{i}{\pi r^n} \int_{-\pi}^{\pi} V(re^{it}) e^{-int} dt \quad (n > 0) \ (0 < r < 1). \tag{6}$$

In an analogous way, we obtain

$$-A_{-n}^* = \frac{i}{\pi r^{|n|}} \int_{-\pi}^{\pi} V(re^{it}) e^{-int} dt \quad (n < 0) \ (0 < r < 1). \tag{7}$$

Letting $r \to 1$ in (6) and (7) we obtain (5) when $n \neq 0$. The case $n = 0$ is trivial.

Finally, relation (4) follows easily from (6).

By T we designate the half-open interval $(-\pi, \pi]$ with the topology given by the distance $d(t, s) = |e^{it} - e^{is}|$, $t, s \in T$. T is a compact space because it is the homeomorphic image of the unit circle of the complex plane. With $\mathcal{B}$ we designate the σ-algebra of the Borel subsets of T.

Lemma 3. *Suppose $1 \leqslant p \leqslant \infty$. Let $F\colon D \to L(H)$ be an analytic operator-valued function which satisfies conditions* (i), (ii) *and* (iii) *of Lemma* 2. *Then there exists a finite positive Borel measure μ on T such that*

$$\|V(z)\|_p \leqslant \frac{1}{2\pi} \int_T \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(t). \tag{8}$$

Proof. We first prove that the continuous function $z \to \|V(z)\|_p$ is subharmonic on D. Indeed, if $z_0 \in D$ and $0 < r < 1 - |z_0|$, then

$$V(z_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(z_0 + re^{it}) dt,$$

and consequently

$$\|V(z_0)\|_p \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \|V(z_0 + re^{it})\|_p dt.$$

Now, by Theorem I, 6.7 of [5], p. 38, there exists a positive harmonic function $u(z)$ on D such that $\|V(z)\|_p \leqslant u(z)$ for each $z \in D$. On the other hand, a well-known theorem ([5], p. 19) affirms that there exists a positive Borel measure μ on T such that

$$u(z) = \frac{1}{2\pi} \int_T \frac{1 - |z|^2}{|e^{it} - z|^2} d\mu(t).$$

This proves the lemma.

To formulate the following theorem, which is an operator-valued version of a well-known theorem of Herglotz, we shall use the theory of integration of scalar functions with respect to a vector measure of finite variation (cf., for example, Chapter II of [3]).

We note that, since T is a metric compact space, any Borel vector measure on T is regular.

THEOREM 1. *Let $F\colon D\to L(H)$ be an analytic operator-valued function. Assume that $1\leqslant p\leqslant\infty$ and that*

(i) $F(0)\in C_p(H)$;

(ii) $V(z)=\operatorname{Im}F(z)\in C_p(H)$ *for each* $z\in D$;

(iii) $\frac{1}{\pi}\int_{-\pi}^{\pi}\|V(re^{it})\|_p\,dt\leqslant C<\infty$.

Then $F(z)$ admits the representation

$$F(z)=A+\frac{i}{2\pi}\int_T\frac{e^{it}+z}{e^{it}-z}\,d\Lambda(t),\tag{9}$$

where $A=\operatorname{Re}F(0)$ and $\Lambda\colon\mathscr{B}\to C_p(H)$ is a unique Borel vector measure of finite variation on T such that $\Lambda(M)$ is a hermitian operator for each $M\in\mathscr{B}$.

Conversely if $A\in C_p(H)$ is a hermitian operator and $\Lambda\colon\mathscr{B}\to C_p(H)$ is a vector measure which verifies the above conditions, then the function $F\colon D\to L(H)$ defined by formula (9) *is analytic in D and satisfies conditions* (i), (ii) and (iii).

Proof. For each $0\leqslant r<1$ we define the linear operator $\varphi_r\colon C(T)\to C_p(H)$ ($C(T)$ is the Banach space of all complex continuous functions with the norm $\|f\|_\infty=\sup|f(t)|$) by the formula

$$\varphi_r(f)=\int_{-\pi}^{\pi}f(t)V(re^{it})\,dt\qquad(f\in C(T)).\tag{10}$$

By condition (iii), we have

$$\|\varphi_r(f)\|_p\leqslant C\pi\|f\|_\infty.\tag{11}$$

By Lemma 2 the limit

$$\lim_{r\to1}\varphi_r(f)$$

exists in the norm of $C_p(H)$ for each trigonometric polynomial f. Since $\|\varphi_r\|\leqslant C\pi$, by (10), and the set $P(T)$ of all trigonometric polynomials is a dense subspace of $C(T)$, there exists, by Theorem 2.11.4 of [7], p. 41, a bounded linear operator $\varphi\colon C(T)\to C_p(H)$ such that

$$\lim_{r\to1}\varphi_r(f)=\varphi(f)$$

for every $f\in C(T)$ (the convergence being in the norm of the space $C_p(H)$).

We shall see that this operator φ satisfies the relation

$$\|\varphi(f)\|_p\leqslant\int_T|f(t)|\,d\mu(t)\qquad(f\in C(T))\tag{12}$$

where μ is a positive Borel measure on T.

In fact, by formula (10) and Lemma 3, we have

$$\|\varphi_r(f)\|_p\leqslant\int_T\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(t-s)+r^2}\,|f(s)|\,ds\right\}d\mu(t).$$

From this formula, passing to the limit for $r\to1$, and using well-known properties of the Poisson kernel, we obtain formula (12). This formula means that φ is a dominated linear operation in the sense of [3], p. 379. Therefore the operator $\varphi\colon C(T)\to C_p(H)$ satisfies the hypothesis of Theorem 2 of [3], p. 380. Consequently, there exists a unique Borel vector measure $\Lambda\colon\mathscr{B}\to C_p(H)$ of finite variation on T such that

$$\varphi(f)=\int_T f(t)\,d\Lambda(t).$$

Hence, from (5) we get

$$A_n=\frac{i}{\pi}\int_T e^{-int}\,d\Lambda(t)$$

and

$$\operatorname{Im}A_0=V(0)=\frac{1}{2\pi}\int_T d\Lambda(t).$$

Therefore, from formula (3), we obtain

$$F(z)=A+\frac{i}{\pi}\int_T\left\{\frac{1}{2}+\sum_{n=1}^{\infty}e^{-int}z^n\right\}d\Lambda(t).$$

Finally, taking into account the relation

$$\frac{1}{2}\,\frac{e^{it}+z}{e^{it}-z} = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-int}z^n \qquad (z \in D)$$

formula (9) follows.

The uniqueness of representation (9) follows easily from the scalar case. The converse part can be easily verified.

Exponential representation.

THEOREM 2. *Assume that* $1 \leqslant p < \infty$. *Let* $F\colon D \to L(H)$ *be an analytic operator-valued function such that*

(i) $\operatorname{Re} F(z) \geqslant 0$ *for each* $z \in D$;

(ii) $F(z)$ *is an operator with a bounded inverse for each* $z \in D$;

(iii) $F(0) - I \in C_p(H)$;

(iv) $\operatorname{Im} F(z) \in C_p(H)$ *for every* $z \in D$;

(v) $\displaystyle\frac{1}{\pi}\int_{-\pi}^{\pi} \|\operatorname{Im}\log F(re^{it})\|_p\,dt \leqslant C < \infty \qquad (0 < r < 1).$

Then the function $F(z)$ *admits the representation*

$$F(z) = \exp\left\{A + \frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,G(t)\,dt\right\}, \tag{13}$$

where A *is a hermitian operator in* $C_p(H)$ *and* $G\colon (-\pi, \pi) \to C_p(H)$ *is a Bochner integrable function such that:*

(a) $G(t)$ *is hermitian for almost every* $t \in (-\pi, \pi)$;

(b) $\|G(t)\| \leqslant \pi/2$ *for almost every* $t \in (-\pi, \pi)$.

Moreover,

$$\lim_{r\to 1} \|\operatorname{Im} F(re^{it}) - G(t)\|_p = 0 \tag{14}$$

for almost every $t \in (-\pi, \pi)$.

Proof. Let $J\colon D \to L(H)$ be the function defined by $J(z) = \log F(z)$, where the logarithm is defined as in Lemma 1. That the function $J(z)$ is analytic is proved as in Theorem 3.1 of [1]. Moreover, we have

$$\|\operatorname{Im} J(z)\| \leqslant \pi/2 \qquad (z \in D) \tag{15}$$

and

$$F(z) = \exp J(z), \tag{16}$$

(see [1], Th. 2.5 and Th. 2.4, respectively). By conditions (iii), (iv) and (v), and Lemma 1, the function $J(z)$ satisfies the hypothesis of Theorem 1; therefore it admits the representation

$$J(z) = A + \frac{i}{2\pi}\int_T \frac{e^{it}+z}{e^{it}-z}\,dA(t), \tag{17}$$

where $A\colon \mathscr{B} \to C_p(H)$ is a Borel vector measure of bounded variation on T such that $A(M)$ is a hermitian operator for every Borel subset M of T, and A is a hermitian operator in $C_p(H)$.

From formula (17), we get

$$\big(\operatorname{Im} J(z)\xi, \eta\big) = \frac{1}{2\pi}\int_T \frac{1-|z|^2}{|e^{it}-z|^2}\,d\lambda_{\xi,\eta}(t) \tag{18}$$

for every pair of vectors ξ and η in H, where $\lambda_{\xi,\eta}$ is the complex Borel measure on T defined by $\lambda_{\xi,\eta}(\cdot) = \big(A(\cdot)\xi, \eta\big)$.

We observe that the function which appears in the right-hand member of (18) is harmonic and bounded for every pair of vectors ξ and η in H. Indeed, from (15) we get

$$\big|\big(\operatorname{Im} J(z)\xi, \eta\big)\big| \leqslant (\pi/2)\|\xi\|\,\|\eta\|.$$

Hence, by a classic theorem of Fatou, the function $\big(\operatorname{Im} J(z)\xi, \eta\big)$ is the Poisson integral of a function in $L^\infty(-\pi, \pi)$. Therefore the measure $\lambda_{\xi,\eta}$ is absolutely continuous with respect to the Lebesgue measure. This implies that $A(N) = 0$ for every Borel subset of $(-\pi, \pi)$ with Lebesgue measure zero. Therefore the vector measure $A\colon \mathscr{B} \to C_p(H)$ is absolutely continuous. Consequently, since the space $C_p(H)$ $(1 \leqslant p < \infty)$ has the Radon–Nikodým property (see [2], Chap. VII, § 7, pp. 218–219), there exists a Bochner integrable function $G\colon (-\pi, \pi) \to C_p(H)$ such that

$$A(M) = \int_M G(t)\,dt \qquad (M \in \mathscr{B}). \tag{19}$$

Now by Theorem 2, p. 169 of [3], formula (17) can be written as

$$J(z) = A + \frac{i}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,G(t)\,dt. \tag{20}$$

It is easy to see from (14) that $G(t)$ is a hermitian operator for almost every $t \in (-\pi, \pi)$.

Taking imaginary parts in (20), we obtain

$$\operatorname{Im}\log F(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(t-s)+r^2}G(s)\,ds \qquad (0\leqslant r<1).$$

Hence (14) follows for almost every $t\in(-\pi,\pi)$ from Lemma 4 below. Formula (13) follows from (16) and (20).

Finally, from formulae (14) and (15) the relation $\|G(t)\|\leqslant\pi/2$, follows for almost every t.

Lemma 4. *Let X be a Banach space, let $G\colon(-\pi,\pi)\to X$ be a Bochner integrable function and let $K\colon D\to X$ be the function defined by*

$$K(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(t-s)+r^2}G(s)\,ds.$$

Then

$$\lim_{r\to1}\|K(re^{it})-G(t)\| = 0$$

for almost every $t\in(-\pi,\pi)$. ($\|\cdot\|$ *denotes the norm of the Banach space X.*)

Proof. We omit the details. The proof follows, as in the scalar case, from the fact that almost every t is a Lebesgue point of G, i.e.

$$\lim_{h\to0}\frac{1}{2h}\int_{t-h}^{t+h}\|G(s)-G(t)\|\,ds = 0.$$

(See Theorem 3.85, p. 87, of [7].)

Exponential representation in the case in which $F(x)$ is a real hermitian operator for every real $x\in(-1,1)$. In this section we assume that the Hilbert space H has a conjugation $\xi\to\bar\xi$ which satisfies the conditions:

$$\bar{\bar\xi} = \xi;\qquad \overline{\xi+\eta} = \bar\xi+\bar\eta;\qquad \overline{c\xi} = \bar c\bar\xi;\qquad \overline{(\xi,\eta)} = (\bar\xi,\bar\eta).$$

For every A, we define the conjugate $\bar A$ of A as the operator defined by $\bar A\xi = \overline{A\bar\xi}$. It is easy to see that $\bar A\in L(H)$ and that $\|\bar A\| = \|A\|$ for every $A\in L(H)$. An operator A is said to be *real* if $\bar A = A$. It is also easy to see that A^* and $\bar A$ belong to $C_p(H)$ if $A\in C_p(H)$.

Theorem 3. *Assume $1\leqslant p<\infty$. Let $F\colon D\to L(H)$ be an operator-valued analytic function which satisfies conditions* (i)–(v) *of Theorem 2 and, besides, the following one:*

(vi) *$F(x)$ is a real hermitian operator for every $x\in(-1,1)$.*

Then the function $F(z)$ admits the representation

$$F(z) = \exp\left\{A+\frac{1}{\pi}\int_0^{\pi}\frac{z\operatorname{sen}t}{1-2z\operatorname{sen}t+z^2}G(t)\,dt\right\}, \tag{21}$$

where $A\in C_p(H)$ is a real hermitian operator and $G\colon(0,\pi)\to C_p(H)$ is a Bochner integrable function such that:

(a) *$G(t)$ is a real hermitian operator for almost every $t\in(0,\pi)$;*

(b) *$\|G(t)\|\leqslant\pi/2$ for almost every $t\in(0,\pi)$.*

Proof. By Theorem 3 the function $F(z)$ admits representation (13). We claim that the function $G(t)$ verifies the relations

$$G(-t) = -G(t), \tag{22, a}$$

$$\overline{G(t)} = G(t). \tag{22, b}$$

Indeed by condition (vi), we can apply Lemma 5.3 of [1], to verify that $F(\bar z) = F(z)^*$. Thus $V(re^{-it}) = -V(re^{it})$ $(0\leqslant r<1)$. Hence, by (14), we obtain (22, a). Likewise, in virtue of condition (vi), we can apply Lemma 5.2 of [1], to obtain $V(re^{-it}) = -\overline{V(re^{it})}$. Hence, by (14) we have $\overline{G(t)} = -G(-t)$ for almost every t. Whence, from (22,a) we obtain (22, b). Thus $G(t)$ is a real hermitian operator for almost every t.

Finally, taking into account relation (22, a) it is easy to see that formula (21) follows from (13).

The determinant of an operator-valued function $F\colon D\to L(H)$. If $A\in C_1(H)$, we define the determinant of $(I-A)$ as in [6], Chap. IV, Sect. 1. If $A\in C_1(H)$, then $I-\exp A\in C_1(H)$. Therefore we can define the determinant of $\exp A$ by the formula

$$\det(\exp A) = \det[I-(I-\exp A)].$$

Lemma 5. (Generalized formula of Jacobi.) *If $A\in C_1(H)$, then*

$$\det(\exp A) = \exp(\operatorname{tr}A). \tag{23}$$

Proof. In [6], p. 163, it is proved that if $K(z)$ is an operator-valued function with values in $C_1(H)$ and holomorphic in some region, then the determinant $\det\big(I-K(z)\big)$ is holomorphic on the same region. Moreover, the formula

$$\frac{d}{dz}\log\big[\det\big(I-K(z)\big)\big] = -\operatorname{tr}\big[\big(I-K(z)\big)^{-1}K'(z)\big] \tag{24}$$

is valid for points z at which the operator $I-K(z)$ has a bounded inverse.

By choosing $K(z) = I-\exp(zA)$ in formula (24), we obtain

$$\frac{d}{dz}\log[\det(\exp zA)] = \operatorname{tr} A.$$

Hence, integrating from $z=0$ to $z=1$, the lemma is proved.

THEOREM 4. *If $F\colon D\to L(H)$ satisfies the hypothesis of Theorem 2 with $p=1$, then*

$$\det F(z) = \exp\left\{a+\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}g(t)\,dt\right\} \quad (z\in D), \tag{25}$$

where $a = \operatorname{tr} A$ and $g(t) = \operatorname{tr} G(t)$ (A and $G(t)$ are as in formula (13) *of Theorem* 2).

Proof. We first note that the function $g(t)$ in formula (25) is integrable since $|g(t)|\leqslant|\operatorname{tr} G(t)|\leqslant\|G(t)\|_1\in L^1(-\pi,\pi)$, because the operator-valued function $G(t)$ is Bochner integrable.

From formulae (13) and (23), we get

$$\det F(z) = \exp\left\{\operatorname{tr} A+\frac{i}{2\pi}\operatorname{tr}\left(\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}G(t)\,dt\right)\right\}. \tag{26}$$

On the other hand, it is easy to prove that

$$\operatorname{tr}\left(\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}G(t)\,dt\right) = \int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\operatorname{tr}\big(G(t)\big)\,dt. \tag{27}$$

Indeed, let $\{\varphi_k\}$ be a orthonormal basis of H. Then

$$\operatorname{tr}\left(\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}G(t)\,dt\right) = \lim_{n\to\infty}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\left\{\sum_{k=1}^{n}\big(G(t)\varphi_k,\varphi_k\big)\right\}dt.$$

Passing to the limit under the integral sign we obtain (27). To justify this procedure we can apply the dominated convergence Theorem of Lebesgue, since

$$\left|\sum_{k=1}^{n}\big(G(t)\varphi_k,\varphi_k\big)\right|\leqslant\|G(t)\|_1\in L^1(-\pi,\pi).$$

Finally, the theorem follows from (26) and (27).

THEOREM 5. *Let $F\colon D\to L(H)$ be an operator-valued function which satisfies the hypothesis of Theorem 2 with $p=1$. Then*

$$\det F(z) = \prod_{k=1}^{\infty}f_k(z) \quad (z\in D), \tag{28}$$

where $f_k(z)$, $k=1,2,\ldots$, are complex functions such that:

(a) *$f_k(z)$ is analytic on D;*

(b) *$\operatorname{Re} f_k(z)\geqslant 0$ for each $z\in D$.*

Proof. Let $\{\varphi_k\}$ ve an orthonormal basis in H. Then, by Theorem 4

$$\det F(z) = \exp\left\{\sum_{k=1}^{\infty}(A\varphi_k,\varphi_k)+\sum_{k=1}^{\infty}\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\big(G(t)\varphi_k,\varphi_k\big)\,dt\right\}.$$

Hence

$$\det F(z) = \prod_{k=1}^{\infty}f_k(z),$$

where

$$f_k(z) = \exp\left\{(A\varphi_k,\varphi_k)+\frac{i}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\big(G(t)\varphi_k,\varphi_k\big)\,dt\right\}.$$

The function $f_k(z)$ is evidently analytic on D. To prove that $\operatorname{Re} f(z)\geqslant 0$, it suffices to see that

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-z|^2}{|e^{it}-z|^2}\big(G(t)\varphi_k,\varphi_k\big)\,dt\right|\leqslant\pi/2.$$

This is easy because $|(G(t)\varphi_k,\varphi_k)|\leqslant\|G(t)\|\leqslant\pi/2$ almost everywhere and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-z|^2}{|e^{it}-z|^2}\,dt = 1.$$

An easy consequence of the previous theorems is

THEOREM 6. *Let $F\colon D\to L(H)$ be an operator-valued function which satisfies the hypothesis of Theorem 3. Then*

$$\det F(z) = \prod_{k=1}^{\infty}f_k(z) \quad (z\in D), \tag{29}$$

where $f_k(z)$, $k=1,2,\ldots$, are complex valued functions such that

(a) $f_k(z)$ *is analytic on* D;

(b) $\operatorname{Re} f(z) \geqslant 0$, *for each* $z \in D$;

(c) $f_k(x)$ *is real for every* $x \in (-1, 1)$.

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