entropy numbers of compact operators in Banach spaces we get on this way again the statement of H. König [8] about the behaviour of eigenvalues of r-nuclear operators in L_p-spaces.

COROLLARY 2. *Let $0 < r < 1$, $1 < p < \infty$. If $S \in \mathfrak{N}_r(L_p, L_p)$, then the sequence of eigenvalues $(\lambda_n(S))$ belongs to the Lorentz sequence space $l_{s,r}$, where $1/s = 1/r - |1/2 - 1/p|$.*

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A theory for ungrounded electrical grids and its application to the geophysical exploration of layered strata*

by

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Abstract. A method is presented for solving the finite-difference approximation to $\nabla \cdot (\sigma \nabla \varphi) = \beta$ over a half-volume, where φ is unknown, σ and β are given, σ varies only in the normal direction to the boundary of the half-volume, and β is nonzero only on that boundary. The method is based on a theory, developed herein, of infinite ungrounded electrical grids; no truncation of any grid is imposed. The solution is given in terms of an infinite continued fraction of Laurent operators and yields some computational procedures that are quite efficient. The variations of σ in the normal direction to the boundary are allowed to be quite arbitrary so long as σ is positive, bounded, and bounded away from zero. The theory has significance for the resistivity method of geophysical exploration. Formulas are developed for the apparent resistivity of the earth under various configurations of current and voltage probes. In addition, it is proven that the obtained solution is the unique solution for which a generalized form of Tellegen's theorem is satisfied.

1. Introduction. It has been some ten years now since the elements of a rigorous and quite general theory of infinite electrical networks were first proposed [7]. Since that time the theory has expanded considerably, but up until quite recently most of the results consisted of existence and uniqueness theorems for the current-voltage regimes in infinite networks. (See the survey articles [19] and [23]). There was not much information on how those current-voltage regimes could be computed. One of the problems is that an infinite electrical network can respond in many different ways to a set of sources supplying a finite total amount of power. However, for certain classes of such networks, only one of those solutions corresponds to finite power dissipation in the networks. It is that unique finite-power solution that is the one of practical interest in most cases.

Starting about two years ago, methods were developed for computing the finite-power regimes in a grounded grid, that is, in a square or cubic grid having a branch connecting each node to a common ground [21], [22].

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Such grids are of practical interest, for their nodal equations represent the finite-difference approximations of the partial-differential equation

$$\nabla^2\varphi - \gamma\varphi = \beta, \tag{1.1}$$

where γ is a positive quantity. This is the steady-state heat-radiation equation [6]. Electrical engineers meet up with it in the theory of solid-state devices [13], [15] and [17], p. 99; its occurrence on a semi-infinite domain arises from the fact that the thickness of a semiconductor chip is very much larger than the region near one of its surfaces where the integrated circuitry is located.

A similar partial-differential equation is

$$\nabla\cdot(\sigma\nabla\varphi) = \beta. \tag{1.2}$$

There is at least one discipline, geophysical exploration, in which practical considerations suggest the application of (1.2) over a semi-infinite domain. For example, one way of searching for mineral deposits is the resistivity method [3], [9], [18]. This consists of the injection of current into the earth and the measurement of the resulting electric potential variations along the earth's surface. The earth appears as a semi-infinite medium—for all practical purposes. Also, β is the extracted current distribution, φ is the electrical potential, and σ is the conductivity of the earth. The variations in φ along the earth's surface can be used to prognosticate about the variations in σ inside the earth, and the latter can be viewed as a clue to the location of mineral ores.

A basic step in this whole process is the computation of φ along the surface for given σ and β. If the variations in σ are sufficiently simple, φ can be determined analytically by using integral equations, series expansions, electrical images, and so forth; see, for example, [3], [8], [9], [10], [11], and [16]. For more complicated variations, approximate numerical methods must be used. One important method involves the replacement of (1.2) by its finite-difference approximation [5], [12]. To solve for the discrete values of φ, the usual procedure is to truncate the grid along a finite boundary that encompasses a region much larger than that being surveyed. At this point, however, another problem arises: Appropriate boundary conditions must be specified along the surfaces of truncation. But it is not at all clear just what those boundary conditions should be.

An alternative procedure, the one that is adopted in this work, is to analyze the infinite resistive grid without truncating it. We do so through an analysis similar to that used for the grounded grids arising from the discretization of (1.1) [21], [22], but there is now a fundamental difference. The grids corresponding to (1.1) are grounded, whereas the grids corresponding to (1.2) are not. This is reflected in the fact that the operator-valued continued fractions for certain driving-point immittances converge readily in the former case, whereas we seem to be working right on the borderline of convergence in the present case. That is, a number of expressions that we would like to use for the ungrounded grids do not converge, and we are therefore forced to employ more delicate arguments in order to bring our analysis to a successful conclusion.

In this work we study the case where the conductivity σ in (1.2) is a function of depth below the earth's surface, that function being positive, bounded, and bounded away from zero. Other than these conditions and perhaps some mild regularity assumptions, no further requirements are imposed on σ. This allows us to study a horizontally layered geophysical structure where the layering is either smooth or discrete and varies in quite an unrestricted fashion.

Our objectives herein are to construct a theory for ungrounded electrical grids and to apply it to the geophysical exploration of layered strata. With regard to the latter objective, we derive formulas for the surface potential (more exactly, for the node voltages along the boundary of the approximating cubic grid) for various patterns of σ and β, and also formulas for apparent resistivity as measured by the Wenner, Schlumberger, and double-dipole configurations of probes [3]. These formulas allow very rapid computation of the stated quantities and do so under quite arbitrary vertical variations in σ. This is in contrast to the customary analysis of vertically varying σ where σ is usually restricted to being a step function with just a few steps or to having some simple mathematical form such as an exponential; see, for example, [1], [2], and [4].

At the end of this work, we prove that the solution obtained herein is the unique solution for which a generalized form of Tellegen's theorem is satisfied or, to put it another way, it is the unique finite-power solution dictated by Theorem 2.1 of [21]. It is important that this point of rigor be resolved, for otherwise there would be no information concerning which one of the infinity of possible solutions our procedure generates.

2. Discretization and the ungrounded electrical grid.

We choose a three-dimensional rectangular coordinate system in the half-volume Ω such that x and y are orthogonal horizontal distances, z is the vertical distance measured positively into the earth, and the origin is on the plane boundary $\partial\Omega$ of Ω. To discretize our boundary-value problem, we choose the increments Δx, Δy, and Δz such that $\Delta x = \Delta y > 0$ and $\Delta z > 0$ and then replace x by $j_1\Delta x$, y by $j_2\Delta y$, and z by $k\Delta z/2$, where $j_1, j_2 = \ldots, -1, 0, 1\ldots$ and $k = 1, 2, 3, \ldots$ Following Mufti [12] and Dey and Morrison [5], we then replace (1.2) by a finite-difference approximation in such a fashion that the increments in x, y, and z are Δx, Δy, and Δz. The result is modelled by an ungrounded resistive rectangular electrical grid whose nodes are indexed by (j_1, j_2, k) or alternatively by (j, k), where $j = (j_1, j_2)$ is a doublet of integers and k is odd and positive. This is shown in Figure 1.

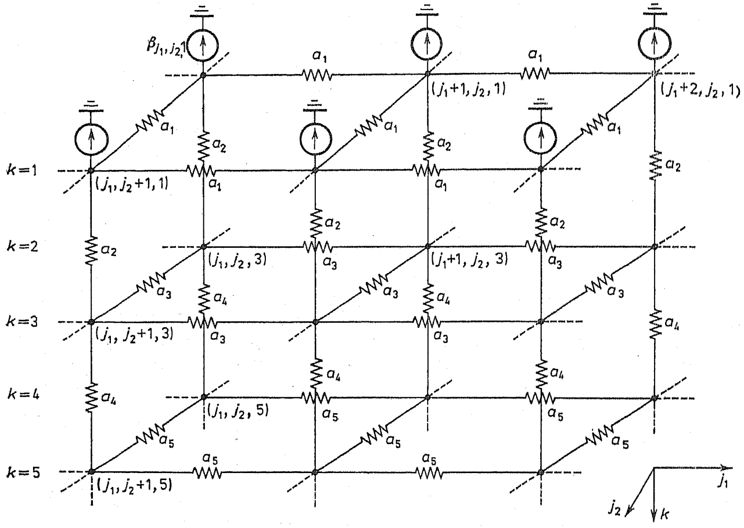

Figure 1

The values of φ and β at the nodes are denoted by

$$\varphi_{j,k} = \varphi(j_1 \Delta x, j_2 \Delta y, k \Delta z)$$

and

$$\beta_{j,k} = \beta(j_1 \Delta x, j_2 \Delta y, k \Delta z),$$

where k is odd. The $\varphi_{j,k}$ are measured in volts. As for the $\beta_{j,k}$ we note that in the resistivity method of geophysical exploration β results from the application of current to the earth through current probes. If I is injected at some point, we represent this as the application of a current source $\beta_{j,k} = I/(\Delta x)^2 \Delta z$ that extracts current from node (j, k) and injects it into a hypothetical ground (in the electrical sense). β has units of amperes/meter³.

Since σ varies only in the z direction, the conductances a_k in Figure 1 are given by

$$(2.1) \qquad a_k = (\Delta x)^{-2} \sigma(k \Delta z/2) \quad \text{for} \quad k = 1, 3, 5, \ldots$$

and

$$(2.2) \qquad a_k = (\Delta z)^{-2} \sigma(k \Delta z/2) \quad \text{for} \quad k = 2, 4, 6, \ldots$$

Since conductivity is measured in mhos/meter, the a_k have units of mhos/meter³. The nodal equations of Figure 1 are precisely the equations generated by our finite-difference approximation to (1.2).

Actually, we could at this point allow the earth to have an anisotropic conductivity, so long as the conductivity tensor with respect to the x, y, and z coordinates is diagonal. This would merely change the values of σ in (2.1) and (2.2) and also the conductance values a_k in Figure 1 in different ways along the j_1, j_2, and k directions. Furthermore, another extension we could now incorporate is unequal increments $(\Delta z)_k$ in the z direction; that is, $k \Delta z/2$ would be replaced by $\sum (\Delta z)_k/2$. This would only change the values of the a_k, where k is even.

Throughout this work we assume that the following conditions are satisfied.

Conditions A. (i) *The conductances a_k vary, if at all, only with k, not with j. Also there exist two numbers m and M with $0 < m < M < \infty$ such that $m < a_k < M$ for every $k = 1, 2, 3, \ldots$*

(ii) *Only a finite number of the current sources $\beta_{j,1}$ are non-zero. Moreover,*

$$\sum \beta_{j,1} = 0, \quad j = (j_1, j_2).$$

There are no other current sources and no voltage sources.

Condition A(i) asserts in effect that the earth's conductivity approaches neither 0 nor ∞ as depth increases indefinitely. Condition A(ii) asserts that as much current is extracted from the earth as is injected into it through the current probes; moreover, all the current probes are at the surface and are finite in number.

Given the a_k and the $\beta_{j,1}$, we shall solve for the node voltages $v_{j,k}$ (k odd) of Figure 1.

3. The equivalent ladder network of operators. The nodal equations of Figure 1 are the following.

For every $j = (j_1, j_2)$ we have, for $k = 1$,

$$(3.1) \qquad (4a_1 + a_2) v_{j_1,j_2,1} - a_1 v_{j_1-1,j_2,1} - a_1 v_{j_1+1,j_2,1} - a_1 v_{j_1,j_2-1,1} - a_1 v_{j_1,j_2+1,1} - a_2 v_{j_1,j_2,3} = -\beta_{j_1,j_2,1}$$

and for $k = 3, 5, 7 \ldots$,

(3.2) $$(4a_k + a_{k-1} + a_{k+1}) v_{j_1,j_2,k} - a_k v_{j_1-1,j_2,k} - a_k v_{j_1+1,j_2,k} - a_k v_{j_1,j_2-1,k} - \\ - a_k v_{j_1,j_2+1,k} - a_{k-1} v_{j_1,j_2,k-2} - a_{k+1} v_{j_1,j_2,k+2} = 0.$$

A more concise way of writing these equations is to suppress the index j by working with the ladder network of Figure 2 wherein the voltages and currents are vectors in the form of a two-dimensional array of numbers and the resistances and conductances are operators of a certain kind. Let D denote the space of all ordered doublets whose entries are integers. Thus $j = (j_1, j_2) \in D$. l_2 will denote the Hilbert space of all two-dimensional arrays $\{u_j \colon j \in D\}$ of complex numbers u_j with the inner product

$$(u, w) = \sum_{j \in D} u_j \overline{w}_j, \qquad u, w \in l_2.$$

A continuous linear mapping F of l_2 into l_2 can be represented by a matrix-like notation, namely, by the four-dimensional arrary $[F_{m,j}]$, where $m, j \in D$. Indeed, for $w = Fu$, we have

$$w_m = \sum_{j \in D} F_{m,j} u_j.$$

(The converse is not true: Only certain four-dimensional arrays will represent continuous linear mappings of l_2 into l_2).

A useful way of manipulating these entities is to exploit double Fourier series. Let C be the unit circle and $C \times C$ the Cartesian product of C with itself. Corresponding to any $u \in l_2$, we have the double Fourier series:

$$\tilde{u}(\omega) = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} u_{j_1,j_2} e^{i(\omega_1 j_1 + \omega_2 j_2)}, \qquad \omega = (\omega_1, \omega_2) \in C \times C.$$

We denote the operator that maps u into $\tilde{u}$ by $\mathscr{F}$. $\mathscr{F}$ is a topological linear ismorphism of l_2 onto the Hilbert space L_2 of (equivalence classes of) quadratically integrable functions on $C \times C$. Every continuous linear mapping F of l_2 into l_2 corresponds to a unique continuous linear mapping of L_2 into L_2 given by $\mathscr{F} F \mathscr{F}^{-1}$.

Among all the four-dimensional matrices $F = [F_{m,j}]$, where $m, j \in D$, are those that are Laurent, that is they satisfy $F_{m+p,j+p} = F_{m,j}$ for every $p \in D$. If F is a continuous linear Laurent mapping of l_2 into l_2, then $\mathscr{F} F \mathscr{F}^{-1}$ is the operation of multiplying the members of L_2 by an essentially bounded function $\tilde{F}(\omega)$ [4]. We denote that operation by $\tilde{F}(\omega)\cdot$. Thus $F = \mathscr{F}^{-1}[\tilde{F}(\omega)\cdot]\mathscr{F}$.

We return now to the matter of getting simpler representations for the network of Figure 1 and Equations (3.1) and (3.2). To each $j = (j_1, j_2) \in D$, we assign the norm $\|j\| = |j_1| + |j_2|$.

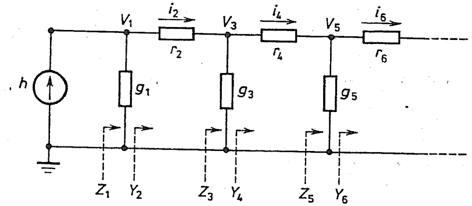

Figure 2

For each $k = 1, 3, 5, \ldots$, we let g_k be the Laurent operator whose (m, j) entry in its matrix representation is

$$(g_k)_{m,j} = \begin{cases} 4a_k & \text{for} \quad \|m-j\| = 0, \\ -a_k & \text{for} \quad \|m-j\| = 1, \\ 0 & \text{for} \quad \|m-j\| > 1. \end{cases}$$

We think of g_k as an operator-valued conductance that acts on the l_2-valued node voltage v_k in Figure 2 to yield the l_2-valued current $g_k v_k$ flowing downward through g_k. It is easy to show that g_k is a continuous linear mapping. However, it is not invertible.

Furthermore, for each $k = 2, 4, 6, \ldots$, we let r_k be the diagonal Laurent operator whose (m, j) entry is

$$(r_k)_{m,j} = \begin{cases} a_k^{-1} & \text{for} \quad \|m-j\| = 0, \\ 0 & \text{for} \quad \|m-j\| > 0. \end{cases}$$

Now, we view r_k as an operator-valued resistance that maps the l_2-valued current i_k in Figure 2 into the l_2-valued voltage $r_k i_k$ in a linear continuous fashion. r_k is invertible.

Finally, h is the vector in l_2 whose jth entry is $h_j = -\beta_{j,1}$.

With these definitions in hand, we can write the nodal equations for Figure 2 as the following relationships between vectors in l_2. For $k = 1$,

(3.3) $$(g_1 + r_2^{-1}) v_1 - r_2^{-1} v_3 = h,$$

and for $k = 3, 5, 7, \ldots$,

(3.4) $$(g_k + r_{k-1}^{-1} + r_{k+1}^{-1}) v_k - r_{k+1}^{-1} v_{k+2} - r_{k-1}^{-1} v_{k-2} = 0.$$

Upon expanding these equations in accordance with their components, we obtain precisely (3.1) and (3.2). Hence Figure 2 truly is a concise representation of Figure 1 so far as the computation of node voltages is concerned.

4. Driving-point and characteristic resistances. Our method of solving for the node voltages in Figure 2 makes use of the driving-point resistances $Z_1, Z_3, Z_5, \ldots$ and the driving-point conductances $Y_2, Y_4, Y_6, \ldots$. We need to determine these quantities.

A formal application of Kirchhoff's and Ohm's laws yields the following for Z_1, namely, an infinite continued fraction whose entries are certain Laurent operators:

$$Z_1 = \frac{1}{g_1 +}\,\frac{1}{r_2 +}\,\frac{1}{g_3 +}\,\frac{1}{r_4 +}\cdots \tag{4.1}$$

However, since the g_k are not invertible, we have to ascertain what sense this expression can possibly have. This is accomplished by using $\mathscr{F}$ to transfer our analysis onto $C\times C$. As was explained in [21], g_k for k odd becomes multiplication by the function $2a_k(2-\cos\omega_1-\cos\omega_2)$ and r_k for k even becomes multiplication by the constant $b_k = a_k^{-1}$. Consequently, Z_1 becomes multiplication by the function

$$\tilde{Z}_1(\omega) = \frac{1}{2a_1(2-\cos\omega_1-\cos\omega_2)\ +}\,\frac{1}{b_2 +}\,\frac{1}{2a_3(2-\cos\omega_1-\cos\omega_2)\ +}\,\frac{1}{b_4 +}\cdots \tag{4.2}$$

This is a continuous positive function except at the point $\omega = (\omega_1, \omega_2) = 0$, where the even truncations of (4.2) converge to $b_2+b_4+b_6+\ldots = \infty$ and the odd truncations of (4.2) are all ∞. Thus, $\tilde{Z}_1(0) = \infty$.

The explanation of this singularity is that Z_1 is an unbounded operator. In fact, it cannot be applied to every member of l_2. However, Conditions A insure that h is in the domain of Z_1 and that $Z_1 h \in l_2$. This allows our analysis to proceed.

To see this, note that $\mathscr{F}$ maps $h = \{h_j : j\in D\}$ into

$$\tilde{h}(\omega) = \sum h_j e^{i(j,\omega)}, \tag{4.3}$$

where $(j,\omega) = j_1\omega_1 + j_2\omega_2$. But, by Condition A(ii), $\sum h_j = 0$, and therefore

$$\tilde{h}(\omega) = \sum h_j[e^{i(j,\omega)}-1].$$

Let $\|\omega\| = (\omega_1^2+\omega_2^2)^{1/2}$. As $\|\omega\|\to 0$,

$$\tilde{h}(\omega) \sim i\omega_1 H_1 + i\omega_2 H_2 \to 0, \tag{4.4}$$

where

$$H_1 = \sum_{j_1}\sum_{j_2} j_1 h_{j_1,j_2}, \quad H_2 = \sum_{j_1}\sum_{j_2} j_2 h_{j_1,j_2}.$$

So, quite possibly, zero in (4.3) might cancel the singularity in (4.2) and thereby allow $\tilde{Z}_1(\omega)\tilde{h}(\omega)$ to remain a member of L_2. This is truly the case, as can be seen by examining the behaviour of $\tilde{Z}_1(\omega)$ near the origin.

Indeed, since $0 < m < a_k < M < \infty$ for all odd k and $0 < M^{-1} < b_k < m^{-1} < \infty$ for all even k, it follows from (4.2) that $\tilde{Z}_1(\omega)$ is bounded by the function obtained by replacing the a_k by m and the b_k by m^{-1} in the right-hand side of (4.2). That function is equal to

$$m^{-1}\left\{-\frac{1}{2} + \left[\frac{1}{4} + \frac{1}{2(2-\cos\omega_1-\cos\omega_2)}\right]^{1/2}\right\}. \tag{4.5}$$

As $\|\omega\|\to 0$, (4.5) is asymptotic to $m^{-1}\|\omega\|^{-1}$. This fact coupled with (4.4) yields the order relation

$$\tilde{Z}_1(\omega)\tilde{h}(\omega) = O\left(\frac{\omega_1 H_1 + \omega_2 H_2}{m\|\omega\|}\right) \tag{4.6}$$

as $\|\omega\|\to 0$. Hence, $\tilde{Z}_1(\omega)\tilde{h}(\omega)$ is quadratically integrable at the origin and is in fact a member of L_2. So truly, when Conditions A are satisfied, h is in the domain of Z_1, and $v_1 = Z_1 h \in l_2$.

If the earth becomes uniform below some depth, these formulas can be made more concise to some extent by using the characteristic resistance of the corresponding uniform operator-valued ladder network [24]. In particular, if $a_k = a$ for $k = 1, 3, 5, \ldots$ and $b_k = b$ for $k = 2, 4, 6, \ldots$, then $Z_1(\omega)$ becomes the characteristic-resistance function

$$\tilde{Z}_0(\omega) = -\frac{b}{2} + \left[\frac{b^2}{4} + \frac{b}{2a(2-\cos\omega_1-\cos\omega_2)}\right]^{1/2}. \tag{4.7}$$

So, if the earth becomes uniform just below the depth where $k = n$, n even, we have

$$\tilde{Z}_1(\omega) = \frac{1}{2a_1(2-\cos\omega_1-\cos\omega_2)\ +}\,\frac{1}{b_2 +}\,\frac{1}{2a_3(2-\cos\omega_1-\cos\omega_2)\ +}\,\frac{1}{b_4 +}\cdots\frac{1}{2a_{n-1}(2-\cos\omega_1-\cos\omega_2)\ +}\,\frac{1}{b_n+\tilde{Z}_0(\omega)}. \tag{4.8}$$

5. Formulas for the node voltages. We have shown that, under Conditions A,

$$\tilde{v}_1(\omega) = \tilde{Z}_1(\omega)\tilde{h}(\omega) \in L_2, \tag{5.1}$$

so that $v_1 = Z_1 h \in l_2$. The components of v_1 are the node voltages at the surface nodes of Figure 1, where $k = 1$, and are in fact the Fourier coefficients of $\tilde{v}_1(\omega)$;

$$v_{j_1,j_2,1} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \tilde{v}_1(\omega_1, \omega_2) e^{-i(\omega_1 j_1 + \omega_2 j_2)} d\omega_1 d\omega_2. \tag{5.2}$$

This expression in conjunction with (5.1), (4.2), and (4.3) yields a formula from which any desired $v_{j,1} = v_{j_1,j_2,1}$ can be computed.

Furthermore, we can compute the voltages at the nodes within the earth (i.e., for $k = 3, 5, 7, \ldots$) either through a limb analysis [20], once the surface node voltages have been determined, or by using the "propagation constants" θ_k of the ladder network. Let us examine the latter method.

For $k = 1, 3, 5, \ldots$, we may write

$$v_{k+2} = v_k - r_{k+1} i_{k+1}, \qquad i_{k+1} = Y_{k+1} v_k.$$

Therefore

$$v_{k+2} = \theta_k v_k, \qquad \theta_k = 1 - r_{k+1} Y_{k+1}, \tag{5.3}$$

where 1 now denotes the identity operation on l_2. Thus

$$v_k = \theta_{k-2}\theta_{k-4}\ldots\theta_1 v_1, \qquad v_1 = Z_1 h, \qquad k = 3, 5, 7, \ldots \tag{5.4}$$

We can again compute the components of v_k by passing to $C \times C$. First of all, $b_{k+1}\tilde{Y}_{k+1}(\omega)\cdot = \mathscr{F} r_{k+1} Y_{k+1} \mathscr{F}^{-1}$, where

$$\tilde{Y}_{k+1}(\omega) = \cfrac{1}{b_{k+1} + \cfrac{1}{2a_{k+2}(2-\cos\omega_1 - \cos\omega_2) + \cfrac{1}{b_{k+3} + \cfrac{1}{2a_{k+2}(2-\cos\omega_1-\cos\omega_2) + \cdots}}}}$$

It follows from this that $b_{k+1}\tilde{Y}_{k+1}(\omega)$ is nonnegative, continuous, and bounded by 1, and it attains the value 0 at $(\omega_1, \omega_2) = (0, 0)$. Consequently,

$$\tilde{\theta}_k(\omega) = 1 - b_{k+1}\tilde{Y}_{k+1}(\omega)$$

is a positive continuous function with values less than 1 except at $\omega = 0$, where it is equal to 1. Thus θ_k is a positive Laurent mapping of l_2 into l_2 and $\|\theta_k\| = 1$. By (5.4), $\|v_k\| \leqslant \|v_1\|$, and so, under Condition A, the node

voltages of Figure 1 comprise a bounded set; in addition, the node voltages along any horizontal plane of Figure 1 are quadratically summable.

The voltage at the node (j_1, j_2, k), where $k = 3, 5, 7, \ldots$, is

$$v_{j_1,j_2,k} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \tilde{v}_k(\omega_1, \omega_2) e^{-i(\omega_1 j_1 + \omega_2 j_2)} d\omega_1 d\omega_2, \tag{5.5}$$

where

$$\tilde{v}_k(\omega_1, \omega_2) = \tilde{v}_k(\omega) = \tilde{\theta}_{k-2}(\omega)\tilde{\theta}_{k-4}(\omega)\ldots\tilde{\theta}_1(\omega)\tilde{Z}_1(\omega)\tilde{h}(\omega).$$

If the earth becomes uniform below the depth where $k = n$, n even, we have for $k = n+1, n+3, n+5, \ldots$

$$\tilde{\theta}_k(\omega) = 1 + ab(2 - \cos\omega_1 - \cos\omega_2) - [a^2b^2(2-\cos\omega_1-\cos\omega_2)^2 + 2ab(2-\cos\omega_1-\cos\omega_2)]^{1/2}. \tag{5.6}$$

6. Apparent resistivity. In the resistivity method of geophysical exploration the customary procedure is to inject and extract a current I into the earth through a pair of current probes and to measure the potential V between a pair of voltage probes. The apparent resistivity ϱ_a of a nonhomogeneous earth corresponding to a given configuration of the current and voltage probes is the resistivity a homogeneous isotropic earth would have to have were it to respond to the given impressed current I with the same voltage V. Thus ϱ_a depends upon the relative positions of the current and voltage probes. The earth is explored by varying those positions. In this section we present formulas from which the apparent resistivity of an earth with vertically varying resistivity can be computed.

The Wenner configuration. The relative positions of the voltage and current probes for the Wenner configuration are shown in Figure 3 (a). The current and voltage probes lie on the same straight line, with the voltage probes straddling the current probes. The distances between adjacent probes are all the same, namely $2x$. For a homogeneous and isotropic earth with a resistivity of ϱ_a, the ratio V/I can be shown to be $\varrho_a/4\pi x$ ([3], p. 9). Hence the apparent resistivity ϱ_a for an earth with a varying resistivity is

$$\varrho_a = \frac{4\pi x V}{I}. \tag{6.1}$$

The Schlumberger configuration. The probe positions are shown in Figure 3 (b). Here the probes are all in line too, but the distance $2x$ between voltage probes is small as compared with the distance $2w$ between the current probes. The apparent resitivity for this arrangement ([3], p. 9) is

$$\varrho_a = \frac{\pi(w^2-x^2)V}{2xI}, \qquad w > x. \tag{6.2}$$

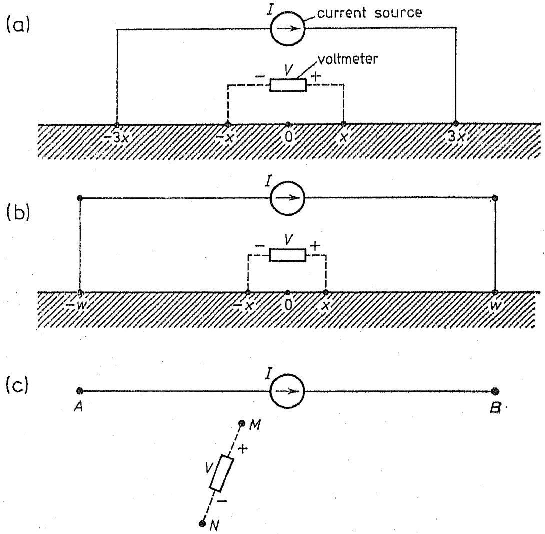

Figure 3

The double-dipole configuration. In this case, the dipoles may have any relative positions, as is illustrated in Figure 3(c). The Wenner and Schlumberger configurations are special cases of the present configuration. Let AM denote the distance between the points A and M, and use a similar notation for the other distances. Then

$$\varrho_a = \frac{2\pi|V|}{|1/AM+1/BN-1/AN-1/BM|I}. \tag{6.3}$$

(See [3], p. 8).

We now employ our prior results to obtain theoretical formulas for these apparent resistivities. Since we are using a grid to model the earth, our formulas will only be approximate ones, but they can be made as accurate as desired by choosing the grid sufficiently fine. We consider first the general case of a double-dipole configuration.

Let us assume that a current of I amperes is injected into the earth at the point $B = (x, y, z) = (\xi\Delta x, 0, \Delta z/2)$, where ξ is a positive integer, and is extracted from the earth at $A = (-\xi\Delta x, 0, \Delta z/2)$. Let us also assume that the points M and N have the coordinates

$$M = (\mu_1\Delta x, \mu_2\Delta y, \Delta z/2),$$
$$N = (\eta_1\Delta x, \eta_2\Delta y, \Delta z/2)$$

where as always $\Delta x = \Delta y$ and the μ's and the η's are integers. Then

$$\tilde{h}(\omega) = (\Delta x)^{-2}(\Delta z)^{-1}I(e^{i\omega_1\xi}-e^{-i\omega_1\xi}). \tag{6.4}$$

Therefore

$$\tilde{v}_1(\omega_1, \omega_2) = (\Delta x)^{-2}(\Delta z)^{-1}\tilde{Z}_1(\omega_1, \omega_2)I2i\sin\omega_1\xi, \tag{6.5}$$

where $\tilde{Z}_1(\omega_1, \omega_2)$ is given by (4.2) or (4.8). But $a_k = \sigma_k(\Delta x)^{-2}$ for k odd and $b_k = \sigma_k^{-1}(\Delta z)^2$ for k even, where σ_k is the earth's conductivity at the depth $k\Delta z/2$. So, with $\sigma_k = \varrho_k^{-1}$ and $\zeta = (\Delta z/\Delta x)^2$, we may write

$$\tilde{Z}_1(\omega_1, \omega_2) = \cfrac{(\Delta x)^2}{2\sigma_1(2-\cos\omega_1-\cos\omega_2)+\cfrac{1}{\varrho_2\zeta+\cfrac{1}{2\sigma_3(2-\cos\omega_1-\cos\omega_2)+\cfrac{1}{\varrho_4\zeta+\dots}}}} \tag{6.6}$$

Equation (5.2) determines the electric potentials at points M and N Upon taking their difference, we obtain

$$\begin{aligned} V &= v_{\mu_1,\mu_2,1} - v_{\eta_1,\eta_2,1} \\ &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\tilde{v}_1(\omega_1, \omega_2)[e^{i(\omega_1\mu_1+\omega_2\mu_2)}-e^{-i(\omega_1\eta_1+\omega_2\eta_2)}]d\omega_1 d\omega_2, \end{aligned} \tag{6.7}$$

where $\tilde{v}_1(\omega_1, \omega_2)$ is given by (6.5) in conjunction with (6.6). Also

$$\begin{aligned} AM &= [(\mu_1+\xi)^2+\mu_2^2]^{1/2}\Delta x, \\ AN &= [(\eta_1+\xi)^2+\eta_2^2]^{1/2}\Delta x, \\ BM &= [(\mu_1-\xi)^2+\mu_2^2]^{1/2}\Delta x, \\ BN &= [(\eta_1-\xi)^2+\eta_2^2]^{1/2}\Delta x. \end{aligned} \tag{6.8}$$

The substitution of (6.7) and (6.8) into (6.3) yields the apparent resistivity of the earth as determined by our resistive-grid model under Conditions A.

Formula (6.7) simplifies for the Wenner and Schlumberger configurations. For the Wenner configuration, set $x = \mu \Delta x$, where μ is a positive integer. Hence $\xi = 3\mu$. Also, $\mu_1 = \mu$, $\eta_1 = -\mu$, $\mu_2 = \eta_2 = 0$. Therefore

$$(6.9)\qquad \varrho_a = \frac{16\mu}{\pi \Delta x \Delta z} \int_0^\pi \int_0^\pi \tilde{Z}_1(\omega_1, \omega_2) \sin 3\omega_1 \mu \sin \omega_1 \mu \, d\omega_1 d\omega_2 .$$

Similarly, for the Schlumberger configuration, set $x = \mu \Delta x$ and $w = \xi \Delta x$. Again $\mu_1 = \mu$, $\eta_1 = -\mu$, $\mu_2 = \eta_2 = 0$. Consequently,

$$(6.10)\qquad \varrho_a = \frac{2(\xi^2 - \mu^2)}{\pi \mu \Delta x \Delta z} \int_0^\pi \int_0^\pi \tilde{Z}_1(\omega_1, \omega_2) \sin \omega_1 \xi \sin \omega_1 \mu \, d\omega_1 d\omega_2 .$$

Numerical computations based upon these formulas for discrete-layer configurations of the earth yield the same resistivity curves one finds in the literature (see, for example, [18], p. 671) except for the fact that for small potential-electrode spacing the discretization has the effect of increasing somewhat the apparent resistivity.

7. The uniqueness of our solution. We shall now prove that, under Conditions A, our solution for the infinite grid of Figure 1 is precisely that solution dictated by Theorem 2.1 of [21]. In other words, it is the unique solution that satisfies a generalized form of Tellegen's theorem, a special consequence of which states that the power dissipated in the grid is equal to the power supplied by the current sources. We now use the notation of [21] and state the definitions we need.

In the following, H_r denotes a real Hilbert space with the inner product $(\cdot, \cdot)$.

CONDITIONS B. *Let N be a connected countably infinite electrical network having no self loops. The currents and voltages of N are members of H_r. Each branch B_j of N is a parallel connection of a (possibly zero) current source $h_j \in H_r$ and a (nonzero) conductance g_j which is a positive invertible operator mapping H_r into H_r. There are no other current sources and no voltage sources.*

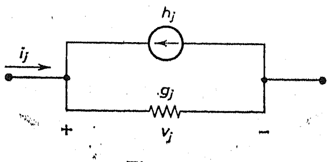

Figure 4

$\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3, \ldots$ denotes the branches in N. The typical branch $\boldsymbol{B}_j$, which we take to be oriented, is illustrated in Figure 4. The branch voltage (drop) $v_j \in H_r$ and the branch current $i_j \in H_r$ are measured with respect to the orientation of $\boldsymbol{B}_j$. Thus $i_j = g_j v_j - h_j$, $\boldsymbol{i} = \sum i_j \boldsymbol{B}_j$ is the 1-chain of branch currents, $\boldsymbol{v}' = \sum v_j \boldsymbol{B}_j'$ is the 1-cochain of branch voltages, and $\boldsymbol{h} = \sum h_j \boldsymbol{B}_j$ is the 1-chain of current sources. Kirchoff's node law states that $\boldsymbol{i}$ is a cycle, and his loop law states that $\boldsymbol{v}'$ is a coboundary.

A 1-cochain $\boldsymbol{w}' = \sum w_j \boldsymbol{B}_j'$, where $w_j \in H_j$, is defined as a functional on a 1-chain $\boldsymbol{x} = \sum x_j \boldsymbol{B}_j$, where $x_j \in H_r$, by $\langle \boldsymbol{w}', \boldsymbol{x} \rangle = \sum (w_j, x_j)$ whenever $\sum (w_j, x_j)$ exists. The latter will certainly be the case when $\boldsymbol{x}$ is a finite 1-chain (i.e., when all but a finite number of x_j's are zero). We now let $\mathscr{V}$ be the Hilbert space of all coboundaries $\boldsymbol{v}' = \sum v_j \boldsymbol{B}_j'$ such that

$$(7.1)\qquad \sum (v_j, g_j v_j) < \infty .$$

The inner product of two coboundaries $\boldsymbol{v}'$ and $\boldsymbol{w}'$ in $\mathscr{V}$ is defined to be $\sum (v_j, g_j w_j)$. Thus the norm of $\boldsymbol{v}'$ is

$$\|\boldsymbol{v}'\| = \left[\sum (v_j, g_j v_j)\right]^{1/2} .$$

Finally, we define the operator G of N to be the mapping of any 1-cochain $\boldsymbol{v}' = \sum v_j \boldsymbol{B}_j'$ into the 1-chain $G\boldsymbol{v}' = \sum g_j v_j \boldsymbol{B}_j$.

THEOREM 2.1 of [21]. *Let N satisfy Conditions* A, *and let its branch parameters satisfy*

$$(7.2)\qquad \sum (g_j^{-1} h_j, h_j) < \infty .$$

Then there exists a unique $\boldsymbol{v}' \in \mathscr{V}$ such that

$$(7.3)\qquad \langle \boldsymbol{w}', \boldsymbol{h} - G\boldsymbol{v}' \rangle = 0,$$

for all $\boldsymbol{w}' \in \mathscr{V}$.

This theorem states in effect that four conditions determine a unique set of branch voltages: Kirchhoff's loop law ($\boldsymbol{v}'$ is a coboundary), the finite-power dissipation condition (7.1), the finite-power-available condition (7.2), and a generalized form of Tellegen's theorem (7.3), which encompasses Kirchhoff's node law and Ohm's law as consequences.

We now examine Figure 2 with $H_r = l_{2r}$ being the real Hilbert space consisting of the real vectors in l_2. We let the first branch of that infinite ladder be the parallel combination of the l_{2r}-valued current source h and the conductance operator $g_1 \in [l_{2r}; l_{2r}]$. The other branches have no current sources. For each branch, the branch current i_k and the branch voltage drop v_k are measured in the same direction; i_1 is the l_{2r}-valued current flowing downward through g_1. Also $\boldsymbol{h} = h\boldsymbol{B}_1$ is the 1-chain of current sources. Note that (7.2) is automatically satisfied because there is only one current source h, it is in the first branch, and it is in the domain of g_1^{-1}. A review of the proofs of [21] shows that for our purposes, we do not need the invertibility of the g_j (the h_j being in the domains of the g_j^{-1} suffices) even though that condition was assumed in Conditions B.

Let $\boldsymbol{v}' = \sum v_k \boldsymbol{B}_k'$ be the coboundary of all branch voltage drops and let $\boldsymbol{i} = \sum i_k \boldsymbol{B}_k$ be the 1-chain of branch currents for Figure 2. We have already shown that, under Conditions A, the v_k and i_k are all members of l_{2r}. Our first objective is to show that $v' \in \mathscr{V}$. To do so, we need merely show that

$$\sum_{k=1}^{\infty} (v_k, i_k) < \infty, \tag{7.4}$$

where $i_k = g_k v_k$ for every k. (Note that, for k even, $g_k = r_k^{-1}$ exists). By Kirchhoff's node and loop laws, we may write

$$\begin{aligned} (v_1, h) &= (v_1, i_1) + (v_1, i_2) \\ &= (v_1, i_1) + (v_2, i_2) + (v_3, i_2) \\ &= (v_1, i_1) + \ldots + (v_{n+1}, i_{n+1}) + (v_{n+1}, i_{n+2}) \\ &= (v_1, i_1) + \ldots + (v_{n+1}, i_{n+1}) + (v_{n+1}, Y_{n+2} v_{n+1}). \end{aligned} \tag{7.5}$$

Since v_1 and h are both members of l_{2r}, (v_1, h) is finite. Also, $(v_1, i_1), \ldots$ $\ldots, (v_{n+1}, i_{n+1})$ are all nonnegative finite quantities because g_k, for k even, and r_k, for k odd, are positive operators. Moreover, for every even n, Y_{n+2} is also a positive operator as can be seen by examining its continued-fraction expansion. It follows that we can let $n \to \infty$ in the right-hand side of (7.5) to conclude that the partial sums of the left-hand side of (7.4) comprise a monotonically increasing bounded sequence and therefore converge. So truly $\boldsymbol{v}' \in \mathscr{V}$.

Next we want to prove that, for every $\boldsymbol{w}' \in \mathscr{V}$,

$$\langle \boldsymbol{w}', \boldsymbol{h} \rangle = \langle \boldsymbol{w}', G\boldsymbol{v}' \rangle, \tag{7.6}$$

where G is the operator for the network of Figure 2, that is

$$G\boldsymbol{v}' = \sum_{k=1}^{\infty} g_k v_k \boldsymbol{B}_k = \sum_{k=1}^{\infty} i_k \boldsymbol{B}_k = \boldsymbol{i}.$$

By the definition of the application of a coboundary to a 1-chain,

$$\langle \boldsymbol{w}', G\boldsymbol{v}' \rangle = (w_1, i_1) + (w_2, i_2) + (w_3, i_3) + \ldots \tag{7.7}$$

On the other hand, the components of $\boldsymbol{w}'$ satisfy Kirchhoff's loop law, whereas the components of $\boldsymbol{h} - \boldsymbol{i}$ satisfy Kirchhoff's node law. Therefore, we may apply the same expansion as that used in (7.5) to write

$$\begin{aligned} \langle \boldsymbol{w}, \boldsymbol{h} \rangle &= (w_1, h) + (w_1, i_1) + (w_1, i_2) \\ &= (w_1, i_1) + (w_2, i_2) + (w_3, i_2) \\ &= \ldots \\ &= (w_1, i_1) + (w_2, i_2) + (w_3, i_3) + \ldots \end{aligned} \tag{7.8}$$

A comparison of (7.7) and (7.8) establishes (7.6).

Thus, our solution for Figure 2 satisfies all the conditions required in Theorem 2.1 of [21] and is therefore the unique solution dictated by that theorem.

Our final objective is to transfer that conclusion to the network of Figure 1. The first difficulty is that the current sources in Figure 1 are not connected across conductances, as required in [21]. But, since $\sum h_j = 0$ and there are only a finite number of h_j, we can replace the h_j by another entirely equivalent finite set of current sources that are connected across the a_1 conductances. This can be accomplished as follows. Choose any node α in the $k = 1$ plane. Then, choose any node β in that plane having a (nonzero) current h_β fed into it. (α and β may be the same node.) Finally, choose a finite path from α to β along the branches of the $k = 1$ plane and connect a current source of value h_β across each conductance in that path with h_β being directed in the direction from α to β. (When $\alpha = \beta$, omit the last step.) Doing this for every nonzero component of h, we obtain the equivalent set of current sources. Since that set is finite, the requirement (7.2) will be satisfied.

Next, consider any $\boldsymbol{w}' \in \mathscr{V}$ for Figure 2. Its first element w_1 is a vector in l_{2r} consisting of node voltages in the first plane ($k = 1$) of Figure 1. It is easy to show that the set of voltage drops for all the branches of Figure 1 due to $\boldsymbol{w}'$ is quadratically summable. Let $\boldsymbol{u}'$ be the corresponding coboundary of voltage drops for Figure 1. Let $\boldsymbol{f}$ denote the equivalent 1-chain of branch current sources for Figure 1 constructed in the preceding paragraph from $\boldsymbol{h}$. It follows from that construction that $\langle \boldsymbol{w}', \boldsymbol{h} \rangle = \langle \boldsymbol{u}', \boldsymbol{f} \rangle$. In other words, we have converted $\langle \boldsymbol{w}', \boldsymbol{h} \rangle$ into a form $\langle \boldsymbol{u}', \boldsymbol{f} \rangle$ appropriate for the application of Theorem 2.1 of [21] to the grid of Figure 1.

We must do the same thing for $\langle \boldsymbol{w}', G\boldsymbol{v}' \rangle$. Consider $i_1 = g_1 v_1$, the first component of $G\boldsymbol{v}'$. Each component of i_1 is in turn the sum of the four currents $\alpha, \beta, \gamma, \delta$ being carried away from some node, say n_1, through the four a_1 conductances incident to that node. Moreover, (w_1, i_1) is a sum of terms of the form $\lambda_1(\alpha + \beta + \gamma + \delta)$, where λ_1 is the component of w_1 at n_1. Let n_2 be a first-plane node adjacent to n_1 and let α be the current flowing through an a_1 conductance from n_1 to n_2. The same analysis at n_2 shows that (w_1, i_1) has a term $\lambda_2(-a)$ corresponding to the current $-a$ flowing from n_2 to n_1, where λ_2 is the component of w_1 at n_2. Thus, the sum (w_1, i_1) contains the term $a(\lambda_1-\lambda_2)$, where a is the current in the branch connecting n_1 to n_2 corresponding to the i_1 vector and $\lambda_1-\lambda_2$ is the voltage drop in that branch corresponding to the w_1 vector. In fact, this decomposition shows that (w_1, i_1) is simply the sum of the products of the branch voltage drops in the $k = 1$ plane due to w_1 with the corresponding branch currents due to i_1. A similar conclusions can be drawn for each (w_k, i_k) when k is odd.

Moreover, the same can be done for each (w_k, i_k), where k is even. In fact, there is nothing to demonstrate in this case, for now the components of w_k are the branch voltage drops across the a_k conductances and the components of i_k are the branch currents in those conductances.

Thus $\langle \boldsymbol{w}', G\boldsymbol{v}'\rangle = \langle \boldsymbol{u}', G_a\boldsymbol{v}'_a\rangle$, where G_a is the operator for Figure 1 and $\boldsymbol{v}'_a$ is the coboundary of voltage drops for Figure 1 corresponding to the branch currents in Figure 1.

Altogether then we have shown that (7.6) can be transferred (with an appropriate change of notation) to the grid of Figure 1; in particular, $\langle \boldsymbol{u}', \boldsymbol{f}\rangle = \langle \boldsymbol{u}', G_a\boldsymbol{v}'_a\rangle$. Consequently, Theorem 2.1 of [21] applies to that grid and asserts that the solution for it obtained herein is the unique solution satisfying a generalized form of Tellegen's theorem, namely (7.3). As a corollary, we have that the power supplied by the current sources is equal to the power dissipated in the grid.

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