Entropy numbers of $r$-nuclear operators between $L_p$ spaces

by

BERND CARL (Jena)

Abstract. We show that the sequence of entropy numbers of $r$-nuclear operators acting from $L_p$ into $L_q$, $0 < r < 1$, $1 < p, q < \infty$, belongs to the Lorentz sequence space $l_r$, where

$$1/r = 1/2 + \min(1/\beta - 1/2; \gamma - 1/2) + \max(1/2; 1/\gamma).$$

Introduction. Since the fundamental work of Grothendieck the $r$-nuclear operators ("opérateurs à puissance r-ième" [6]) were intensively investigated. A representation of the theory of these operators can be found in the book Operator ideals of Pietsch [14]. A remarkable fact about the distribution of eigenvalues of $r$-nuclear operators was proved by H. König [8].

The aim of this paper is to determine the "degree of compactness" of $r$-nuclear operators in terms of entropy numbers. As an application we also get once more König's result about the behaviour of eigenvalues of $r$-nuclear operators acting in $L_p$ spaces.

Let $0 < r < 1$. An operator $S \in \mathcal{L}(E, F)$ from a Banach space $E$ into a Banach space $F$ is called $r$-nuclear if it admits a representation

$$S = \sum_{n=1}^{\infty} a_n \otimes y_n,$$

with $\sum_{n=1}^{\infty} \|a_n\| \|y_n\| < \infty$. Let

$$N_r(S) = \inf \left( \sum_{n=1}^{\infty} \|a_n\| \|y_n\| \right)^r,$$

where the infimum is taken over all possible representations of $S$. The class of these nuclear operators is denoted by $\mathcal{N}_r(E, F)$. $[\mathcal{N}_r, \mathcal{N}_p]$ forms an $r$-
normed operator ideal (cf. [14], (18.5)). For every operator \( S \in \mathcal{L}(E, F) \) the \( n \)-th entropy number \( e_n(S) \) is defined to be the infimum of all \( \varepsilon > 0 \) such that there exist \( x_1, \ldots, x_{n-1} \in F \) for which
\[
e_n(S) = \sup\left\{ \| T \| \mid T \in \mathcal{L}(E, F),\; S \cdot T \leq \varepsilon \right\}.
\]

Here \( U_p \) and \( U_p \) are the closed unit balls of \( E \) and \( F \), respectively. Roughly speaking, the asymptotic behaviour of \( e_n(S) \) characterizes the "degree of compactness" of \( S \). In particular, \( S \) is compact if and only if \( \lim_{n \to \infty} e_n(S) = 0 \). Furthermore, let us mention the multiplicativity of the \( e_n \) s:
\[
e_n + m - 1(SM) \leq e_n(S)e_m(T) \quad \text{for} \quad T \in \mathcal{L}(E, F), \; S \in \mathcal{L}(F, G).
\]

Put
\[
\mathcal{L}_{pa} = \{ S \in \mathcal{L} \mid e_n(S) \leq L_{pa} \}
\]

and
\[
L_{pa}(S) = \inf \{ e_n(S) \mid S \in \mathcal{L}_{pa} \}
\]

where \( L_{pa} \) stands for the quasi-normed Lorentz sequence space (cf. [15]) and \( e_n(S) \) is a norming constant (cf. [14], (14.3)). Then \( \mathcal{L}_{pa} \) becomes a quasi-normed operator ideal ([14], (14.3.5)). From the multiplicativity of the entropy numbers we get the useful product formula
\[
\mathcal{L}_{pa} \cdot \mathcal{L}_{pa} \subseteq \mathcal{L}_{pa} \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} = 1, \; \frac{1}{p} + \frac{1}{q} = 1.
\]

The definition of the product of quasi-normed operator ideals is taken from [14], (7.1).

Moreover, we need the following notions. The \( n \)-th approximation number of an operator \( S \in \mathcal{L}(E, F) \) is defined by
\[
a_n(S) = \inf \{ \| S - A \| : \text{rank}(A) < n \}.
\]

Define for \( S \in \mathcal{L}(U, E), \; n = 1, 2, \ldots, \)
\[
\mathcal{L}_n(S) = \left\{ \sum \left\| S(h) \right\| d_n(h) \right\|^{1/q}, \quad q = \frac{1}{n}
\]

where \( d_n \) is the \( n \)-dimensional standard normal distribution on the \( n \)-dimensional space \( E \). For \( S \in \mathcal{L}(E, F) \) we put
\[
\Pi_n(S) = \sup \{ \Pi_n(SX) : \| X \| \leq 1, \; n = 1, 2, \ldots \}
\]

An operator \( S \in \mathcal{L}(E, F) \) is called \( \varepsilon \)-Banachifying if it is the limit of finite rank operators with respect to \( \Pi_n \). The class of these operators, denoted by \( \mathcal{B}_\varepsilon \), is a normed operator ideal.

The definition of an absolutely \( (p, q) \)-summing operator can be found in [14], (17.2). The ideals consisting of these operators are denoted by \( \mathcal{B}_{pq} \). For the definition of Banach spaces of type \( p \) we refer to [12]. It should be mentioned that the spaces \( L_q = L_q(\mu) \) have the type \( \min(p, 2) \) if \( 1 \leq p < \infty \).

The eigenvalues \( \lambda_n(\mathcal{K}) \) of a compact operator \( S \in \mathcal{L}(E, F) \) are ordered in non-increasing absolute values and counted according to their algebraic multiplicities. By \( e, e_0, \ldots \) we mean always positive constants not depending on the natural numbers \( n, m \) and on the operators. The main theorem. We start our considerations with a series of lemmas.

The first result goes back to Chevet et al. [4].

**Lemma 1.** Let \( T \in \mathcal{L}(H, E) \) be an operator from a Hilbert space into a Banach space. Then
\[
T \in \mathcal{B}_p \quad \text{if and only if} \quad T \in \mathcal{B}_p.
\]

The next statement due to Dudley [5] and Sudakov [16] was translated by Kühn [10] into the "operator language".

**Lemma 2.** Let \( T \in \mathcal{L}(H, E) \) be an operator from a Hilbert space into a Banach space. Then
\[
T \in \mathcal{B}_p \quad \text{implies} \quad T \in \mathcal{L}_{1,\infty}.
\]

These two lemmas are used in order to prove

**Lemma 3.** (i) Let \( E \) be a Banach space such that \( E' \) is of type 2. Then
\[
\mathcal{L}_p(E, F) \subseteq \mathcal{L}_{1,\infty}(E, F).
\]

(ii) Let \( 2 \leq p < \infty \); then
\[
\mathcal{L}_p(L_p, F) \subseteq \mathcal{L}_{P_{pq}}(L_p, F).
\]

**Proof.** Given \( S \in \mathcal{L}_p(E, F) \). By Pietsch's factorization theorem [14], (17.3.7), we have \( S = X E \) with \( X \in \mathcal{B}_p(E, H) \) and \( Y \in \mathcal{L}(F, E) \), where \( H \) is a Hilbert space. From [13] we know that \( X' \in \mathcal{B}_p \) too. Since \( E' \) is of type 2, Lemma 1 implies \( X \in \mathcal{B}_p \), and Lemma 2 implies \( X' \in \mathcal{B}_p \), which proves (i). Let us turn to the second assertion. For \( S \in \mathcal{L}_p(L_p, F) \) we have again a factorization \( S = X E \) through a Hilbert space \( H \) with \( X \in \mathcal{B}_p(L_p, H) \) and \( Y \in \mathcal{L}(H, F) \).
Using once more Pietsch’s factorization theorem we get for the dual operator of $X$ the factorization

$$
\begin{array}{c}
H \\
\downarrow \hspace{1cm} A \hspace{1cm} B \\
X' \\
\uparrow \hspace{1cm} C_{1} \hspace{1cm} C_{2} \\
L_p' \\
\end{array}
$$

A result of Kwapien [11] states that if $B \in \mathscr{B}_{L_p}$, and therefore $X' \in \mathscr{B}_{L_p}$. Since $X'$ is defined on a Hilbert space, a result of König [7] yields that the sequence of approximation numbers $\{a_n(X')\}$ belongs to $l_p$ and hence $\{a_n(X')\} \in l_p$ (cf. [14], (11.7.4)). Consequently, by Theorem 2 of [1] $\{a_n(X)\} \in l_p$, which implies $\{a_n(S)\} \in l_p$.

Finally, the following statement is a recent result of [3].

**Lemma 4.** Let $E$ be a Banach space of type $q$ and $S \in \mathscr{L}(l_1, F)$ an operator admitting a factorization $S = T \Xi$, where $D \in \mathscr{L}(l_1, l_1)$. Then $\{a_n(S)\} \in \mathscr{L}_{l_1}$, $0 < r < \infty$, $0 < t \leq \infty$, and $X \in \mathscr{L}(l_1, F)$. Then

$$
S \in \mathscr{L}_{|s|}(l_1, F) \quad \text{for} \quad |s| = 1/r + 1 - 1/q.
$$

Now we are able to prove the main theorem.

**Theorem.** Let $0 < r < 1$, $1 < p, q < \infty$, $1/\alpha = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. Then

$$
\mathcal{R}_{\alpha}(l_p, l_q) \subseteq \mathcal{R}_{\alpha}(l_p, l_q).
$$

**Proof.** Given $S \in \mathcal{R}_{\alpha}(l_p, l_q)$. It is well-known that the $r$-nuclear operators may be factorized

$$
\begin{array}{c}
L_p \\
\downarrow \hspace{1cm} D \hspace{1cm} \uparrow \\
X \\
\downarrow \hspace{1cm} S \\
L_q \\
\end{array}
$$

where $X$, $Y$ are bounded operators, and $D$ is a diagonal operator, $D(\xi) = (\sigma, \xi)$, with a generating sequence $(\sigma_i) \in l_p$. We may write the diagonal operator $D$ in the form

$$
\begin{array}{c}
L_p \\
\downarrow \hspace{1cm} D \hspace{1cm} \uparrow \\
X \\
\downarrow \hspace{1cm} S \\
L_q \\
\end{array}
$$

where the generating sequences $(\sigma_i')$ of $D_1$ and $(\sigma_i')$ of $D_1$ belong to $l_1$, $1 < q < \infty$, $1 < p$, $q < \infty$, and $l_{\alpha_{1-\alpha}}$ respectively. Obviously, $D_1 \in \mathscr{L}(l_1, l_1)$, hence $D_1 \in \mathscr{L}(l_1, l_1)$. Lemma 3 implies $D_1 \in \mathscr{L}(l_1, l_1)$. Since $|s| = 1/r + 1 - 1/\alpha = 1/r + \min(1/2, 1/\alpha) \geq 1/\alpha$. Thus

$$
|s| = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q).
$$

Finally, this result can be improved by real interpolation: Combining the well-known interpolation formulas (cf. [9], [11])

$$
\mathcal{R}_{\alpha}(l_p, l_q) \subseteq \mathcal{R}_{\alpha}(l_p, l_q) \subseteq \mathcal{R}_{\alpha}(l_p, l_q),
$$

where $0 < r < 1$, $1 < p, q < \infty$, $1/\alpha = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$, $\alpha = 1, 2$, we arrive at

$$
\mathcal{R}_{\alpha}(l_p, l_q) \subseteq \mathcal{R}_{\alpha}(l_p, l_q) \subseteq \mathcal{R}_{\alpha}(l_p, l_q),
$$

where $1/\alpha = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. This completes the proof.

Finally, it remains to show that the conclusion of the theorem cannot be improved.

**Supplement.** Let $0 < r < 1$, $1 < p, q < \infty$, $1/\alpha = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. If $r < \alpha$, then $\mathcal{R}_{\alpha}(l_p, l_q) \not\subseteq \mathcal{R}_{\alpha}(l_p, l_q)$.

**Proof.** For arbitrary $r, p, q$ with $0 < r < 1$, $1 < p, q < \infty$, we construct operator $S$ such that $S \in \mathcal{R}_{\alpha}(l_p, l_q)$ and $S \not\in \mathcal{R}_{\alpha}(l_p, l_q)$, where $1/\alpha$.
\[ S := \sum_{n=0}^{\infty} S_n : l_p^m \to l_q^m \]

of operators \( S_n : l_p^m \to l_q^m \) defined by

\[
S_n := \begin{cases} 
|\alpha_n(2^n)^{-1/p} I_p | & 1 < q < 2 \leq p < \infty, \\
|\alpha_n(2^n)^{-1/p+1/q} I_q | & 1 \leq p < 2 \leq q < \infty, \\
\alpha_n(2^n)^{-1/p} A_{n-1} & 2 \leq p, q < \infty, \\
\alpha_n(2^n)^{-1/p+1/q} A_q & 1 \leq p, q \leq 2,
\end{cases}
\]

where \( (\alpha_n) \) is a sequence belonging to \( \mathbb{L} \setminus \mathbb{L}_p \) and is ordered in non-increasing absolute values, \( I_p : l_p^m \to l_p^m \) is the identity operator, and \( A_n : l_p^m \to l_q^m \) stands for the Littlewood (Walsh) matrix of rank \( 3^n \). These matrices are defined inductively by

\[
A_0 := (1) \quad \text{and} \quad A_{n+1} = \begin{bmatrix} A_n & A_n \\
A_n & -A_n \end{bmatrix}.
\]

From

\[
N_r(I_p : l_p^m \to l_q^m) \leq \begin{cases} 
(2^{nm})^{1/p} & 1 < q < 2 \leq p < \infty, \\
(2^{nm})^{1/p+1/q} & 1 \leq p < 2 \leq q < \infty
\end{cases}
\]

and

\[
N_r(A_n : l_p^m \to l_q^m) \leq \begin{cases} 
(2^{nm})^{1/p+1/q} & 2 \leq q, p < \infty, \\
(2^{nm})^{1/p+1/q} & 1 \leq p, q \leq 2
\end{cases}
\]

it follows

\[
N_r(S_n : l_p^m \to l_q^m) \leq |\alpha_n|, \quad 1 < p, q < \infty.
\]

Since \( N_r \) is an \( r \)-norm, we have

\[
N_r(S) \leq \sum_{n=0}^{\infty} N_r(S_n) \leq \sum_{n=0}^{\infty} |\alpha_n|^r < \infty.
\]

Consider the operator \( T_m \) defined by the following diagram:

\[
x_m(l_p^m) \xrightarrow{T_m} l_p^m(l_q^m) \]

where \( J \) and \( P \) are the injection and projection, respectively, and \( T_m \) is the identity operator. Since \( |\lambda_n(T_m)| = |\lambda_n(S_n)| \) and \( |\lambda_n(A_{n+1})| = 2^{nm} \), \( i = 1, \ldots, 2^n \), we get the eigenvalues of \( S_n \):

\[
|\lambda_i(S_n)| = |\alpha_n(2^n)^{-1/p+1/q}|, \quad i = 1, \ldots, 2^n,
\]

\[
1/p = 1/p + \min(1, \frac{1}{2}, 1/p) - \max(1/2, 1/p) = 1/p + 1 - 1/p - 1/q.
\]

Using a well-known inequality between eigenvalues and entropy numbers \([4]\), and

\[
e_{nm}(T_m : l_p^m \to l_q^m) \leq \alpha_n(2^n)^{1/p-1/q},
\]

we obtain

\[
|\alpha_n|^{(2^{nm})^{-1/p+1/q}} = |\alpha_n(S_m)| = |\alpha_n(T_m)| \leq 2e_{nm+1}(T_m)
\]

\[
\leq 2 \cdot \sum_{n=0}^{\infty} \| \lambda_n(\mathcal{S}_n) \|^{2^{nm}} \cdot e_{nm}(T_m)
\]

\[
\leq 2e_{nm}(S_m) \alpha_n(2^n)^{1/p-1/q}.
\]

Hence \( e_{nm}(S) \leq o(\alpha_n(2^n)^{1/p-1/q}) \). This yields

\[
L_{p,q}^r(S) \leq \left\{ e_n(S) \right\}^r \leq \sum_{n=0}^{\infty} e_n(S)^{2^{nm} \cdot r} \cdot 2^{nm} \cdot r^{2^{nm}}
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} e_{nm-1+k}(S)^{2^{nm-1+k} \cdot r} \cdot 2^{nm} \cdot r^{2^{nm}}
\]

\[
\leq \sum_{n=0}^{\infty} e_{nm}(S)^{2^{nm} \cdot r} \cdot 2^{nm} \cdot r^{2^{nm}}
\]

\[
\leq \sum_{n=0}^{\infty} |\lambda_n|^{r \cdot 2^{nm}} = \infty
\]

\[
if r_1 < r, \text{ which completes the proof.}
\]

**Remarks.** In another paper we will show that the preceding theorem can be generalized as follows:

**Corollary 1.** Let \( E' \) be of type \( p \) and \( F \) of type \( q \). If \( 1/p > 1/p + 1/2, \)

\[
1/p = 1/p + 1/2 - 1/p - 1/q, \text{ then}
\]

\[
R_n(E, F) \subseteq \mathcal{L}_{nm}(E, F).
\]

However, in the above statement there is the additional condition \( 1/p > 1/p + 1/2, \) we do not know whether it is necessary. Finally, from our theorem and the inequality \( |\lambda_n(S)| \leq \sqrt{2} e_n(S) \) between eigenvalues and
entropy numbers of compact operators in Banach spaces we get on this way again the statement of H. König [8] about the behaviour of eigenvalues of \( r \)-nuclear operators in \( L_p \)-spaces.

**Corollary 2.** Let \( 0 < r < 1, 1 < p < \infty \). If \( S \in \mathcal{B}_r(L_p, L_q) \), then the sequence of eigenvalues \( \|\lambda_n(S)\| \) belongs to the Lorentz sequence space \( l_{r,p} \), where \( 1/p = 1/r - 1/q - 1/p \).

**References**


Received August 4, 1982 (1781)