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STUDIA MATHEMATICA, T. LXXVII. (1983)

Entropy numbers of r-nuclear operators between L_p spaces

by

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Abstract. We show that the sequence of entropy numbers of r-nuclear operators acting from L_p into L_q, $0 < r < 1$, $1 < p, q < \infty$, belongs to the Lorentz sequence space $l_{s,r}$, where

$$1/s = 1/r + \min(1/2; 1/p) - \max(1/2; 1/q).$$

Introduction. Since the fundamental work of Grothendieck the r-nuclear operators ("operateurs á puissance r-ième" [6]) were intensively investigated. A representation of the theory of these operators can be found in the book *Operator ideals* of Pietsch [14]. A remarkable fact about the distribution of eigenvalues of r-nuclear operators was proved by H. König [8].

The aim of this paper is to determine the "degree of compactness" of r-nuclear operators in terms of entropy numbers. As an application we also get once more König's result about the behaviour of eigenvalues of r-nuclear operators acting in L_p spaces.

Let $0 < r < 1$. An operator $S \in \mathscr{L}(E, F)$ from a Banach space E into a Banach space F is called *r-nuclear* if it admits a representation

$$S = \sum_{n=1}^{\infty} a_n \otimes y_n, \qquad a_n \in E', \; y_n \in F$$

with $\sum_{n=1}^{\infty} \|a_n\|^r \|y_n\|^r < \infty$. Let

$$N_r(S) := \inf \Big(\sum_{n}^{\infty} \|a_n\|^r \|y_n\|^r \Big)^{1/r},$$

where the infimum is taken over all possible representations of S. The class of these nuclear operators is denoted by $\mathfrak{N}_r(E, F)$. $[\mathfrak{N}_r, N_r]$ forms an r-

normed operator ideal (cf. [14], (18.5)). For every operator $S \in \mathscr{L}(E, F)$ the nth entropy number $e_n(S)$ is defined to be the infimum of all $\varepsilon \geqslant 0$ such that there exist $y_1, \ldots, y_{2^{n-1}} \in F$ for which

$$S(U_E) \subseteq \bigcup_{1}^{2^{n-1}} \{y_i + \varepsilon U_F\}.$$

Here U_E and U_F are the closed unit balls of E and F, respectively. Roughly speaking, the asymptotic behaviour of $e_n(S)$ characterizes the "degree of compactness" of S. In particular, S is compact iff $\lim_n e_n(S) = 0$. Furthermore, let us mention the multiplicativity of the e_n's:

$$e_{n+m-1}(ST) \leqslant e_n(S) e_m(T) \quad \text{for} \quad T \in \mathscr{L}(E, F),\ S \in \mathscr{L}(F, G).$$

Put

$$\mathscr{L}_{p,q} := \{S \in \mathscr{L}: (e_n(S)) \in l_{p,q}\}$$

and

$$L_{p,q}(S) := \varepsilon_{p,q} \| (e_n(S)) \|_{p,q} \quad \text{for} \quad S \in \mathscr{L}_{p,q},$$

where $[l_{p,q}, \|\cdot\|_{p,q}]$, $0 < p, q \leqslant \infty$, $l_p := l_{p,p}$, stands for the quasinormed Lorentz sequence spaces (cf. [15]) and $\varepsilon_{p,q}$ is a norming constant (cf. [14], (14.3)). Then $[\mathscr{L}_{p,q}; L_{p,q}]$ becomes a quasinormed operator ideal ([14], (14.3.5)). From the multiplicativity of the entropy numbers we get the useful product formula

$$\mathscr{L}_{p_1,q_1} \circ \mathscr{L}_{p_0,q_0} \subseteq \mathscr{L}_{p,q} \quad \text{for} \quad 1/p = 1/p_0 + 1/p_1,\ 1/q = 1/q_0 + 1/q_1.$$

The definition of the product of quasinormed operator ideals is taken from [14], (7.1).

Moreover, we need the following notions. The *n-th approximation number* of an operator $S \in \mathscr{L}(E, F)$ is defined by

$$a_n(S) := \inf\{\|S - A\| : \operatorname{rank}(A) < n\}.$$

Define for $S \in \mathscr{L}(l_2^n, E)$, $n = 1, 2, \ldots$,

$$\Pi_s(S) := \Big(\int_{l_2^n} \|Sh\|^2 d\mathfrak{s}_n(h)\Big)^{1/2},$$

where $\mathfrak{s}_n$ is the n-dimensional standard normal distribution on the n-dimensional space l_2^n. For $S \in \mathscr{L}(E, F)$ we put

$$\Pi_s(S) := \sup\{\Pi_s(SX) : \|X: l_2^n \to E\| \leqslant 1,\ n = 1, 2, \ldots\}.$$

An operator $S \in \mathscr{L}(E, F)$ is called *$\mathfrak{s}$-Radonifying* if it is the limit of finite rank operators with respect to Π_s. The class of these operators, denoted by $\mathscr{R}_s$, is a normed operator ideal.

The definition of an absolutely (p, q)-summing operator can be found in [14], (17.2). The ideals consisting of these operators are denoted by $\mathscr{P}_{p,q}$ $(\mathscr{P}_p := \mathscr{P}_{p,p})$. For the definition of Banach spaces of type p we refer to [12]. It should be mentioned that the spaces $L_p := L_p(\mu)$ have the type $\min(p, 2)$ if $1 \leqslant p < \infty$.

The eigenvalues $\lambda_k(S)$ of a compact operator $S \in \mathscr{L}(E, E)$ are ordered in non increasing absolut values and counted according to their algebraic multiplicities. By $c, c_0, \ldots$ we mean always positive constants not depending on the natural numbers n, m and on the operators.

The main theorem. We start our considerations with a series of lemmas. The first result goes back to Chevet et al. [4].

LEMMA 1. *Let $T \in \mathscr{L}(H, E)$ be an operator from a Hilbert space into a Banach space. Then*

$$T \in \mathscr{R}_s \quad \textit{if and only if} \quad T' \in \mathscr{P}_2.$$

The next statement due to Dudley [5] and Sudakov [16] was translated by Kühn [10] into the "operator language".

LEMMA 2. *Let $T \in \mathscr{L}(H, E)$ be an operator from a Hilbert space into a lBanach space. Then*

$$T \in \mathscr{R}_s \quad \textit{implies} \quad T' \in \mathscr{L}_{2,\infty}.$$

These two lemmas are used in order to prove

LEMMA 3. (i) *Let E be a Banach space such that E' is of type 2. Then*

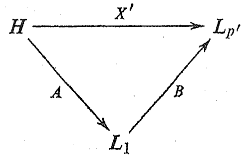
$$\mathscr{P}_2(E, F) \subseteq \mathscr{L}_{2,\infty}(E, F).$$

(ii) *Let $2 \leqslant p < \infty$; then*

$$\mathscr{P}_2(L_p, F) \subseteq \mathscr{L}_{p,p}(L_p, F).$$

Proof. Given $S \in \mathscr{P}_2(E, F)$. By Pietsch's factorization theorem [14], (17.3.7), we have $S = YX$ with $X \in \mathscr{P}_2(E, H)$ and $Y \in \mathscr{L}(H, F)$, where H is a Hilbert space. From [13] we know that $X'' \in \mathscr{P}_2$, too. Since E' is of type 2, Lemma 1 implies $X' \in \mathscr{R}_s$, and Lemma 2 implies $X'' \in \mathscr{L}_{2,\infty}$. The injectivity of the ideal $\mathscr{L}_{2,\infty}$ yields $X \in \mathscr{L}_{2,\infty}$, and thus $S \in \mathscr{L}_{2,\infty}$ which proves (i). Let us turn to the second assertion. For $S \in \mathscr{P}_2(L_p, F)$ we have again a factorization $S = YX$ through a Hilbert space H with $X \in \mathscr{P}_2(L_p, H)$ and $Y \in \mathscr{L}(H, F)$.

Using once more Pietsch's factorization theorem we get for the dual operator of X the factorization



A result of Kwapień [11] states that $B \in \mathscr{P}_{p,2}$ and therefore $X' \in \mathscr{P}_{p,2}$. Since X' is defined on a Hilbert space, a result of König [7] yields that the sequence of approximation numbers $\big(a_n(X')\big)$ belongs to l_p and hence $\big(a_n(X)\big) \in l_p$ (cf. [14], (11.7.4)). Consequently, by Theorem 2 of [1] $\big(e_n(X)\big) \in l_p$, which implies $\big(e_n(S)\big) \in l_p$.

Finally, the following statement is a recent result of [3].

LEMMA 4. *Let F be a Banach space of type q and $S \in \mathscr{L}(l_1, F)$ an operator admitting a factorization $S = TD$, where $D \in \mathscr{L}(l_1, l_1)$ is a diagonal operator, $D(\xi_i) := (\sigma_i \xi_i)$, generated by a sequence $(\sigma_i) \in l_{r,t}$, $0 < r < \infty$, $0 < t \leqslant \infty$, and $T \in \mathscr{L}(l_1, F)$. Then*

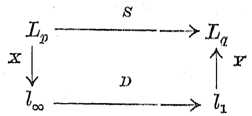
$$S \in \mathscr{L}_{s,t}(l_1, F) \quad \text{for} \quad 1/s = 1/r + 1 - 1/q.$$

Now we are able to prove the main theorem.

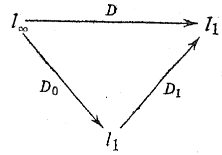
THEOREM. *Let $0 < r < 1$, $1 < p, q < \infty$, $1/s = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. Then*

$$\mathfrak{N}_r(L_p, L_q) \subseteq \mathscr{L}_{s,r}(L_p, L_q).$$

Proof. Given $S \in \mathfrak{N}_r(L_p, L_q)$. It is well-known that the r-nuclear operators may be factorized



where X, Y are bounded operators, and D is a diagonal operator, $D(\xi_i) = (\sigma_i \xi_i)$, with a generating sequence $(\sigma_i) \in l_r$. We may write the diagonal operator D in the form



where the generating sequences (σ_i^0) of D_0 and (σ_i^1) of D_1 belong to l_1 and $l_{r/(1-r)}$, respectively. Obviously, $D_0 \in \mathscr{P}_2(l_\infty, l_1)$, hence $D_0 X \in \mathscr{P}_2(L_p, l_1)$. Lemma 3 implies $D_0 X \in \mathscr{L}_{\max(2,p),\infty}(L_p, l_1)$. Since L_q, $1 < q < \infty$, is of type $\min(2, q)$, by Lemma 4 we have $YD_1 \in \mathscr{L}_{s_1, r/(1-r)}(l_1, L_q) \subseteq \mathscr{L}_{s_1,\infty}(l_1, L_q)$ with $1/s_1 = 1/r - 1 + 1 - \max(1/2, 1/q) = 1/r - \max(1/2, 1/q)$. Thus

$$S = YD_1 D_0 X \in \mathscr{L}_{,s\infty}(L_p, L_q),$$

$$1/s = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q).$$

Finally, this result can be improved by real interpolation: Combining the well-known interpolation formulas (cf. [9], [1])

$$\mathfrak{N}_r(E, F) \subseteq \big(\mathfrak{N}_{r_0}(E, F), \mathfrak{N}_{r_1}(E, F)\big)_{\theta,r},$$

$1/r = (1-\theta)/r_0 + \theta/r_1$, $0 < \theta < 1$, $0 < r_0 < r < r_1 < 1$, and

$$\big(\mathscr{L}_{s_0,t_0}(E, F), \mathscr{L}_{s_1,t_1}(E, F)\big)_{\theta,t} \subseteq \mathscr{L}_{s,t}(E, F),$$

$1/s = (1-\theta)/s_0 + \theta/s_1$, $0 < \theta < 1$, $0 < t_0, t_1, t < \infty$, with the inclusions

$$\mathfrak{N}_{r_i}(L_p, L_q) \subseteq \mathscr{L}_{s_i,\infty}(L_p, L_q)$$

proved above, $1/s_i = 1/r_i + \min(1/2, 1/p) - \max(1/2, 1/q)$, $i = 1, 2$, we arrive at

$$\mathfrak{N}_r(L_p, L_q) \subseteq \big(\mathfrak{N}_{r_0}(L_p, L_q), \mathfrak{N}_{r_1}(L_p, L_q)\big)_{\theta,r}$$
$$\subseteq \big(\mathscr{L}_{s_0,\infty}(L_p, L_q), \mathscr{L}_{s_1,\infty}(L_p, L_q)\big)_{\theta,r} \subseteq \mathscr{L}_{s,r}(L_p, L_q),$$

where $1/s = (1-\theta)/s_0 + \theta/s_1 = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. This completes the proof.

Finally, it remains to show that the conclusion of the theorem cannot be improved.

SUPPLEMENT. *Let $0 < r < 1$, $1 < p, q < \infty$, $1/s = 1/r + \min(1/2, 1/p) - \max(1/2, 1/q)$. If $r_0 < r$, then $\mathfrak{N}_r(l_p, l_q) \nsubseteq \mathscr{L}_{s,r_0}(l_p, l_q)$.*

Proof. For arbitrary r, p, q with $0 < r < 1$, $1 < p, q < \infty$, we construct operator S such that $S \in \mathfrak{N}_r(l_p, l_q)$ and $S \notin \mathscr{L}_{s,r_0}(l_p, l_q)$, where $1/s$

$= 1/r+\min(1/2, 1/p)-\max(1/2, 1/q)$ and $r_0 < r$. For this purpose we write the desired operator $S\colon l_p \to l_q$ as a telescopic sum

$$S := \sum_{n=0}^{\infty} \oplus S_{2^n}\colon l_p(l_p^{2^n}) \to l_q(l_q^{2^n})$$

of operators $S_{2^n}\colon l_p^{2^n} \to l_q^{2^n}$ defined by

$$S_{2^n} := \begin{cases} \sigma_n (2^n)^{-1/r} I_{2^n}, & 1 < q \leqslant 2 \leqslant p < \infty, \\ \sigma_n (2^n)^{-1/r+1/p-1/q}\, I_{2^n}, & 1 < p \leqslant 2 \leqslant q < \infty, \\ \sigma_n (2^n)^{-1/r-1/q} A_{2^n}, & 2 \leqslant p, q < \infty, \\ \sigma_n (2^n)^{-1/r-1+1/p} A_{2^n}, & 1 < p, q \leqslant 2, \end{cases}$$

where (σ_n) is a sequence belonging to $l_r \setminus l_{r,r_0}$ and is ordered in non-increasing absolute values, $I_{2^n}\colon l_p^{2^n} \to l_q^{2^n}$ is the identity operator, and $A_{2^n}\colon l_p^{2^n} \to l_q^{2^n}$ stands for the Littlewood (Walsh) matrix of rank 2^n. These matrices are defined inductively by

$$A_{2^0} := (1) \quad \text{and} \quad A_{2^{n+1}} = \begin{bmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{bmatrix}.$$

From

$$N_r(I_{2^n}\colon l_p^{2^n} \to l_q^{2^n}) \leqslant \begin{cases} (2^n)^{1/r}, & 1 < q \leqslant 2 \leqslant p < \infty, \\ (2^n)^{1/r-1p+1/q}, & 1 < p \leqslant 2 \leqslant q < \infty \end{cases}$$

and

$$N_r(A_{2^n}\colon l_p^{2^n} \to l_q^{2^n}) \leqslant \begin{cases} (2^n)^{1/r+1/q}, & 2 \leqslant q, p < \infty, \\ (2^n)^{1/r+1-1/p}, & 1 < p, q \leqslant 2 \end{cases}$$

it follows

$$N_r(S_{2^n}\colon l_p^{2^n} \to l_q^{2^n}) \leqslant |\sigma_n|, \quad 1 < p, q < \infty.$$

Since N_r is an r-norm, we have

$$N_r^r(S) \leqslant \sum_{n=0}^{\infty} N_r^r(S_{2^n}) \leqslant \sum_{n=0}^{\infty} |\sigma_n|^r < \infty.$$

Consider the operator T_m defined by the following diagram:

$$\begin{array}{ccc} l_p^m(l_p^{2^n}) & \xrightarrow{\ T_m\ } & l_p^m(l_p^{2^n}) \\ \Big\downarrow J^{(m)} & & \Big\uparrow I^{(m)} \\ & & l_q^m(l_q^{2^n}) \\ & & \Big\uparrow P^{(m)} \\ l_p(l_p^{2^n}) & \xrightarrow{\ S\ } & l_q(l_q^{2^n}) \end{array}$$

where $J^{(m)}$ and $P^{(m)}$ are the natural injection and projection, respectively, and $I^{(m)}$ is the identity operator. Since $|\lambda_{2^n-1+i}(T_m)| = |\lambda_i(S_{2^n})|$ and $|\lambda_i(A_{2^n})| = 2^{n/2}$, $i = 1, \ldots, 2^n$, we get for the eigenvalues of the S_{2^n}'s:

$$|\lambda_i(S_{2^n})| = |\sigma_n|(2^n)^{-1/s-1/q+1/p}, \quad i = 1, \ldots, 2^n,$$

$1/s = 1/r+\min(1/2, 1/p)-\max(1/2, 1/q)$. Using a well-known inequality between eigenvalues and entropy numbers [1], and

$$e_{2^m}(I^{(m)}\colon l_p^{2^m} \to l_q^{2^m}) \leqslant c_0 (2^m)^{1/p-1/q},$$

[2], we obtain

$$\begin{aligned} |\sigma_m|(2^m)^{-1/s-1/q+1/p} &= |\lambda_m(S_{2^m})| = |\lambda_{2^m}(T_m)| \leqslant 2e_{2^{m+1}}(T_m) \\ &\leqslant 2\|J^{(m)}\|\, e_{2^m}(S) \|P^{(m)}\|\, e_{2^m}(I^{(m)}) \\ &\leqslant 2e_{2^m}(S)\, c_0 (2^m)^{1/p-1/q}. \end{aligned}$$

Hence $e_{2^m}(S) \geqslant c|\sigma_m|(2^m)^{-1/s}$. This yields

$$\begin{aligned} L_{s,r_0}^{r_0}(S) &\gtrsim \|(e_k(S))\|_{s,r_0}^{r_0} = \sum e_k^{r_0}(S) k^{r_0/s-1} \\ &\gtrsim \sum_m \sum_{k=1}^{2^{m-1}} e_{2^{m-1}+k}^{r_0}(S)(2^{m-1}+k)^{r_0/s-1} \\ &\gtrsim \sum_m e_{2^m}^{r_0}(S) 2^{mr_0/s} \\ &\gtrsim \sum_m |\sigma_m|^{r_0} 2^{m(-r_0/s)} 2^{mr_0/s} \\ &\gtrsim \sum_m |\sigma_m|^{r_0} = \infty \end{aligned}$$

if $r_0 < r$, which completes the proof.

Remarks. In another paper we will show that the preceding theorem can be generalized as follows:

COROLLARY 1. *Let E' be of type p and F of type q. If $1/r > 1/p+1/2$, $1/s = 1/r+1-1/p-1/q$, then*

$$\mathfrak{N}_r(E, F) \subseteq \mathscr{L}_{s,r}(E, F).$$

However, in the above statement there is the additional condition $1/r > 1/p+1/2$, we do not know wether it is necessary. Finally, from our theorem and the inequality $|\lambda_n(S)| \leqslant \sqrt{2}\, e_n(S)$ between eigenvalues and entropy numbers of compact operators in Banach spaces we get on this way again the statement of H. König [8] about the behaviour of eigenvalues of r-nuclear operators in L_p-spaces.

COROLLARY 2. *Let* $0 < r < 1$, $1 < p < \infty$. *If* $S \in \mathfrak{N}_r(L_p, L_p)$, *then the sequence of eigenvalues* $(\lambda_n(S))$ *belongs to the Lorentz sequence space* $l_{s,r}$, *where* $1/s = 1/r - |1/2 - 1/p|$.

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A theory for ungrounded electrical grids and its application to the geophysical exploration of layered strata*

by

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Abstract. A method is presented for solving the finite-difference approximation to $\nabla\cdot(\sigma\nabla\varphi) = \beta$ over a half-volume, where φ is unknown, σ and β are given, σ varies only in the normal direction to the boundary of the half-volume, and β is nonzero only on that boundary. The method is based on a theory, developed herein, of infinite ungrounded electrical grids; no truncation of any grid is imposed. The solution is given in terms of an infinite continued fraction of Laurent operators and yields some computational procedures that are quite efficient. The variations of σ in the normal direction to the boundary are allowed to be quite arbitrary so long as σ is positive, bounded, and bounded away from zero. The theory has significance for the resistivity method of geophysical exploration. Formulas are developed for the apparent resistivity of the earth under various configurations of current and voltage probes. In addition, it is proven that the obtained solution is the unique solution for which a generalized form of Tellegen's theorem is satisfied.

1. Introduction. It has been some ten years now since the elements of a rigorous and quite general theory of infinite electrical networks were first proposed [7]. Since that time the theory has expanded considerably, but up until quite recently most of the results consisted of existence and uniqueness theorems for the current-voltage regimes in infinite networks. (See the survey articles [19] and [23]). There was not much information on how those current-voltage regimes could be computed. One of the problems is that an infinite electrical network can respond in many different ways to a set of sources supplying a finite total amount of power. However, for certain classes of such networks, only one of those solutions corresponds to finite power dissipation in the networks. It is that unique finite-power solution that is the one of practical interest in most cases.

Starting about two years ago, methods were developed for computing the finite-power regimes in a grounded grid, that is, in a square or cubic grid having a branch connecting each node to a common ground [21], [22].

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