The first equation of (24) means that

$$(\tfrac{1}{3}h^3 - ch^2 + c^2h)' = 0.$$

References

[1] P. Antosik and J. Ligęza, *Products of measures and functions of finite variations*, in: Proceed. Conf. 'Generalized Functions and Operational Calculus' (Varna 1975), ed. by I. Dimovski, Sofia 1979, pp. 20–26.

[2] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of Distributions, the Sequential Approach*, Amsterdam/Warszawa 1973.

[3] L. Berg, *On the multiplication of distributions*, in: V. S. Vladimirov, M. K. Polivanov, A. P. Prudnikov and J. A. Bryčkov, *Generalized Functions and their Applications in Mathematical Physics* (Russian), Proceed. Int. Conf. Moscow Nov., 24–28, 1980, Moscow 1981, pp. 71–79.

[4] — *Differentiationsalgebren mit einem kubischen Element*, Z. Anal. Anwend. 1 (4) (1982), 25–30.

[5] I. Kaplansky, *An Introduction to Differential Algebra*, Paris 1957.

[8] L. Berg, *Some products of distributions with several variables*, in: Proceed. Conf. 'Topology and Measure II' (Rostock–Warnemünde 1977), Part 2, ed. by J. Flachsmeyer, Z. Frolík and F. Terpe, Wiss. Beiträge EMA-Universität Greifswald 1980, pp. 7–9.

[9] — *Values in a distribution algebra*, Demonstratio Math. 12 (1979), 639–644.

[10] — *Distribution algebras with reflexive inverses*, Resultate Math. 3 (1980), 7–16.

[11] — *Distributionenalgebren mit Wurzeln*, Rostocker Math. Koll. 13 (1979), 73–80.

[12] — *Differentiationsalgebren für Distributionen*, Math. Nachr. 97 (1980), 223–231.

[13] — *Taylor's expansion in a distribution algebra*, Anal. Anwend. (to appear).

[14] — *The division of distributions by the independent variable*, Math. Nachr. 78 (1977), 327–338.

[15] V. K. Ivanov, *The algebra generated by the Heaviside function and the delta-functions* (Russian), Izv. Vysš. Učebn. Zaved. Matematica 1977, no. 10 (185), 65–69.

[16] Ma Kyin Myint, *Distribution Algebra*, M. Sc. Thesis, Arts and Science University, Rangoon 1979.

WILHELM-PIECK-UNIVERSITÄT, SEKTION MATHEMATIK
DDR-2500 Rostock, Universitätsplatz 1

Received June 23, 1982 (1773)

STUDIA MATHEMATICA, T. LXXVII. (1983)

Modular spaces over a field with valuation generated by a (ω,ϑ)-convex modular

by

RYSZARD URBAŃSKI (Poznań)

Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. Let X be a vector space over a field K with valuation. The form of an (ω, ϑ)-convex modular on X is given, generalizing the existing definitions of modulars on vector spaces. A simple formula for an F-norm on the modular space X_ϱ generated by modular on X is proved.

1. Introduction. In the literature on modular spaces, nonconvex, convex and s-convex modulars are considered (e.g., Musielak–Orlicz [3], Orlicz [5]). In each of these cases the functional $\|\cdot\|_\varrho$ generating the linear topology in modular space X_ϱ has been defined separately and it was F-norm, norm and s-norm, respectively, in a real vector space.

In this paper we introduce (ω, ϑ)-convex modular for some functions $\omega, \vartheta\colon K \to \boldsymbol{R}^+$, when K is a valued field. The (ω, ϑ)-convex modulars include all cases of modulars considered so far, by selecting appropriate functions ω, ϑ, and at the same time give generalizations of modular spaces to spaces over an arbitrary field with valuation. It is possible now to consider modulars not known before, namely such that they are not any of the three types mentioned above.

Moreover, even in the classical case the definition of $\|\cdot\|_\varrho$ is new because it allows to present all three separate definitions in one form.

2. Preliminary remarks.

2.1. Let X be a vector space over a field K with nontrivial valuation $|\cdot|\colon K \to \boldsymbol{R}$ (where $\boldsymbol{R}$ denotes the real numbers). The set of real number, $|a|$, $a \in K$, will be called the *set of values of* K and will be denoted by $|K|$. The values $|K^*|$, where K^* denotes the set of nonzero elements of K, form a multiplicative subgroup of the positive reals $(\boldsymbol{R}^+, \cdot)$. However, as is well known, $\boldsymbol{R}^+$, has only two types of subgroups, they are either cyclic groups or groups dense in $\boldsymbol{R}^+$ ([4]). If $|K^*|$ is an infinite cyclic group, the valuation is called *discrete*; equivalently, K is said to be *discretly valued.*

In this case, the generator of $|K^*|$ is $r \in |K|$ such that $r > 1$ and the distance from r to 1 is minimal, i.e. $r = \inf\{|a| > 1\colon\ a \in K\}$. The valuation is either discrete or "dense" in $\boldsymbol{R}^+$. If instead of the strong triangle inequality $|\alpha+\beta| \leqslant |\alpha|+|\beta|$, we have $|\alpha+\beta| \leqslant \max(|\alpha|, |\beta|)$, the valuation is said to be *nonarchimedean*, otherwise it is called *archimedean*. The discrete valuation must be nonarchimedean and if K is archimedean valued, $|K|$ must be dense in $\boldsymbol{R}^+$. In this paper $\boldsymbol{N}$ denotes the set of natural numbers and $\boldsymbol{Z}$ denotes the set of integers.

2.2. Let X be an Abelian group. A *quasi-norm* on X is a real-valued function $\|\cdot\|$ defined on X such that

(F1) $\|x\| \geqslant 0$ and $\|x\| = 0$ iff $x = 0$,

(F2) $\|x\| = \|-x\|$,

(F3) $\|x+y\| \leqslant c(\|x\|+\|y\|)$ for any $x, y \in X$,

with an absolute constant $c \geqslant 1$. Inequality (F3) is called *c-triangle inequality* and the function $\|\cdot\|$ a *c-norm* (see [1]).

2.3. Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and let $\|\cdot\|$ be a c-norm on X satisfying the following properties:

(F4) $\|\lambda_n x\| \to 0$ provided $\lambda_n \to 0$,

(F5) for $(a_n) \subset K$ a bounded sequence and $\|x_n\| \to 0$ it follows that $\|a_n x_n\| \to 0$ as $n \to \infty$, then $\|\cdot\|$ is called an *F-quasi norm*. For $c = 1$ we say that $\|\cdot\|$ is an *F-norm*.

2.4. Let f—a real-valued function defined on X—be such that $f(\lambda x) \leqslant f(x)$ for all $\lambda \in K$, $|\lambda| \leqslant 1$; a mapping f with this property is called *monotone*.

2.5. If $\|\cdot\|$ is a monotone c-norm on a vector space X which satisfies condition (F4), then we say that $\|\cdot\|$ is an *$\tilde{F}$-quasi norm*. For $c = 1$ we say that $\|\cdot\|$ is an *$\tilde{F}$-norm* (see Rolewicz [6]).

2.6. Let $f\colon X \to \boldsymbol{R}^+$. We say that the mapping f has *property* (B) if

(B) for every $\lambda \in K$ there exists a $d > 0$ such that

$$f(\lambda x) \leqslant d f(x) \quad \text{for every } x \in X.$$

2.7. Remark. If the field K is archimedean valued, then every $\tilde{F}$-quasi norm satisfies condition (B) (see Urbański [7]). From this it follows that in this case every $\tilde{F}$-quasi norm is an F-quasi norm (see Köthe [2]). But in the case when K is nonarchimedean valued there exist examples of $\tilde{F}$-norms not satisfying condition (F5). In some cases there exist examples of F-norms not satisfying (B) (see Urbański [7]).

The topology in X generated by an F-quasi norm is linear.

2.8. In the case when $\|\cdot\|$ is an $\tilde{F}$-quasi norm condition (F5) is equivalent to the following

(F′5) $\|\lambda x_n\| \to 0$ if $\|x_n\| \to 0$.

2.9. Remark. If the field K is sequentially compact in the topology generated by the valuation $|\cdot|$, then for every F-quasi norm on X conditions (F4) and (F5) are equivalent to the condition:

(G4) if $\|x_n - x\| \to 0$ and $|\lambda_n - \lambda| \to 0$, then $\|\lambda_n x_n - \lambda x\| \to 0$.

2.10. If a c-norm $\|\cdot\|$ on X satisfies condition (G4), then we say that $\|\cdot\|$ is a *c-paranorm*. For $c = 1$ we say that $\|\cdot\|$ is a *paranorm* on X (e.g., Wilansky [8]). Every F-quasi norm on X is a c-paranorm.

2.11. Remark. For every topology generated by the F-quasi norm (c-norm) there exists an equivalent topology generated by the F-norm (1-norm), (see, e.g., Bergh and Löfström [1]). But even in the case of topological equivalence, the geometric properties of the space X need not be equivalent, and hence the constant $c \geqslant 1$ is essential.

3. (ω,ϑ)-Convex modular. Let X be a vector space over a field K with nontrivial valuation $|\cdot|$. Given the mappings $\omega, \vartheta\colon K \to \boldsymbol{R}^+$ satisfying the following conditions:

(a1) there are constants $k, l > 0$ for which

$$\omega(\alpha\beta) \geqslant k\,\omega(\alpha)\,\omega(\beta), \quad \vartheta(\alpha\beta) \geqslant l\,\vartheta(\alpha)\,\vartheta(\beta) \qquad \text{for all } \alpha, \beta \in K,$$

(a2) $\lim\limits_{\alpha\to 0} \omega(\alpha) = 0$, and there exists $q > 0$ such that $\lim\limits_{|\alpha|\to\infty} \omega(\alpha)/\vartheta(\alpha) \geqslant q$,

(a3) $\omega(\alpha), \vartheta(\alpha) > 0$ for $\alpha \in K^*$, $\omega(0) = 0$,

(a4) there exist an $m > 0$ such that for every $b > 0$ there exists $a, c_0 \in \omega(K^*) = \{\omega(\alpha)\colon\ \alpha \in K^*\}$ such that $0 < a \leqslant b < c_0 \leqslant ma$.

3.1. Definition. The functional ϱ with values in $\boldsymbol{R}^+$ defined on X will be called *(ω, ϑ)-convex modular* on X if it satisfies the following conditions:

(M1) if $\varrho(\alpha x) = 0$ for every $\alpha \in K^*$, then $x = 0$ and $\varrho(0) = 0$,

(M2) $\varrho(\alpha x + \beta y) \leqslant \vartheta(\alpha)\varrho(x) + \vartheta(\beta)\varrho(y)$ for every vectors $x, y \in X$ and every $\alpha, \beta \in K$ with $\omega(\alpha)+\omega(\beta) \leqslant 1$,

(M3) $\varrho(x) = \varrho(-x)$.

Let ϱ be an (ω, ϑ)-convex modular on X. The set

$$X_\varrho = \{x\colon\ \lim_{\alpha\to 0} \varrho(\alpha x) = 0,\ x \in X\}$$

will be called a *modular space*. Note that X_ϱ is a vector subspace of the space X, but for the moment we forget about multiplication by scalars. Thus we consider X_ϱ as an Abelian group.

3.2. THEOREM. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and* let ϱ *be a (ω, ϑ)-convex modular on X. Then the functional*

$$\|x\|_\varrho = \inf\{\omega(a) > 0 : \varrho(x/a) \leqslant \omega(a)/\vartheta(a)\}$$

is a c-norm on the Abelian group X_ϱ with $c = m\min^{-1}(k, l)$.

Proof. For $x \in X_\varrho$ the set $\{\omega(a) > 0 : \varrho(x/a) \leqslant \omega(a)/\vartheta(a)\}$ is nonempty. So $\|x\|_\varrho \in [0, \infty)$.

(F1): If $\|x\|_\varrho = 0$, then there exists a sequence $(\beta_n) \subset K^*$ such that $\omega(\beta_n) \to 0$ and $\varrho(x/\beta_n) \leqslant \omega(\beta_n)/\vartheta(\beta_n)$. Let now $a \in K^*$. For sufficiently large n we have $\omega(a\beta_n) \leqslant 1$ and

$$\begin{aligned}\varrho(ax) &= \varrho(a\beta_n x/\beta_n) \leqslant \vartheta(a\beta_n)\varrho(x/\beta_n)\\ &\leqslant l^{-1}\vartheta^{-1}(a^{-1})\vartheta(\beta_n)\omega(\beta_n)\vartheta^{-1}(\beta_n) \leqslant l^{-1}\vartheta^{-1}(a^{-1})\omega(\beta_n).\end{aligned}$$

This implies $\varrho(ax) = 0$ for every $a \in K^*$. Hence, by (M1), we have $x = 0$. Converse implication is obvious.

(F2): It follows from (M3), immediately.

(F3): Let $x, y \in X_\varrho$. Denote $\eta = \min(k, l)$. Given any $\varepsilon > 0$, by property (a4), there exist elements $a, \beta, \gamma, \delta \in K$ such that

$$\|x\|_\varrho \leqslant \omega(a) < \|x\|_\varrho + 2^{-1}m^{-1}\varepsilon\eta \quad \text{and} \quad \varrho(x/a) \leqslant \omega(a)/\vartheta(a),$$

$$\|y\|_\varrho \leqslant \omega(\beta) < \|y\|_\varrho + 2^{-1}m^{-1}\varepsilon\eta \quad \text{and} \quad \varrho(y/\beta) \leqslant \omega(\beta)/\vartheta(\beta),$$

$$\omega(\gamma) \leqslant \eta^{-1}(\|x\|_\varrho + \|y\|_\varrho + m^{-1}\varepsilon\eta) < \omega(\delta), \quad \text{where} \quad \omega(\delta) \leqslant m\omega(\gamma).$$

We now observe that

$$\begin{aligned}\omega(a/\delta) + \omega(\beta/\delta) &\leqslant k^{-1}\omega^{-1}(\delta)\big(\omega(a) + \omega(\beta)\big)\\ &\leqslant k^{-1}\omega^{-1}(\delta)(\|x\|_\varrho + \|y\|_\varrho + m^{-1}\varepsilon\eta) \leqslant \eta k^{-1}\omega^{-1}(\delta)\omega(\delta) \leqslant 1,\end{aligned}$$

and by (M2) we obtain

$$\begin{aligned}\varrho\big((x+y)/\delta\big) &= \varrho(a\delta^{-1}xa^{-1} + \beta\delta^{-1}y\beta^{-1})\\ &\leqslant \vartheta(a\delta^{-1})\varrho(x/a) + \vartheta(\beta\delta^{-1})\varrho(y/\beta)\\ &\leqslant l^{-1}\vartheta(a)\omega(a)\vartheta^{-1}(\delta)\vartheta^{-1}(a) + l^{-1}\vartheta(\beta)\omega(\beta)\vartheta^{-1}(\delta)\vartheta^{-1}(\beta)\\ &\leqslant l^{-1}\vartheta^{-1}(\delta)\big(\omega(a) + \omega(\beta)\big) \leqslant \eta l^{-1}\omega(\delta)\vartheta^{-1}(\delta) \leqslant \omega(\delta)/\vartheta(\delta).\end{aligned}$$

It follows that

$$\|x+y\|_\varrho \leqslant \omega(\delta) \leqslant m\omega(\gamma) \leqslant m\eta^{-1}(\|x\|_\varrho + \|y\|_\varrho) + \varepsilon.$$

Hence

$$\|x+y\|_\varrho \leqslant m\min^{-1}(k, l)(\|x\|_\varrho + \|y\|_\varrho).$$

3.3. Remark. In the case when $\omega(K^*)$ is dense in $\boldsymbol{R}^+$, for every $m > 1$ the mapping $\omega(\cdot)$ satisfies condition (a4). Hence by the above Theorem it follows that the triangle inequality for $\|\cdot\|_\varrho$ is satisfied with the constant $c = \min^{-1}(k, l)$.

3.4. LEMMA. *Let ϱ be an (ω, ϑ)-convex modular on X. Then $\|\lambda_n x\|_\varrho \to 0$ provided $\lambda_n \to 0$ for $x \in X_\varrho$.*

Proof. (F4): Let $x \in X_\varrho$ and $\lambda_n \to 0$. Given any $a \in K^*$, we have $\varrho(\lambda_n x/a) \to 0$. Hence $\varrho(\lambda_n x/a) \leqslant \omega(a)/\vartheta(a)$ for sufficiently large index n. But 0 is the limit point of $\omega(K^*)$, and so $\|\lambda_n x\|_\varrho \to 0$.

3.5. EXAMPLES. (1) Let $K = \boldsymbol{Q}$, where $\boldsymbol{Q}$ denotes the set of rational numbers and let $X = \boldsymbol{R}$. We consider a function f on $\boldsymbol{R}^+$ defined by

$$f(u) = \begin{cases} 2 & \text{for} \quad u \in (\boldsymbol{R}\setminus\boldsymbol{Q}) \cap [0, 1],\\ 2u & \text{for} \quad u \in \boldsymbol{Q} \cap [0, 1],\\ -u+1+2i & \text{for} \quad u \in (2i-1, 2i],\ i \in \boldsymbol{N},\\ u+1-2i & \text{for} \quad u \in (2i, 2i+1],\ i \in \boldsymbol{N}.\end{cases}$$

Define now for $x \in X$

$$\varrho(x) = f(|x|),$$

ϱ is an (ω, ϑ)-convex modular on X with $\omega(a) = |a|$ and $\vartheta(a) = 2$ $(a \in K)$. In this case $X_\varrho = Q$.

(2) Let $X = K$ be a vector space over a field K with nontrivial valuation $|\cdot|$, and let n_0 be a fixed natural number. We introduce a function f on $\boldsymbol{R}^+$,

$$f(u) = \begin{cases} u & \text{for} \quad u \leqslant n_0,\\ [u] & \text{for} \quad u > n_0,\end{cases}$$

where $[u]$ denotes the entire part of the number u. We define $\varrho(x) = f(|x|)$; then ϱ is an (ω, ϑ)-convex modular on X with $\omega(a) = |a|$ and $\vartheta(a) = (1+n_0^{-1})|a|$. Moreover, $\|\lambda x\|_\varrho = |\lambda|\,\|x\|_\varrho$ for $\lambda \in K$, but ϱ is not a convex modular in the classical sense.

(3) Let g be a real-valued function defined on $\boldsymbol{R}^+ \times \boldsymbol{R}^+ \times \ldots$, satisfying the following conditions: $g(x) = 0$ is equivalent to $x = 0$; $g(x+y) \leqslant g(x) + g(y)$ for $x, y \in \boldsymbol{R}^+ \times \boldsymbol{R}^+ \times \ldots$, and $g(x) \leqslant g(y)$ for $x \leqslant y$.

Moreover, let $X^1, X^2, \ldots$ be vector spaces with (ω, ϑ)-convex modulars $\varrho_1(x_1), \varrho_2(x_2), \ldots$, respectively. For $x = (x_1, x_2, \ldots)$, we define

$$\varrho(x) = g\big(\varrho_1(x_1), \varrho_2(x_2), \ldots\big),$$

then $\varrho(\cdot)$ is an (ω, ϑ)-convex modular on $X = X^1 \times X^2 \times \ldots$

3.6. Remark. We observe that in Example (2) if $K = \boldsymbol{R}$, then the modular ϱ was not known before, because it is neither a nonconvex, nor convex, nor s-convex modular in the clasical sense (see, e.g. [3], [5]). Similarly in (1) the modular ϱ is a new type of modular.

4. F-quasi norm generated by the (ω,ϑ)-convex modular. From this section we will consider the functions ω, ϑ with the additional assumption

(a5) if $(\alpha_n) \subset K^*$ is a bounded sequence, then $\vartheta^{-1}(\alpha_n^{-1})$ and $\omega^{-1}(\alpha_n^{-1})$ are bounded.

4.1. Lemma. *For every $k_0 > k^{-1}$ there exists $\delta \in K^*$ such that*

$$\omega(\delta) > k_0 \quad \text{and} \quad |\delta| > 1.$$

Proof. Condition (a4) implies existence of a $\delta \in K^*$ such that $\omega(\delta) > k_0$. Then

$$(*) \qquad \omega(\delta^n) \geqslant k^{-1}(kk_0)^n \quad \text{for every } n \in \boldsymbol{N}.$$

We now consider two cases:

(i) $|\delta| < 1$. This implies that $\delta^n \to 0$. Then from $(*)$ we have $\omega(\delta^n) \to \infty$ as $n \to \infty$. This is a contradiction with $\omega(\delta^n) \to 0$.

(ii) $|\delta| = 1$. Then $\omega(\delta^{-n}) \leqslant k^{-1}\omega(e)\omega^{-1}(\delta^n)$ (where e denote an identity element of K) and we obtain $\omega(\delta^{-n}) \to 0$ as $n \to \infty$. This is a contradiction with the boundednes of $\omega^{-1}(\delta^{-n})$. Hence it must be $|\delta| > 1$.

4.2. Lemma. *Let $(\alpha_n) \subset K$ be a bounded sequence and $\|x_n\|_\varrho \to 0$. Then $\|\alpha_n x_n\|_\varrho \to 0$ as $n \to \infty$.*

Proof. From condition (a5) follows the existence of a constant $k_0 > (1 + k^{-1}l^{-2})k^{-1}$ such that $\vartheta^{-1}(\alpha_n^{-1}) \leqslant k_0$ for every $\alpha_n \neq 0$.

Now from Lemma 4.1 there exists $\delta \in K^*$, such that

$$(1) \qquad \omega(\delta) > k_0 \quad \text{and} \quad |\delta| > 1.$$

But

$$\omega(\alpha_n/\delta^m) \leqslant k^{-1}\omega^{-1}(\alpha_n^{-1})\omega(\delta^{-m}) \quad \text{for every } \alpha_n \in K^* \text{ and } m \in \boldsymbol{N}.$$

From this follows that for some $m \in \boldsymbol{N}$, $m > 1$,

$$(2) \qquad \omega(\alpha_n/\delta^m) \leqslant 1 \quad \text{for every } n \in \boldsymbol{N}.$$

By assumption, for every $\varepsilon > 0$ there exists $n_0 \in \boldsymbol{N}$ such that

$$(3) \qquad \|x_n\|_\varrho < \varepsilon k\omega(\delta^{-m}) \quad \text{for every } n \in \boldsymbol{N} \text{ and } n \geqslant n_0.$$

This implies that for every $n \geqslant n_0$ there exist $\beta_n \in K^*$ such that

$$(4) \qquad \omega(\beta_n) < \varepsilon k\omega(\delta^{-m}) \quad \text{and} \quad \varrho(x_n/\beta_n) \leqslant \omega(\beta_n)/\vartheta(\beta_n).$$

Then

$$\omega(\delta^m\beta_n) \leqslant k^{-1}\omega(\beta_n)\omega^{-1}(\delta^{-m}) < \varepsilon \quad \text{for every } n \geqslant n_0,$$

and from (2) and (M2) we have

$$\varrho(\alpha_n x_n/\delta^m\beta_n) \leqslant \vartheta(\alpha_n/\delta^m)\varrho(x_n/\beta_n).$$

Hence from (4) we obtain

$$(5) \qquad \varrho(\alpha_n x_n/\delta^m\beta_n) \leqslant \vartheta(\alpha_n/\delta^m)\omega(\beta_n)/\vartheta(\beta_n) \quad \text{for all } n \geqslant n_0.$$

Denote now $\gamma_n = \delta^m\beta_n$. Then

$$\begin{aligned}\omega(\beta_n)/\vartheta(\beta_n) &= \omega(\gamma_n\delta^{-m})/\vartheta(\gamma_n\delta^{-m})\\ &\leqslant k^{-1}l^{-1}\omega^{-1}(\delta^m)\vartheta^{-1}(\delta^{-m})\omega(\gamma_n)\vartheta^{-1}(\gamma_n),\end{aligned}$$

and we obtain for $\alpha_n \in K^*$, $n \geqslant n_0$,

$$\begin{aligned}\varrho(\alpha_n x_n/\gamma_n) &\leqslant k^{-1}l^{-1}\vartheta(\alpha_n\delta^{-m})\vartheta^{-1}(\delta^{-m})\omega^{-1}(\delta^m)\omega(\gamma_n)\vartheta^{-1}(\gamma_n)\\ &\leqslant k^{-1}l^{-2}\vartheta^{-1}(\alpha_n^{-1})\omega^{-1}(\delta^m)\omega(\gamma_n)\vartheta^{-1}(\gamma_n)\\ &\leqslant k^{-1}l^{-2}k_0k(kk_0)^{-m}\omega(\gamma_n)\vartheta^{-1}(\gamma_n)\\ &\leqslant l^{-2}k^{-m}k_0^{1-m}\omega(\gamma_n)\vartheta^{-1}(\gamma_n)\\ &\leqslant l^{-2}k^{-m}k^{m-1}(1+k^{-1}l^{-2})^{1-m}\omega(\gamma_n)\vartheta^{-1}(\gamma_n)\\ &\leqslant k^{-1}l^{-2}(1+k^{-1}l^{-2})^{1-m}\omega(\gamma_n)\vartheta^{-1}(\gamma_n) \leqslant \omega(\gamma_n)\vartheta^{-1}(\gamma_n).\end{aligned}$$

Hence

$$\varrho(\alpha_n x_n/\delta^m\beta_n) \leqslant \omega(\delta^m\beta_n)/\vartheta(\delta^m\beta_n),$$

and

$$\omega(\delta^m\beta_n) < \varepsilon \quad \text{for every } n \geqslant n_0.$$

Consequently, $\|\alpha_n x_n\|_\varrho \to 0$ as $n \to \infty$.

From Theorem 3.2 and Lemmas 3.4 and 4.2 follows

4.3. Theorem. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and let ϱ be an (ω, ϑ)-convex modular on X. Then the functional*

$$\|x\|_\varrho = \inf\{\omega(a) > 0:\ \varrho(x/a) \leqslant \omega(a)/\vartheta(a)\}$$

is an F-quasi norm on X_ϱ with the constant $c = m\min^{-1}(k, l)$.

Now we observe that there exist functions ω, ϑ satisfying conditions (a1)–(a4) but non of them satisfies condition (a5).

4.4. EXAMPLES. (1) Let $K = \boldsymbol{R}$. We define

$$\omega(a) = \begin{cases} |a| & \text{for} \quad a \in \boldsymbol{Z} \cap (-1, 1), \\ |a|^{-1} & \text{for} \quad a \in (n, n+1),\ n \in \boldsymbol{Z} \setminus \{-1, 0\}. \end{cases}$$

This function satisfies conditions (a1)–(a4) but does not satisfy condition (a5). Since, for example, for a bounded sequence $a_n = 2^{-n/2}$ $(n \in \boldsymbol{N})$ we have $\omega(a_n) = 2^{-n/2}$, $\omega(a_n)$ is a bounded sequence. But $\omega^{-1}(a_n^{-1}) = \omega^{-1}(2^{n/2}) = 2^{n/2} \to \infty$ as $n \to \infty$. For $\vartheta(a)$ we may take $\vartheta(a) = \omega(a)$ or

$$\vartheta_0(a) = \min\{1, |a|^{-1}\}, \quad a \in \boldsymbol{R} \setminus \{0\},\ \vartheta_0(0) = 1.$$

For $a_n = n^{-1}$ $(n \in \boldsymbol{N})$ we have $\vartheta_0^{-1}(a_n^{-1}) = \vartheta_0^{-1}(n) = n \to \infty$ as $n \to \infty$. Note that the function ω is not continuous on $\boldsymbol{R}$ but ϑ_0 is continuous on $\boldsymbol{R}$.

(2) Let $K = \boldsymbol{R}$, and $\omega(a) = c_1|a|^s$ $(s > 0,\ c_1 > 0)$,

$$\vartheta(a) = \begin{cases} 1 & \text{if} \quad a \in Q, \\ 1/2 & \text{if} \quad a \in \boldsymbol{R} \setminus Q. \end{cases}$$

The functions ω, ϑ satisfy conditions (a1)–(a5) and ϑ is not continuous at any point of $\boldsymbol{R}$.

5. Application to (s, t)-convex modulars. In this section we shall consider special cases of (ω, ϑ)-convex modulars, however, sufficiently general in order to include all classical cases of modulars.

5.1. Let s, t be real numbers such that $0 \leqslant t \leqslant s$, $s > 0$. We take $\omega(a) = |a|^s$, $\vartheta(a) = |a|^t$ for $a \in K$. In this case for simplificity the (ω, ϑ)-convex modular on X will be called an *(s, t)-convex modular*.

Hence the functional ϱ with values in $\boldsymbol{R}^+$ defined on X will be called an *(s, t)-convex modular* on X if it satisfies the following conditions:

(M1) if $\varrho(ax) = 0$ for every $a \in K$, then $x = 0$ and $\varrho(0) = 0$,

(M2) $\varrho(ax + \beta y) \leqslant |a|^t \varrho(x) + |\beta|^t \varrho(y)$ for every vectors $x, y \in X$ and every $a, \beta \in K$ with $|a|^s + |\beta|^s \leqslant 1$.

5.2. THEOREM. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and let ϱ be an (s, t)-convex modular on X. Then the functional*

$$\|x\|_{s,t} = \inf\{|a|^s > 0 : \varrho(x/a) \leqslant |a|^{s-t}\}$$

is an F-quasi norm on X_ϱ with the constant $c = r^s$, where

$$r = \inf\{|a| > 1 : a \in K\}.$$

Proof. From the forms of ω and ϑ it follows that $k = l = 1$ and $\omega(K^*) = \{|a|^s : a \in K^*\} = |K^*|^s$.

I. In the case when the valuation is discrete, $|K^*| = \{r^n : n \in \boldsymbol{Z}\}$ and the generator $r = \inf\{|a| > 1 : a \in K\}$. Hence $\omega(K^*) = \{r^{ns} : n \in \boldsymbol{Z}\}$ and $m = r^s$.

II. If the valuation is nondiscrete, then the set $|K^*|$ is dense in $\boldsymbol{R}^+$. This implies that $\omega(K^*)$ is dense in $\boldsymbol{R}^+$, too.

Applying Theorem 4.3 and Remark 3.3 we now obtain our theorem.

5.3. Remark. Constant $c = r^s$ in the triangle inequality has a specific character because in the case when the field K is nondiscretly valued we have $c = 1$. In the case when K is discretly valued we have $c = r^s > 1$ and r is the generator of a cyclic group $|K^*|$.

For the case of the field of p-adic numbers there exist examples of $(1, 0)$-convex modulars showing that the constant $c = p > 1$ cannot be dropped (see Urbański [7]). In this example the triangle inequality with $c = 1$ is false. From this it follows that the assumption that K is nonarchimedean valued does not imply that the $\tilde{F}$-quasi norm $\|\cdot\|_{s,t}$ is nonarchimedean.

5.4. LEMMA. *Let ϱ be a (s, t)-convex modular on X and let $\lambda, \gamma \in K$ be such that $|\lambda| > 1$, $|\gamma| < 1$ and $|\lambda^2\gamma| < 1$. Then we have*

$$|\gamma|^s \|\lambda x\|_{s,t} \leqslant \|x\|_{s,t}$$

for every $x \in X_\varrho$.

Proof. Let $x \in X_\varrho$; then

$$\begin{aligned} \|\lambda^2\gamma x\|_{s,t} &= \inf\{|a|^s : \varrho(\lambda x / a\lambda^{-1}\gamma^{-1}) \leqslant |a|^{s-t}\} \\ &\geqslant \inf\{|\lambda\gamma|^s : \varrho(\lambda x / a\lambda^{-1}\gamma^{-1}) \leqslant |a\lambda^{-1}\gamma^{-1}|^{s-t}\} \\ &= |\lambda\gamma|^s \|\lambda x\|_{s,t}. \end{aligned}$$

Hence

$$|\lambda\gamma|^s \|\lambda x\|_{s,t} \leqslant \|\lambda^2\gamma x\|_{s,t}.$$

This implies

$$|\gamma|^s\ \lambda x\|_{s,t} \leqslant |\lambda|^{-s} \|\lambda^2\gamma x\|_{s,t} \leqslant \|\lambda^2\gamma x\|_{s,t} \leqslant \|x\|_{s,t}.$$

5.5. THEOREM. *Let ϱ be an (s, t)-convex modular on X; then an F-quasi norm $\|\cdot\|_{s,t}$ has property* (B).

Proof. Suppose that condition (B) is not satisfied. Then there exists $\lambda \in K^*$, $|\lambda| > 1$ such that for every $n \in \boldsymbol{N}$ there exists $x \in X_\varrho$ such that $n\|x\|_{s,t} < \|\lambda x\|_{s,t}$.

We now choose $n \in \boldsymbol{N}$, $\beta \in K$ such that

$$|\lambda| < n, \quad 1 < |\beta|^s < n \quad \text{and} \quad |\lambda^2/\beta| < 1.$$

By $\sim$ (B), there exists an element $x_0 \in X_\varrho$ such that

$$|\beta|^s \| x_0 \|_{s,t} < n \| x_0 \|_{s,t} < \| \lambda x_0 \|_{s,t}.$$

Hence

$$(*) \qquad |\beta|^s \| x_0 \|_{s,t} < \| \lambda x_0 \|_{s,t}.$$

We denote $\gamma = \beta^{-1}$; then $|\gamma| < 1$ and $|\lambda^2 \gamma| < 1$. So, by Lemma 5.4, it follows $\| \lambda x_0 \|_{s,t} |\beta|^{-s} \leqslant \| x_0 \|_{s,t}$, and we get a contradiction with (*).

From Theorems 5.2, 5.5 and from monotonity of $\| \cdot \|_{s,t}$ follows

5.6. THEOREM. *Let X be a vector space over a field K with nontrivial valuation $|\cdot|$ and let ϱ be an (s, t)-convex modular on X, where $s \geqslant t \geqslant 0$, $s > 0$. Then the functional $\| \cdot \|_{s,t}$ is an $\tilde{F}$-quasi norm on X_ϱ satisfying condition* (B). *Moreover, if $s = t$, then the functional $\| \cdot \|_{s,t}$ is s-homogeneous.*

5.7. Remarks. In the case when

(a) $s = 1$, $t = 0$, (1, 0)-convex modular ϱ is a nonconvex modular defined by Musielak and Orlicz in [3]. For $K = \boldsymbol{R}$ or $\boldsymbol{C}$ (complex number) the functional $\| \cdot \|_{1,0}$ is the F-norm introduced in [3].

(b) $s = 1$, $t = 1$, (1,1)-convex modular ϱ is the convex modular considered in [5]. For $K = \boldsymbol{R}$ or $\boldsymbol{C}$ the functional $\| \cdot \|_{1,1}$ is the homogeneous norm given in [5].

(c) $s = t$, $0 < s < 1$, (s, s)-convex modular ϱ is the s-convex modular defined in [5]. For $K = \boldsymbol{R}$ or $\boldsymbol{C}$, $\| \cdot \|_{s,s}$ is the s-homogeneous norm (see [5]).

In [3] and [5] in each of the above cases the functional $\| \cdot \|_\varrho$ has been defined separately.

5.8. EXAMPLES. (1) Let g be a nonnegative convex function on $\boldsymbol{R}^+$, vanishing only at $u = 0$. Then $f(u) = g(|u|^s)$, $0 < s \leqslant 1$ is an (s, s)-convex modular on $\boldsymbol{R}$.

(2) If g is an (s, s)-convex modular on $\boldsymbol{R}$, then the function $f(u) = g^{t/s}(u)$, $0 < t \leqslant s$, is an (s, t)-convex modular on $\boldsymbol{R}$.

(3) Let K be a field with nontrivial valuation $|\cdot|$. Moreover, let f be an (s, t)-convex modular on $\boldsymbol{R}$, where $0 \leqslant t \leqslant s \leqslant 1$, $s > 0$. Given $X = K$. Then the functional $\varrho(x) = f(|x|)$ is an (s, t)-convex modular on X.

References

[1] J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, Berlin, Heidelberg, New York 1976.
[2] G. Köthe, *Topological vector spaces* I, Springer-Verlag, Berlin, Heidelberg, New York 1969.
[3] J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. 18 (1959), 49–65.

[4] L. Narici, E. Beckenstein and G. Bachman, *Functional analysis and valuation theory*, Marcel Dekker, Inc., New York 1971.
[5] W. Orlicz, *A note on modular spaces* I, Bull. Acad. Polon. Sci. 9, 3 (1961), 157–162.
[6] S. Rolewicz, *Linear metric spaces*, Monografie Matematyczne 56, PWN, Warszawa 1972.
[7] R. Urbański, *A modular space over a field with valuation*, in: Proceedings of the International Conference on Constructive Function Theory, Varna 1981, 1983, pp. 584–589.
[8] J. Wilansky, *Functional Analysis*, Blaisdell 1964.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Mielżyńskiego 27/29, 61-725 Poznań

Received June 23, 1982 (1774)