Thus (see for example [4]) the fact that x_0 is a minimum of the function $\alpha(F(x))+\beta\|F(x)-F(x_0)\|$ implies that it is a local Pareto minimum of problem (VP).

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Commutative differential algebras with an algebraic element

by

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Dedicated to Jan Mikusiński on the 70th birthday

Abstract. There are constructed commutative differential algebras containing an element h, which satisfies a polynomial equation. For the commutativity it is necessary that the polynomial possesses a double zero. In one case the algebras contain also an integral of h.

P. Antosik, J. Mikusiński and R. Sikorski in [2] suggested the study of associative differential algebras containing an element h with the properties

$$h = h^2 \tag{1}$$

and $h' \neq 0$. Since h should be interpreted as Heaviside's jump function, the derivative of h was denoted by

$$\delta = h'. \tag{2}$$

Article [3] gives a survey of such algebras. Afterwards article [4] was written, where (1) was replaced by

$$h = 3h^2-2h^3 \tag{3}$$

corresponding to the property $h(0) = 1/2$. However, the results obtained so far are not satisfactory; particulary, all differential algebras with (1) or (3) are noncommutative.

In what follows we construct commutative differential algebras (cf. [5]) where h satisfies other algebraic relations than (1) and (3). At the end of this article we list the references [8]–[16], which were omitted in [3] in printing (the numbering of references refers to this in paper [3]). In [12] one finds similar differential algebras without an algebraic relation for h.

1. Analysis. To begin with we assume the existence of a commutative and associative differential algebra containing an element h with the property

$$h\delta = c\delta \tag{4}$$

for a certain constant $c \neq 0$ and, as before, $\delta \neq 0$ with (2).

Differentiating (4) we obtain

$$h\delta' = c\delta' - \delta^2, \quad h\delta'' = c\delta'' - 3\delta\delta' \tag{5}$$

and by induction

$$h\delta^{(n)} = c\delta^{(n)} - \frac{1}{2}\sum_{i=1}^{n}\binom{n+1}{i}\delta^{(i-1)}\delta^{(n-i)}. \tag{6}$$

Multiplying (4) by δ' and (5) by δ etc., and comparing the results, we find:

$$\delta^3 = \delta^2\delta' = \delta^2\delta'' = \delta\delta'^2 = \delta^2\delta''' = \delta\delta'\delta'' = \delta'^3 = 0. \tag{7}$$

If h is an algebraic element, then there exists a polynomial $P(h)$ with

$$P(h) = 0. \tag{8}$$

According to (4) this implies $P(h)\delta = P(c)\delta = 0$. Similarly, differentiation of (8) implies $P'(h)\delta = 0$ and therefore $P'(c)\delta = 0$. Hence, in view of $\delta \neq 0$ we obtain the equations

$$P(c) = P'(c) = 0. \tag{9}$$

Equation (4) means that $(h^2-2ch)' = 0$, which suggests the choice $c = 1/2$ as in [1]. However, in view of $h^n\delta = c^n\delta$ for all natural numbers n we also have

$$(h^n - nc^{n-1}h)' = 0.$$

2. Synthesis. We now show the existence of a commutative and associative differential algebra C with the basis elements

$$1, h, h^2, \ldots, h^{p-1}, \delta^{(n)}, \delta^{(n)}\delta^{(m)} \tag{10}$$

for a fixed integer $p \geq 2$ and all integers $m, n \geq 0$ as well as (2). The relations in C will be (6) with $c \neq 0$, (8) and

$$\delta^{(n)}\delta^{(m)}\delta^{(k)} = 0 \tag{11}$$

for all integers $k, m, n \geq 0$, where $P(h)$ is an arbitrary polynomial of degree p with (9).

Multiplying (6) by $\delta^{(m)}$ and using (11) we obtain

$$h\delta^{(n)}\delta^{(m)} = c\delta^{(n)}\delta^{(m)} \tag{12}$$

for all integers $m, n \geq 0$. Relations (6) and (12) imply by induction

$$h^m\delta^{(n)} = c^m\delta^{(n)} - \frac{m}{2}c^{m-1}\sum_{i=1}^{n}\binom{n+1}{i}\delta^{(i-1)}\delta^{(n-i)} \tag{13}$$

for all natural numbers m. Hence equations (9) show that relations (6) and (8) are compatible.

Now it is easy to check the associative law for the basis elements (10) in all possible cases. Since by differentiation of (6), (8) and (11) we always obtain valid relations, C is really a differential algebra.

Let us mention that (6) and (12) imply

$$(h-c)^2\delta^{(n)} = 0 \tag{14}$$

for all n.

3. Extension. In the case $P(0) = 0$, i.e., $P(h) = Q(h)h$ with $Q(c) = Q'(c) = 0$, we can adjoin to C an element t with the properties

$$t' = h, \tag{15}$$

$$Q(h)t = 0. \tag{16}$$

The adjoining of t requires the extension of the basis (10) by the new basis elements

$$t^k, t^k h, \ldots, t^k h^{p-2}, t^k\delta^{(n)}, t^k\delta^{(n)}\delta^{(m)} \tag{17}$$

for all natural numbers k. From (16) we obtain by differentiation

$$Q(h)h + Q'(h)\delta t = 0$$

or

$$P(h) + Q'(c)\delta t = 0,$$

but in view of (8) and $Q'(c) = 0$ this is not a new equation.

Thus we are left with the problem of introducing between the basis elements new relations which make sense. Relations like $t\delta = 0$ or $t^2\delta = 0$ lead to $\delta^2 = 0$ and therefore they are unsuitable. A possible relation would be $t\delta^2 = 0$ with the consequences

$$2t\delta\delta' + c\delta^2 = 0,$$

$$t(\delta\delta'' + \delta'^2) + 2c\delta\delta' = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

From (13) or (14) we find $Q(h)\,\delta^{(n)} = 0$ for all n. Hence, according to (8) and (16), we obtain

$$Q(h)x = 0$$

for all elements x, though in view of the basis elements (10) we have $Q(h) \neq 0$.

4. Generalization. Finally we want to sketch the possible conditions under which there seems to exist a commutative differential algebra without (4) and (11). We start from the relation

$$(h-c)^q h = 0 \tag{18}$$

with $q \geqslant 1$, $c \neq 0$, which implies

$$(h-c)^{q-1}h^2 = c(h-c)^{q-1}h. \tag{19}$$

Differentiating (18) we obtain

$$(h-c)^q\,\delta + q(h-c)^{q-1}h\,\delta = 0;$$

multiplying this equation by h and considering (19) we obtain $(h-c)^{q-1}h\delta = 0$, $(h-c)^q\delta = 0$ and analogously

$$(h-c)^{q-1}\,\delta = 0. \tag{20}$$

From this we find inductively for $n \leqslant q$

$$(h-c)^{q-n}\delta^{2n-1} = 0 \tag{21}$$

and therefore for $n = q$

$$\delta^{2q-1} = 0. \tag{22}$$

On the other hand, (20) implies inductively

$$(h-c)^q\,\delta^{(n)} = 0 \tag{23}$$

for all n.

For $q = 2$ relations (20) and (23) are equal to (4) and (14), respectively, and relation (22) is a special case of (11). This means that in the case $P(h) = (h-c)^2 h$ relation (4) is not independent but a consequence of (8).

5. The case $q = 3$. For $q = 3$ equation (21) reads for $n = 1, 2$ and 3, respectively,

$$(h-c)^2\,\delta = 0, \quad (h-c)\,\delta^3 = 0, \quad \delta^5 = 0. \tag{24}$$

Differentiating the first equation of (24), we obtain the equations

$$(h-c)^2\delta' + 2(h-c)\,\delta^2 = 0,$$

$$(h-c)^2\delta'' + 6(h-c)\,\delta\delta' + 2\delta^3 = 0,$$

$$(h-c)^2\delta''' + (h-c)(8\,\delta\delta'' + 6\,\delta'^2) + 12\,\delta^2\delta' = 0,$$

$$(h-c)^2\delta^{(4)} + (h-c)(10\,\delta\delta''' + 20\,\delta'\delta'') + 20\,\delta^2\delta'' + 30\,\delta\delta'^2 = 0,$$

etc. and differentiating the second equation of (24) we obtain

$$3(h-c)\,\delta^2\delta' + \delta^4 = 0, \tag{25}$$

$$3(h-c)(\delta^2\delta'' + 2\,\delta\delta'^2) + 7\,\delta^3\delta' = 0, \tag{26}$$

$$3(h-c)(\delta^2\delta''' + 6\,\delta\delta'\delta'' + 2\,\delta'^3) + 10\,\delta^3\delta'' + 27\,\delta^2\delta'^2 = 0. \tag{27}$$

Calculating $(h-c)^2\,\delta^{(n)}\,\delta^{(m)}$ in different ways, we obtain for $n = 2$, $m = 0$ equation (25). For $n = 3$, $m = 0$ and $n = 2$, $m = 1$ we obtain, respectively,

$$(h-c)(4\,\delta^2\delta'' + 3\,\delta\delta'^2) + 6\,\delta^3\delta' = 0,$$

$$(h-c)(3\,\delta\delta'^2 - \delta^2\delta'') + \delta^3\delta' = 0$$

and from these equations

$$(h-c)\,\delta^2\delta'' + \delta^3\delta' = 0, \quad 3(h-c)\,\delta\delta'^2 + 2\,\delta^3\delta' = 0 \tag{28}$$

corresponding to (26). For $n = 4$, $m = 0$ and $n = 3$, $m = 1$ we obtain, respectively,

$$(h-c)(\delta^2\delta''' + 2\,\delta\delta'\delta'') + 2\,\delta^3\delta'' + 3\,\delta^2\delta'^2 = 0,$$

$$(h-c)(4\,\delta\delta'\delta'' + 3\,\delta'^3 - \delta^2\delta''') + 6\,\delta^2\delta'^2 = 0.$$

But (27) is linearly dependent on these equations, and so it is not possible to determine, as in (28), the elements

$$(h-c)\,\delta^2\delta''', \quad (h-c)\,\delta\delta'\delta'', \quad (h-c)\,\delta'^3.$$

Of course we can try to choose one of these elements in a suitable way; however, no reason can be seen for a natural choice.

Further consequences of the foregoing equations are

$$\delta^4\delta' = \delta^3\delta'^2 = \delta^4\delta'' = 0$$

instead of (7) and besides (23) with $q = 3$

$$(h-c)\,\delta^2\delta'^2 = (h-c)\,\delta\delta'^3 = (h-c)\,\delta^2\delta'\delta'' = 0$$

as well as

$$(h-c)^2\delta'^3 = (h-c)^2\,\delta'^2\delta'' = 0.$$

The first equation of (24) means that

$$(\tfrac{1}{3}h^3 - ch^2 + c^2h)' = 0.$$

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Modular spaces over a field with valuation generated by a (ω,ϑ)-convex modular

by

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Dedicated to Professor Jan Mikusiński on the occasion of his 70th birthday

Abstract. Let X be a vector space over a field K with valuation. The form of an (ω,ϑ)-convex modular on X is given, generalizing the existing definitions of modulars on vector spaces. A simple formula for an F-norm on the modular space X_ϱ generated by modular on X is proved.

1. Introduction. In the literature on modular spaces, nonconvex, convex and s-convex modulars are considered (e.g., Musielak–Orlicz [3], Orlicz [5]). In each of these cases the functional $\|\cdot\|_\varrho$ generating the linear topology in modular space X_ϱ has been defined separately and it was F-norm, norm and s-norm, respectively, in a real vector space.

In this paper we introduce (ω,ϑ)-convex modular for some functions $\omega,\vartheta\colon K\to\boldsymbol{R}^+$, when K is a valued field. The (ω,ϑ)-convex modulars include all cases of modulars considered so far, by selecting appropriate functions ω, ϑ, and at the same time give generalizations of modular spaces to spaces over an arbitrary field with valuation. It is possible now to consider modulars not known before, namely such that they are not any of the three types mentioned above.

Moreover, even in the classical case the definition of $\|\cdot\|_\varrho$ is new because it allows to present all three separate definitions in one form.

2. Preliminary remarks.

2.1. Let X be a vector space over a field K with nontrivial valuation $|\cdot|\colon K\to\boldsymbol{R}$ (where $\boldsymbol{R}$ denotes the real numbers). The set of real number, $|a|$, $a\in K$, will be called the *set of values of* K and will be denoted by $|K|$. The values $|K^*|$, where K^* denotes the set of nonzero elements of K, form a multiplicative subgroup of the positive reals $(\boldsymbol{R}^+,\cdot)$. However, as is well known, $\boldsymbol{R}^+$, has only two types of subgroups, they are either cyclic groups or groups dense in $\boldsymbol{R}^+$ ([4]). If $|K^*|$ is an infinite cyclic group, the valuation is called *discrete*; equivalently, K is said to be *discretly valued*.