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Sufficient condition for Pareto optimization in Banach spaces

by

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Abstract. The paper contains sufficient conditions for vector optimization in Banach spaces.

The aim of this paper is an extension of the results of [3] concerning sufficient conditions for vector optimization to Banach spaces.

Let X, Y_1, Y_2, Z, P be Banach spaces over reals. Let Y_1, Y_2, P be ordered. Let F, G_1, G_2, H be continuously differentiable operators mapping a domain $U \subset X$ into

$$F\colon U \to P,$$
$$G_1\colon U \to Y_1,$$
$$G_2\colon U \to Y_2,$$
$$H\colon U \to Z.$$

We are looking for a local Pareto minimum of the following problem

$$\text{(VP)} \qquad \begin{aligned} &F(x) \to \inf, \\ &G_1(x) \leqslant 0, \\ &G_2(x) \leqslant 0, \\ &H(x) = 0. \end{aligned}$$

We recall that $x_0 \in U$ is a *local Pareto minimum* of problem (VP) if there is a neighbourhood Q of x_0 such that for $x' \in Q$ satysfying constraints of problem (VP)

$$F(x) \leqslant F(x_0) \quad \text{implies} \quad F(x_0) \leqslant F(x).$$

THEOREM 1. *Suppose that*

(i) *there are continuous linear functionals*

$$\alpha \in P^*, \quad \lambda_1 \in Y_1^*, \quad \lambda_2 \in Y_2^*, \quad \gamma \in Z^*$$

such that

$$\alpha(\nabla F)+\lambda_1(\nabla G_1)+\lambda_2(\nabla G_2)+\gamma(\nabla H) = 0,$$

where ∇F, ∇G_1, ∇G_2, ∇H *are the differentials of* F, G_1, G_2, H *taken at the point* x_0 *(this is called a necessary condition of optimality of the Kuhn–Tucker type)*;

(ii) *the functionals* $\alpha, \lambda_1, \lambda_2$ *are uniformly positive, i.e., there are positive constant* C_α, C_1, C_2 *such that for* $p \geqslant 0$, $y_1 \geqslant 0$, $y_2 \geqslant 0$

$$\|p\| \leqslant C_\alpha \alpha(p),$$

$$\|y_1\| \leqslant C_1\lambda_1(y_1),$$

$$\|y_2\| \leqslant C_2\lambda_2(y_2);$$

(iii) *the constraints are active at* x_0, *i.e.*,

$$G_1(x_0) = 0, \quad G_2(x_0) = 0, \quad H(x_0) = 0;$$

(iv) ∇F *is a surjection on* P *and* $(\nabla G, \nabla G_2, \nabla H)$ *is a surjection on* $Y_1 \times Y_2 \times H$;

(v) *the space* $L_1 = \ker \nabla F$ *and the halfsubspace*

$$L_2 = \ker\nabla G_1 \cap \ker \nabla H \cap \{x\colon\ \nabla G_2(x) \leqslant 0\}$$

have a positive gap d, *i.e.*,

$$d = \max(\inf\{\|x-y\|,\ x \in L_1,\ y \in L_2,\ \|x\| = 1\},$$
$$\inf\{\|x-y\|,\ x \in L_1,\ y \in L_2,\ \|y\| = 1\}) > 0.$$

Then x_0 *is a local Pareto minimum of problem* (VP).

Proof. The prof will be conducted in 3 steps. In the first step we shall show that x_0 is a local minimum of the following scalar problem:

$$\text{(MPe)}\qquad \begin{aligned} &\alpha(F(x))+\beta\|F(x)-F(x_0)\| \to \inf,\\ &G_1(x) = 0,\\ &G_2(x) \leqslant 0,\\ &H(x) = 0 \end{aligned}$$

for all $\beta > 0$. In the second step, using results of [3] we shall show that x_0 is a local solution of the following problem

$$\text{(MP)}\qquad \begin{aligned} &\alpha(F(x))+\beta\|F(x)-F(x_0)\| \to \inf,\\ &G_1(x) \leqslant 0,\\ &G_2(x) \leqslant 0,\\ &H(x) = 0. \end{aligned}$$

The third step consists of the showing that for sufficiently small β each local solution of the problem (MP) is a local Pareto solution of problem (VP).

We shall start with the first step. By assumption (iv) and by the Ljusternik theorem [1] for each $\varepsilon > 0$ there is Q such that for $x \in Q$, such that $G_1(x) = 0$, $G_2(x) \leqslant 0$, $H(x) = 0$, there is $h \in L_2$ such that

$$\|x_0+h-x\| \leqslant \varepsilon\|h\|. \tag{1}$$

Basing ourselves on (v) and (iv), we find that there is a $k > 0$ such that

$$\|h\| \leqslant k\|\nabla F(h)\|. \tag{2}$$

Now we shall estimate $\alpha(F(x)-F(x_0))$ from below. By assumption (i), $\alpha(\nabla F(h)) \geqslant 0$. Thus by (1) and (2),

$$\alpha(F(x)-F(x_0)) \geqslant -\varepsilon\|h\| \geqslant -\varepsilon k\|\nabla F(h)\|.$$

By (1) and by the definition of a differential there is a neighbourhood $Q_1 \subset Q$ such that for $x \in Q_1$ satisfying the constraint there is an $h \in L_2$ such that (1) holds and

$$\|\nabla F(h)\| \leqslant 2\,\|(F(x)-F(x_0))\|.$$

Hence

$$\alpha(F(x))-\alpha(F(x_0)) \geqslant -2\varepsilon h\ \|F(x)-F(x_0)\|.$$

Taking $\beta > 2\varepsilon h$, we find that x_0 is a local solution of the problem (MPe).

Taking β sufficiently small, by Theorem 1 of [3] we conclude that x_0 is a local minimum of problem (MP). Here we use the fact that λ_1 is uniformly positive. To finish the proof we shall take $\beta < 1/C$. Then the function $\alpha(F(x))+\beta\|F(x)-F(x_0)\|$ is an ordering function at the point x_0, which means that $F(x) \leqslant F(x_0)$ implies that

$$\alpha(F(x))+\beta\|F(x)-F(x_0)\| \leqslant \alpha(F(x_0)).$$

In fact,

$$\alpha(F(x_0)-F(x)) \geqslant (1/C)\|F(x)-F(x_0)\| \geqslant \beta\|F(x)-F(x_0)\|.$$

Thus (see for example [4]) the fact that x_0 is a minimum of the function $\alpha(F(x))+\beta\|F(x)-F(x_0)\|$ implies that it is a local Pareto minimum of problem (VP).

References

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Commutative differential algebras with an algebraic element

by

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Dedicated to Jan Mikusiński on the 70th birthday

Abstract. There are constructed commutative differential algebras containing an element h, which satisfies a polynomial equation. For the commutativity it is necessary that the polynomial possesses a double zero. In one case the algebras contain also an integral of h.

P. Antosik, J. Mikusiński and R. Sikorski in [2] suggested the study of associative differential algebras containing an element h with the properties

$$h = h^2 \tag{1}$$

and $h' \neq 0$. Since h should be interpreted as Heaviside's jump function, the derivative of h was denoted by

$$\delta = h'. \tag{2}$$

Article [3] gives a survey of such algebras. Afterwards article [4] was written, where (1) was replaced by

$$h = 3h^2-2h^3 \tag{3}$$

corresponding to the property $h(0) = 1/2$. However, the results obtained so far are not satisfactory; particulary, all differential algebras with (1) or (3) are noncommutative.

In what follows we construct commutative differential algebras (cf. [5]) where h satisfies other algebraic relations than (1) and (3). At the end of this article we list the references [8]–[16], which were omitted in [3] in printing (the numbering of references refers to this in paper [3]). In [12] one finds similar differential algebras without an algebraic relation for h.