Our *complement* says that, in spite of the above example, we can prove that *the function y given by* (14) *is n-times continuously differentiable so that it is the unique solution of* (11).

Proof. Multiplying both sides of (11)′ by h^n we obtain

$$a_n y + a_{n-1} hy + \ldots + a_0 h^n y = h^n f + \beta_{n-1} h + \ldots + \beta_0 h^n.$$

Then $F(t) = h^n f + \beta_{n-1} h + \ldots + \beta_0 h^n$ is surely n-times continuously differentiable. Thus, by $y \in \mathscr{C}'$ and by (2), we have the result:

$$y = -a_n^{-1}(a_{n-1} hy + a_{n-2} h^2 y + \ldots + a_0 h^n y) + a_n^{-1}\{F(t)\}$$

is once continuously differentiable and its derivative satisfies

$$(15) \quad y' = -a_n^{-1}(a_{n-1} hy' + a_{n-2} h^2 y' + \ldots + a_0 h^n y') + a_n^{-1}\{F'(t)\} + \text{a polynomial in } t,$$

because, e.g.,

$$(h^3 y)' = h^2 y = h^2(hy' + y(0)) = h^3 y' + h^2 y(0)$$

by (9). Thus y' given by (15) is continuously differentiable in t and satisfies

$$y'' = -a_n^{-1}(a_{n-1} hy'' + a_{n-2} h^2 y'' + \ldots + a_0 h^n y'') + a_n^{-1}\{F''(t)\} + \text{a polynomial in } t$$

and so forth.

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A remark on Yosida's complement to Mikusiński's operational calculus

by

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Dedicated to Professor J. Mikusiński on his 70th birthday

Abstract. According to the Mikusiński theory of operational calculus, the Cauchy problem for the nth order ordinary differential equation with complex coefficiants and with inhomogeneous term $f \in C[0, \infty)$ is transformed into the operational equation:

$$(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) y = f + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \ldots + c_0.$$

As a complement to the theory, Prof. K. Yosida showed the fact which states that the solution y of the above operational equation is n-times continuously differentiable so that y is the true solution of the original equation. In this paper, a remark on the above complement is made by giving a direct proof.

It is well known, in the Mikusiński theory of operational calculus, that the Cauchy problem:

$$(1) \quad \begin{gathered} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = f, \\ y(0) = b_0, \quad y'(0) = b_1, \ldots, y^{(n-1)}(0) = b_{n-1}, \\ a_i \in C, \quad i = 0, \ldots, n, \quad b_j \in C, \ j = 0, \ldots, n-1 \quad \text{and} \quad f \in C[0, \infty) \end{gathered}$$

is transformed into the operational equation:

$$(2) \quad \begin{gathered} (a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0) y = f + c_{n-1} s^{n-1} + c_{n-2} s^{n-2} + \ldots + c_0, \\ c_m = a_{m+1} b_0 + a_{m+2} b_1 + \ldots + a_n b_{n-m-1}, \quad m = 0, 1, \ldots, n-1, \end{gathered}$$

where $s = 1/h$ $(= 1/\{1\})$ (cf. [1], [2] and [3]). Therefore we have

$$(3) \quad y = \frac{f}{p(s)} + \frac{q(s)}{p(s)}$$

with $p(s) = a_n s^n + \ldots + a_0 = a_n (s - \alpha_1)(s - \alpha_2) \ldots (s - \alpha_n)$

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and

$$q(s) = c_{n-1}s^{n-1} + \ldots + c_0 = c_{n-1}(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_{n-1}).$$

In [4], K. Yosida proved, in the framework of Mikusiński's operational calculus, that the solution of equation (1) or (2) has the n-times continuous differentiability.

In the present paper we shall give another proof to it by making the following

Remark. *Let f be continuous and g_i continuously differentiable with $i = 1, 2, \ldots, n$. Then the n-times iterated convolution $g_1 g_2 \ldots g_n f$ is an exactly n-times continuously differentiable function, where the product is taken in the sense of convolution, and we have*

$$(4) \qquad (g_1 g_2 \ldots g_n f)^{(n)} = \Big(\prod_{i=1}^{n} \big(g_i' + g_i(0)\big)\Big) f$$

and more generally

$$(5) \qquad (g_1 g_2 \ldots g_n f)^{(m)} = \Big(\prod_{i=1}^{m} \big(g_{\sigma(i)}' + g_{\sigma(i)}(0)\big)\Big)\Big(\prod_{i=m+1}^{n} g_{\sigma(i)}\Big) f$$

for $m \leqslant n$ and any permutation σ. Here in (4) and (5), $g_i(0)$ is identified with the constant operator $\{g_i(0)\}/h$ and it is to be noted that $g_i(0)f = \{g_i(0)f(t)\}$.

Proof. *The case $n = 1$.* By an elementary calculus,

$$(g_1 f)' = \Big\{\Big(\int_0^t g_1(t-u)f(u)\,du\Big)'\Big\} = \Big\{\int_0^t g_1'(t-u)f(u)\,du + g(0)f(t)\Big\}$$
$$= \big(g_1' + g_1(0)\big)f.$$

The general case. Recalling the commutativity of the convolution product, we have

$$\begin{aligned}(g_1 g_2 \ldots g_n f)^{(n)} &= \big(g_1(g_2 g_3 \ldots g_n f)\big)^{(n)} \\ &= \Big(\big(g_1' + g_1(0)\big)(g_2 g_3 \ldots g_n f)\Big)^{(n-1)} \\ &= \Big(g_2\Big(\big(g_1' + g_1(0)\big) g_3 \ldots g_n f\Big)\Big)^{(n-1)} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \Big(\prod_{i=1}^{n} \big(g_i' + g_i(0)\big)\Big) f.\end{aligned}$$

The proof of general formula (5) is ommited.

Hence we can show that $\big(1/p(s)\big)f$ of (3) is n-times continuously differentiable on account of $1/(s-a_i) = \{\exp(a_i t)\}$, and so is the solution of equation (1) or (2), because $q(s)$ is only a differential operator.

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