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The Bergman and Schiffer transforms on weighted norm spaces

by

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Abstract. We characterize the weighted measures λ on a plane region $D \notin O_G$ for which the Bergman projection and the Bergman–Schiffer transform are bounded on $L_p(D:\lambda)$, $1 < p < \infty$.

§1. Introduction. Let $D \notin O_G$ (i.e., D has a nontrivial Green's function) be a plane region and let λ be a non-negative locally integrable function on D. This paper is concerned with the question of the boundedness of the Bergman projection P_D and the Bergman–Schiffer transform Q_D in $L_p(D:\lambda)$, $1 < p < \infty$. This is done by introducing two classes of weights $M_p(D)$ and $M_p(\partial D)$ in terms of a universal cover mapping of D and non-euclidean (hyperbolic) sectors. In this way we are able to generalize the results of Bekollé and Bonami [1] and also extend our previous results in [3], [4]. The question of the boundedness of the Bergman projection has been considered in the past by several authors. The reader is referred to the articles [1], [2]–[4], [6]–[7], [9], [10] for details.

§2. Preliminaries. Let $\Delta = \{z\colon |z| < 1\}$ denote the unit disk in the plane. We recall several known facts about the noneuclidean geometry on Δ which is known as the Poincare's model of hyperbolic geometry.

The Bergman and the Hilbert kernel of Δ are given by

$$k(z, \zeta) = \frac{1}{\pi}(1-z\bar{\zeta})^{-2}; \qquad z, \zeta \in \Delta,$$

and by

$$h(z, \zeta) = \frac{1}{\pi}(z-\zeta)^{-2}; \qquad z, \zeta \in \Delta,\ z \neq \zeta,$$

respectively. The line element in this Poincaré geometry is given by

$$dl(z) = \sqrt{\pi k(z, \bar{z})}\,|dz|$$

and the surface element is

$$ds(z) = \pi k(z, \bar{z})\, d\sigma(z),$$

where $d\sigma(z) = dx\,dy$, $z = x+iy$, denotes the area Lebesgue measure. The above line and surface elements are invariant under the action of the group Möb(Δ), where each $A \in$ Möb(Δ) is given by

$$A(z) = \alpha\frac{z-\zeta}{1-z\bar{\zeta}}; \quad |\alpha| = 1, \ \zeta \in \Delta.$$

If L is a circle orthogonal to the boundary of Δ, $\partial\Delta$, then the part of L in Δ is called a *noneuclidean line* (or *geodesic*) in the Poincaré geometry. Naturally, a diameter of Δ is also a noneuclidean line and every noneuclidean line separates the noneuclidean plane Δ into two noneuclidean half planes. Any part of a noneuclidean line is called a *noneuclidean segment*.

By a *noneuclidean sector* in Δ we shall mean a subset S of Δ bounded by a simple curve consisting of finitely many noneuclidean segments and possibly a part of $\partial\Delta$. We distinguish two special families $N(\partial\Delta)$ and $N(\Delta)$ of noneuclidean sectors in Δ as follows:

$S \in N(\partial\Delta)$ if S is bounded by a simple curve consisting of (i) an arc of $\partial\Delta$ with the possibility that this arc can be reduced to a point, (ii) from the end points of this arc in $\partial\Delta$ we have a finite number of noneuclidean segments arranged in conjugated pairs, (iii) the last conjugated pair is joined by a noneuclidean segment inside Δ.

The family $N(\Delta)$ consists of all $S \in N(\partial\Delta)$ and also similar sectors as in $N(\partial\Delta)$ but, where the "first" conjugated pair is joined by a noneuclidean segment inside Δ. Evidently, these families remain invariant under the action of any $A \in$ Möb(Δ) and they cover Δ.

Let D be a measurable subset of the plane and let λ be a non-negative locally integrable function on D. The space $L_p(D:\lambda)$ stands for the class of functions f on D for which

$$\|f\|_{L_p(D:\lambda)} = \Big\{\int_D |f(z)|^p \lambda(z)\,d\sigma(z)\Big\}^{1/p} < \infty$$

is finite. We write $L_p(D)$ for $L_p(D:1)$ and $\|f\|_p$ for $\|f\|_{L_p(D)}$. We shall always assume that $1 < p < \infty$ and that $q = p/(p-1)$.

Let λ be a non-negative locally integrable function on Δ. λ is said to *belong to* $M_p(\Delta)$ if it satisfies the Muckenhoupt condition:

$$\operatorname*{Sup}_S \Big\{\frac{1}{|S|}\int_S \lambda(z)\,d\sigma(z)\Big\}\cdot\Big\{\frac{1}{|S|}\int_S \lambda(z)^{-1/(p-1)}d\sigma(z)\Big\}^{p-1} < \infty,$$

where the supremum is taken over all $S \in N(\Delta)$. Here $|S| = \sigma(S)$. When the family $N(\Delta)$ is replaced by the subfamily $N(\partial\Delta)$, λ is then said to *belong to* $M_p(\partial\Delta)$. Clearly, $M_p(\Delta) \subset M_p(\partial\Delta)$.

The Bergman projection on Δ is defined by

$$(P_\Delta f)(\zeta) = \int_\Delta \overline{k(z,\zeta)}f(z)\,d\sigma(z)$$

and the Hilbert transform on Δ is given by

$$(T_\Delta f)(\zeta) = \int_\Delta \overline{h(z,\zeta)}f(z)\,d\sigma(z),$$

where the last integral is taken in the principal value sense. It is well known that these operators are bounded on $L_p(\Delta)$. For future reference we shall record the following two propositions:

PROPOSITION 1. *The Hilbert transform T_Δ is a bounded operator from $L_p(\Delta:\lambda)$ into $L_p(\Delta:\lambda)$ if and only if $\lambda \in M_p(\Delta)$.*

PROPOSITION 2. *The Bergman projection P_Δ is a bounded operator from $L_p(\Delta:\lambda)$ into $L_p(\Delta:\lambda)$ if and only if $\lambda \in M_p(\partial\Delta)$.*

Proposition 1 is essentially due to Coifman and Fefferman [5] while the proof of Proposition 2 is analogous to one given in Bekollé and Bonami [1]. The families $N(\Delta)$ and $N(\partial\Delta)$ can be, of course, replaced by a variety of other families (see for example [1] and [5]). It is, however, more advantageous, as we shall see later, to use the present families of noneuclidean sectors instead.

§3. The Bergman–Schiffer transform. Let $D \notin O_G$ be a plane region and let $G = G_D(z,\zeta)$ be its Green's function. Thus

$$G_D(z,\zeta) = H(z,\zeta) - \log|z-\zeta|,$$

where $H(z,\zeta)$ is symmetric and harmonic in $D\times D$. The Bergman kernel of D, $K(z,\zeta) = K_D(z,\zeta)$, is given by

$$K(z,\zeta) = -\frac{2}{\pi}\,\partial_z\partial_{\bar{\zeta}}G$$

and its "adjoint" $L(z,\zeta) = L_D(z,\zeta)$ by

$$L(z,\zeta) = -\frac{2}{\pi}\,\partial_z\partial_\zeta G.$$

In analogy to the Bergman projection

$$(P_D f)(\zeta) = \int_D \overline{K(z,\zeta)}f(z)\,d\sigma(z),$$

we also consider the Bergman–Schiffer transform

$$(Q_D f)(\zeta) = \int_D \overline{L(z,\zeta)}f(z)\,d\sigma(\zeta),$$

where the integral is taken in the principal value sense. The operators P_D and Q_D are always bounded on $L_2(D)$ and, moreover, the following relationships

(3.1) $$Q_D = T_D - T_D P_D$$

and

(3.2) $$I_D - P_D = Q_D^* Q_D$$

hold on $L_2(D)$. Here I_D is the identity operator on $L_2(D)$, Q_D^* is the adjoint operator of Q_D and T_D is the Hilbert transform

(3.3) $$(T_D f)(\zeta) = \frac{1}{\pi}\int_D \overline{(z-\zeta)}^{-2} f(z)\,d\sigma(z)$$

on D. The above relationships and other related results were established in [4].

Since $D \notin O_G$, D has the unit disk $\Delta = \{w\colon |w| < 1\}$ as its universal cover. Let $\pi\colon \Delta \to D$ be a universal covering map for D. Let Γ be the covering group of π, that is, Γ consists of all those $\gamma \in \text{Möb}(\Delta)$ for which $\pi\circ\gamma = \pi$. Then, as is well known,

$$G_D(z, \zeta) = \sum_{\gamma\in\Gamma} \log\left|\frac{1-\gamma(w)\bar\tau}{\gamma(w)-\tau}\right|$$

with $z = \pi(w)$, $\zeta = \pi(\tau)$; $w, \tau \in \Delta$. On the unit disk Δ,

$$G_\Delta(w, \tau) = \log\left|\frac{1-w\bar\tau}{w-\tau}\right|,$$

$$K_\Delta(w, \bar\tau) = k(w, \bar\tau)$$

and

$$L_\Delta(w, \tau) = h(w, \tau).$$

Therefore,

$$G_D(z, \zeta) = \sum_{\gamma\in\Gamma} G_\Delta\big(\gamma(w), \tau\big),$$

(3.4) $$K_D(z, \zeta)\pi'(w)\overline{\pi'(\tau)} = \sum_{\gamma\in\Gamma} k\big(\gamma(w), \bar\tau\big)\gamma'(w)$$

and

(3.5) $$L_D(z, \zeta)\pi'(w)\pi'(\tau) = \sum_{\gamma\in\Gamma} h\big(\gamma(w), \tau\big)\gamma'(w).$$

The above representations are evidently well defined and they are independent of the choice of the projection map π.

§4. The classes $M_p(D)$ and $M_p(\partial D)$. Let $\pi\colon \Delta\to D$ be a universal cover map for the region $D \notin O_G$ and let Γ be the covering group of π. Let $\Omega = \Delta/\Gamma$ be a normal fundamental region of Γ (see Siegel [8], p. 38). Thus, $\Delta = \bigcup_{\gamma\in\Gamma} \gamma(\Omega)$, where this union is disjoint and the boundary of Ω consists of finite or denumerable number of noneuclidean segments arranged in pairs conjugated by elements of Γ and touches $\partial\Delta$, and so it has zero plane measure. In this case $\pi|_\Omega$ is (modulo a set of measure zero) a homeomorphism of Ω onto D and we write $\phi = (\pi|_\Omega)^{-1}$.

We shall write $N(\Omega) = \{S\cap\Omega\colon S \in N(\Delta)\}$ and $N(\partial\Omega) = \{S\cap\Omega\colon S \in N(\partial\Delta)\}$. Evidently, $N(\Omega)$ is a subfamily of $N(\Delta)$ and it covers Ω. Similarly, $N(\partial\Omega)$ is a subfamily of $N(\partial\Delta)$ and, of course, $N(\partial\Omega) \subset N(\Omega)$. We define $N(D) = \pi[N(\Omega)]$ and $N(\partial D) = \pi[N(\partial\Omega)]$. This definition of the families $N(D)$ and $N(\partial D)$ is independent of the projection map π as the first part of the next theorem shows. Clearly, $N(\partial D) \subset N(D)$.

Let λ be a non-negative locally integrable function on D. For a measurable subset U of D we write

$$\|\phi'\|_{L_p(U:\lambda)} = \Big\{\int_U |\phi'(z)|^p \lambda(z)\,d\sigma(z)\Big\}^{1/p}$$

and

$$\|\phi'\|_{p:U} = \|\phi'\|_{L_p(U:1)}.$$

The weight function λ is said to *belong to class* $M_p(D)$ if

$$\sup_U \left\{\frac{\|\phi'\|_{L_p(U:\lambda)}}{\|\phi'\|_{2:U}}\right\}\cdot\left\{\frac{\|\phi'\|_{L_p(U:\lambda^{1-q})}}{\|\phi'\|_{2:U}}\right\} < \infty,$$

where the supremum is taken over all $U \in N(D)$. When the supremum is taken over the subfamily $N(\partial D)$, λ is then said to *belong to* $M_p(\partial D)$. Again, these definitions are independent of the projection π as the following theorem shows:

THEOREM 1. *Let $\pi_1\colon \Delta\to D$ be another covering map and let Γ_1 be its cover group. Let also $\Omega_1 = \Delta/\Gamma_1$ be a (normal) fundamental region of Γ_1 with $\psi = (\pi_1|_{\Omega_1})^{-1}$. Then,*

$$N(D) = \pi[N(\Omega)] = \pi_1[N(\Omega_1)]$$

and

$$N(\partial D) = \pi[N(\partial\Omega)] = \pi_1[N(\partial\Omega_1)].$$

Moreover, for any non-negative locally integrable function μ on D,

$$\frac{1-r}{1+r}\|\phi'\|_{L_p(U:\mu)} \leqslant \|\psi'\|_{L_p(U:\mu)} \leqslant \frac{1+r}{1-r}\|\phi'\|_{L_p(U:\mu)}$$

for all measurable subsets U of D. Here, r is a constant in $[0, 1)$ which is independent of U.

Proof. There exists an $A \in \text{Möb}(\Delta)$ so that $\pi_1 \circ A = \pi$ and hence $\Gamma = A^{-1}\Gamma_1 A$. Therefore, $\Omega_1 = A(\Omega)$ and $\psi = A \circ \phi$. Evidently, $A[N(\Omega)] = N(\Omega_1)$ and $A[N(\partial\Omega)] = N(\partial\Omega_1)$. Consequently, $\pi[N(\Omega)] = \pi_1[N(\Omega_1)]$ and $\pi[N(\partial\Omega)] = \pi_1[N(\partial\Omega_1)]$, proving the first part of the theorem. Next, $\psi'(z) = A'(\phi(z))\phi'(z)$ and

$$\|\psi'\|^p_{L_p(U:\mu)} = \int_U |A'(\phi(z))|^p |\phi'(z)|^p \mu(z)\, d\sigma(z)$$

for a subset U of D. Since $A \in \text{Möb}(\Delta)$, A is given by

$$A(w) = a\frac{w-\tau_0}{1-\bar{\tau}_0 w}; \quad |a| = 1, \quad |\tau_0| < 1.$$

Consequently

$$A'(w) = a(1-|\tau_0|^2)(1-\bar{\tau}_0 w)^{-}$$

and, therefore,

$$\frac{1-|\tau_0|}{1+|\tau_0|} \leqslant |A'(w)| \leqslant \frac{1+|\tau_0|}{1-|\tau_0|}$$

for each $w \in \Delta$. Using the fact that $w = \phi(z) \in \Omega \subset \Delta$ this, therefore, yields

$$\frac{1-|\tau_0|}{1+|\tau_0|} \|\phi'\|_{L_p(U:\mu)} \leqslant \|\psi'\|_{L_p(U:\mu)} \leqslant \frac{1+|\tau_0|}{1-|\tau_0|} \|\phi'\|_{L_p(U:\mu)}.$$

The theorem now follows by setting $r = |\tau_0|$.

It is evident that $M_p(D) \subset M_p(\partial D)$ and that if $\lambda \in M_p(D)$, then $\lambda^{1-q} \in M_q(D)$. Similarly, if $\lambda \in M_p(\partial D)$, then $\lambda^{1-q} \in M_q(\partial D)$.

We now prove:

THEOREM 2. *The Bergman projection P_D is a bounded operator from $L_p(D:\lambda)$ into $L_p(D:\lambda)$ if and only if $\lambda \in M_p(\partial D)$.*

Proof. By definition and by virtue of (3.4) we have

$$\begin{aligned}
(P_D f)(\zeta) &= \int_D \overline{K_D(z,\xi)} f(z)\, d\sigma(z) \\
&= \int_\Omega \overline{K_D(z,\xi)} f(z) |\pi'(w)|^2 d\sigma(w) \\
&= \pi'(\tau)^{-1} \int_\Omega \sum_{\gamma\in\Gamma} \overline{k(\lambda(w),\bar{\tau})}\ \overline{\gamma'(w)} f(\pi w)\pi'(w)\, d\sigma(w) \\
&= \pi'(\tau)^{-1} \int_\Omega \sum_{\gamma\in\Gamma} \overline{k(\gamma(w),\bar{\tau})}\ \overline{\gamma'(w)}\, g(w)\, d\sigma(w)
\end{aligned}$$

with $g = (f\circ\pi)\pi'$. Here, g is an automorphic form satisfying

$$g(w) = g(\gamma w)\gamma'(w)$$

for each $w \in \Delta$ and all $\gamma \in \Gamma$. Also

$$\|g\|_{L_p(\Delta:\mu)} = \|f\|_{L_p(D:\lambda)}; \qquad \mu(w) = |\pi'(w)|^{2-p}\lambda(\pi w)\chi_\Omega(w),$$

where χ_Ω is the characteristic function of Ω. In what follows, the interchange of sum and integral is justified by virtue of Fubini's theorem, and, we find that

$$\begin{aligned}
(P_D f)(\zeta) &= \pi'(\tau)^{-1} \sum_{\gamma\in\Gamma} \int_\gamma \overline{k(\gamma(w),\bar{\tau})}\ \overline{\gamma'(w)}\, g(w)\, d\sigma(w) \\
&= \pi'(\tau)^{-1} \sum_{\gamma\in\Gamma} \int_{\gamma\Omega} \overline{k(v,\tau)}\, g(v)\, d\sigma(v) \\
&= \pi'(\tau)^{-1} \int_\Delta \overline{k(v,\tau)}\, g(v)\, d\sigma(v).
\end{aligned}$$

Consequently,

$$(P_D f)(z) = \pi'(w)^{-1}(P_\Delta g)(w); \qquad z = \pi(w) \in D. \tag{4.1}$$

Therefore,

$$\begin{aligned}
\|P_D f\|^p_{L_p(D:\lambda)} &= \int_D |\pi'(w)|^{-p} |(P_\Delta g)(w)|^p \lambda(z)\, d\sigma(z) \\
&= \int_\Omega |(P_\Delta g)(w)|^p |\pi'(w)|^{2-p} \lambda(\pi w)\, d\sigma(w).
\end{aligned}$$

The norm inequality $\|P_D f\|^p_{L_p(D:\lambda)} \leqslant C\|f\|^p_{L_p(D:\lambda)}$ is therefore equivalent to

$$\int_\Delta |(P_\Delta g)(w)|^p \mu(w)\, d\sigma(w) \leqslant C \int_\Delta |g(w)|^p \mu(w)\, d\sigma(w)$$

with the weight

$$\mu(w) = |\pi'(w)|^{2-p}\lambda(\pi w)\chi_\Omega(w).$$

This, in view of Proposition 2, is equivalent to

$$\sup_S \left\{\frac{1}{|S|}\int_S \mu(w)\, d\sigma(w)\right\} \cdot \left\{\frac{1}{|S|}\int_S \mu(w)^{-1/(p-1)} d\sigma(w)\right\}^{p-1} < \infty$$

or

$$\sup_S \frac{1}{|S|^p} \left\{\int_{S\cap\Omega} |\pi'(w)|^{2-p}\lambda(\pi w)\, d\sigma(w)\right\} \times$$

$$\times \left\{\int_{S\cap\Omega} |\pi'(w)|^{(p-2)/(p-1)} \lambda(\pi w)^{-1/(p-1)} d\sigma(w)\right\}^{p-1} < \infty,$$

where the supremum is taken over all $S \in N(\partial\Delta)$. This evidently is equivalent to

$$\text{(4.2)}\qquad \sup_V \frac{1}{|V|^p}\left\{\int_V |\pi'(w)|^{2-p}\lambda(\pi w)\,d\sigma(w)\right\}\times \times\left\{\int_V |\pi'(w)|^{(p-2)/(p-1)}\lambda(\pi w)^{-1/(p-1)}\,d\sigma(w)\right\}^{p-1} < \infty,$$

where the supremum is taken over all $V \in N(\partial\Omega)$. We write $U = \pi(V)$, $V \in N(\partial\Omega)$. Then $U \in N(\partial D)$ and

$$|V| = \int_V d\sigma(w) = \int_U |\phi'(z)|^2\,d\sigma(z) = \|\phi'\|^2_{2:U}.$$

Also

$$\int_V |\pi'(w)|^{2-p}\lambda(\pi w)\,d\sigma(w) = \|\phi'\|^p_{L_p(U:\lambda)}$$

and

$$\left\{\int_V |\pi'(w)|^{(p-2)/(p-1)}\lambda(\pi w)^{-1/(p-1)}\,d\sigma(w)\right\}^{p-1} = \|\phi'\|^p_{L_q(U:\lambda^{1-q})}.$$

Therefore, condition (4.2) is equivalent to the condition that $\lambda \in M_p(\partial D)$. This concludes the proof.

Similarly to (4.1) we can also show that

$$(Q_D f)(z) = \overline{\pi'(w)}^{-1}(T_\Delta g)(w); \qquad z = \pi(w) \in D$$

with $g = (f\circ\pi)\pi'$. Using this relationship, (3.5) and Proposition 1 we can show, exactly as in Theorem 2, the following theorem:

THEOREM 3. *The Bergman–Schiffer transform Q_D is a bounded operator from $L_p(D:\lambda)$ into $L_p(D:\lambda)$ if and only if $\lambda \in M_p(D)$.*

§5. The class W_p. We now make some applications of Theorems 2 and 3 when the weight function λ is identically 1. Let $D \notin O_G$ be a plane region, D is said to *belong to class W_p* if

$$\sup_U \left\{\frac{\|\phi'\|_{p:U}}{\|\phi'\|_{2:U}}\right\}\cdot\left\{\frac{\|\phi'\|_{q:U}}{\|\phi\|_{2:U}}\right\} < \infty,$$

where the supremum is taken over all $U \in N(D)$. The region $D \notin O_G$ is said to *belong to class W_p^1* when the supremum is taken over all $U \in N(\partial D)$. According to Theorem 1, these definitions are independent of the projection map π. We shall also show that in fact $W_p^1 = W_p$. Clearly, $D \in W_p$ if and only if $D \in W_q$ and always $D \in W_2$. Using Theorems 2 and 3 we obtain (see also [4], [9]):

COROLLARY 1. *P_D is a bounded operator from $L_p(D)$ into $L_p(D)$ if and only if $D \in W_p^1$.*

COROLLARY 2. *Q_D is a bounded operator from $L_p(D)$ into $L_p(D)$ if and only if $D \in W_p$.*

We now refer to the formulae (3.1)–(3.3). It is well known that the Hilbert transform T_D is a bounded operator from $L_p(D)$ into $L_p(D)$. Consequently, using (3.1) and (3.2) we conclude (see also [4]) the following proposition:

PROPOSITION 3. *On $L_p(D)$ the boundedness of P_D is equivalent to the boundedness of Q_D.*

Using this proposition and the previous corollaries we obtain:

COROLLARY 3. *The classes W_p^1 and W_p are identical, and the following statements are equivalent:*

(1) *$D \in W_p$.*

(2) *P_D is a bounded operator from $L_p(D)$ into $L_p(D)$.*

(3) *Q_D is a bounded operator from $L_p(D)$ into $L_p(D)$.*

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