STUDIA MATHEMATICA, T. LXXVI. (1983)

Power series in locally convex algebras

by

GÜNTER CZICHOWSKI (Greifswald)

Abstract. In this paper we consider the class of AE-algebras. The main theorem shows that this class is the most general in a certain sense, having good convergence properties for power series.

A classical result concerning the operation of entire holomorphic functions on locally convex algebras is due to B. Mitjagin, S. Rolewicz and W. Żelazko [2]. It can be formulated as follows:

Let A be a commutative and associative locally convex algebra. Suppose that A is metrizable and complete and that every entire holomorphic function operates (by its power series) on the whole algebra. Then A is m-convex.

Clearly, this result is a partial converse to the obvious fact that m-convexity of A implies the operation of entire functions on the whole algebra A.

We shall give here in an analogous way a characterization of locally convex algebras having "good" convergence properties for power series. The motivation is given by infinite dimensional Lie theory starting with the Campbell–Hausdorff series as a power series in a suitable locally convex Lie algebra (this concept is realized in [1]). With respect to this application it should be mentioned that the algebras occurring in this paper are not associative in general, but the multiplication is bilinear and jointly continuous. In fact, we regard a class of locally convex algebras—the so called AE-algebras—with an estimation rule for long products. We show that the convergence of power series in AE-algebras may be regarded analogously to that in the complex plane. Many classical algebras are AE-algebras and examples are given.

Our main theorem shows that the class of AE-algebras is in a certain sense the most general class of locally convex algebras with the "usual" convergence properties of power series. The proof uses a polarization formula to compute products by powers. Formulas of this kind are well known [3], [4], but here various types are given by a simple proof.

1. AE-algebras. Let A be a locally convex algebra and p, q seminorms on A. We shall call q an *asymptotical estimate for* p if there exists a natural number $m = m(p, q)$ such that

$$p(x_1 \ldots x_n) \leqslant q(x_1) \ldots q(x_n)$$

if $n \geqslant m$ and $x_1, \ldots, x_n \in A$.

Since A is not assumed to be associative, it should be noticed that $x_1 \ldots x_n$ stands here for a product of these elements with respect to arbitrary parentheses. Further, for any subset B of A we denote by B^n the set of all products with n factors from B with respect to arbitrary parentheses.

It is easy to see that q is an asymptotical estimate for p if and only if for $n \geqslant m$ we have $V^n \subseteq U$, where $U = \{x;\ p(x) < 1\}$, $V = \{x;\ q(x) < 1\}$.

DEFINITION. A locally convex algebra A is called an AE-*algebra* (AE—Asymptotical Estimate) if there exists a distinguished seminorm p_0 on A and for every seminorm p on A there exists a system $\{q_i;\ i \in I\}$ of asymptotical estimates for p satisfying

(1) $\{q_i;\ i \in I\}$ is directed downwards,

(2) $\inf\{q_i(x);\ i \in I\} = p_0(x)$ for every $x \in A$.

Any such system is called an AE-*system for* p.

Replacing the seminorms by the corresponding absolutely convex neighborhoods of zero, we get

LEMMA 1. *A locally convex algebra A is an AE-algebra if and only if there is a distinguished absolutely convex neighborhood U_0 of zero in A and for every neighborhood U of zero there exists a covering $\{V_i;\ i \in I\}$ of U_0 by absolutely convex neighborhoods of zero satisfying*

(1) *$\{V_i;\ i \in I\}$ is directed upwards,*

(2) *$V_i^n \subseteq U$ for $n \geqslant m_i$, where m_i depends on V_i.*

Obviously, subalgebras of AE-algebras are again of this type, and by Lemma 1 it is easy to see that quotient algebras of AE-algebras have the AE-property, too.

PROPOSITION 1. *Every AE-algebra A is m-convex.*

Proof. If U is an absolutely convex neighborhood of zero in A, then by Lemma 1 there exists an absolutely convex neighborhood V of zero such that $V^n \subseteq U$ if $n \geqslant m$. Taking V sufficiently small, we may assume $V^n \subseteq U$ for all n. Take $W = \bigcup_{n \in N} V^n$; then $W^2 \subseteq W \subseteq U$, and the same inclusion holds for the convex closure of W.

As a consequence we may assume that the distinguished seminorm p_0 of an AE-algebra satisfies the condition $p_0(xy) \leqslant p_0(x) \cdot p_0(y)$ (we say that p_0 is *submultiplicative*).

We remark that many classical locally convex algebras are AE-algebras, for instance Banach algebras, the well-known algebras of test functions or rapidly decreasing functions and free algebras with a finite number of generators. Here we shall give a typical example, for more details see [1].

Let I denote a compact interval in $\boldsymbol{R}$ and let A be the algebra of all complex valued smooth functions f on I with support in I, equipped with the topology given by the seminorms

$$p_m(f) = \sup\{|f^{(k)}(x)|;\ x \in I,\ k \leqslant m\}.$$

Then A is an AE-algebra with the distinguished seminorm p_0. To prove this, let m be fixed and define for real numbers $s, t \in \boldsymbol{R}$, $0 < s < 1$, $t > 1$, the seminorms q_{st} by

$$q_{st}(f) = \max\{s \cdot p_m(f),\ t \cdot p_0(f)\}.$$

Obviously, the system $\{q_{st}\}$ is directed downwards and

$$\inf\{q_{st}(f);\ 0 < s < 1,\ t > 1\} = p_0(f).$$

Let now s, t be fixed and $f_1, \ldots, f_n \in A$ satisfy $q_{st}(f_i) \leqslant 1$ for $i = 1, \ldots, n$. Then the Leibniz formula

$$(f_1 \ldots f_n)^{(k)} = \sum_{i_1 + \ldots + i_n = k} \binom{k}{i_1, \ldots, i_n} \cdot f_1^{(i_1)} \ldots f_n^{(i_n)}$$

implies in the case $k \leqslant m \leqslant n$ the estimate

$$|(f_1 \ldots f_n)^{(k)}(x)| \leqslant n^k \cdot s^{-k} \cdot t^{k-n} \leqslant n^m \cdot s^{-m} \cdot t^{m-n}.$$

Since the right-hand side tends to zero as $n \to \infty$, it follows that $p_m(f_1 \ldots f_n) \leqslant 1$ for sufficiently large n and q_{st} is an asymptotical estimate for p_m. Therefore A is an AE-algebra.

2. Power series in AE-algebras. Let A be a complete AE-algebra with the distinguished seminorm p_0. To regard the convergence of power series in A we restrict our considerations to the essential case of two variables and remark once more that A is not assumed to be associative or commutative.

A *monomial $M(x, y)$ of length $l(M) = m$ in the variables x, y* is by definition a product (with arbitrary parentheses) of m factors each of which equals x or y. If there is a unit in A, then it is by definition the unique monomial of length zero.

DEFINITION. A series $\sum_{k=0}^{\infty} a_k \cdot M_k(x, y)$, $a_k \in \boldsymbol{C}$, is called a *power series in x, y* if the $M_k(x, y)$ are monomials in x, y and for every $m = 0, 1, 2, \ldots$ we have $a_m^* = \sum_{k;\, l(M_k) = m} |a_k| < \infty$. The real number $R = (\overline{\lim}\, (a_m^*)^{1/m})^{-1}$ is called the *radius of convergence* of the power series.

To prove some results concerning the convergence of power series, we need the following two lemmas.

LEMMA 2. *If A is an AE-algebra and p a seminorm on A, then there exists an AE-system $\{q_i;\ i\in I\}$ for p such that $p\leqslant C_i\cdot q_i$ with certain constants C_i.*

Proof. Let $\{q_i';\ i\in I\}$ be an AE-system for p and define for every positive real number $s>0$, $q_{is}(x)=\max\{q_i'(x), s\cdot p(x)\}$. Obviously, $\{q_{is};\ i\in I, s>0\}$ is an AE-system for p, too and $p\leqslant s^{-1}\cdot q_{is}$ holds.

LEMMA 3. *Let p denote a multiplicative seminorm on A and q an asymptotical estimate for p such that $p\leqslant C\cdot q$ with $C\geqslant 1$. Let further $m=m(p,q)$ and assume $q(x_i)\leqslant r$, $i=1,\dots,n$.*

If $x_1\dots x_n$ denotes a product of these elements with respect to arbitrary parentheses, then

$$p(x_1\dots x_n)\leqslant C^m\cdot r^n.$$

Proof. If $n\geqslant m$, we have $p(x_1\dots x_n)\leqslant q(x_1)\dots q(x_n)\leqslant C^m\cdot r^n$. If $n<m$, we use the fact that p is multiplicative and get

$$p(x_1\dots x_n)\leqslant p(x_1)\dots p(x_n)\leqslant C^n\cdot r^n\leqslant C^m\cdot r^n.$$

PROPOSITION 2. *Let A be a complete AE-algebra and let $\sum_{k=0}^{\infty}a_kM_k(x,y)$ denote a power series in $x,y\in A$ with radius of convergence $R>0$. Then the power series converges for all x,y with $p_0(x)<R$, $p_0(y)<R$, and the convergence is absolute and locally uniform with respect to an arbitrary seminorm of A.*

Proof. Suppose $p_0(x)<r$, $p_0(y)<r$ with $0<r<R$.

Further, let p denote a seminorm on A, which we may assume to be multiplicative, and q an asymptotical estimate for p satisfying $q(x)<r$, $q(y)<r$. By Lemma 2 we can assume $p\leqslant C\cdot q$ with $C\geqslant 1$ and by Lemma 3 we get

$$\sum_{k=0}^{\infty}|a_k|\cdot p\big(M_k(x,y)\big)\leqslant\sum_{k=0}^{\infty}|a_k|\cdot C^m\cdot r^{l(M_k)}\leqslant C^m\cdot\sum_{n=0}^{\infty}a_n^*\cdot r^n,$$

where $m=m(p,q)$. This implies the absolute convergence with respect to p. Moreover, the estimate proved above holds uniformly for all x,y with $q(x)\leqslant r$, $q(y)\leqslant r$. That means that if x_0,y_0 are points with $p_0(x_0)<r$, $p_0(y_0)<r$, $0<r<R$, and p is an arbitrary seminorm, then there is a neighborhood U of (x_0,y_0) in $A\times A$ such that the power series converges absolutely and uniformly with respect to p on U.

COROLLARY. *The mapping f: $f(x,y)=\sum_{k=0}^{\infty}a_k\cdot M_k(x,y)$, defined for all x,y with $p_0(x)<R$, $p_0(y)<R$, is continuous.*

Remark. It can be proved that f is in fact smooth with respect to the Michal–Bastiani differentiability. Moreover, the usual statements concerning multiplication or composition of power series hold for power series in complete AE-algebras [1].

3. Polarization formulas. Let A be a commutative and associative locally convex algebra over $\boldsymbol{K}=\boldsymbol{R}$ or $\boldsymbol{K}=\boldsymbol{C}$. We shall prove formulas expressing the product $x_1\dots x_n$ by powers.

PROPOSITION 3. *Let $T=\{t\in\boldsymbol{K};\ |t|=1\}$ and μ a measure on T, which is invariant under the transformation $t\to -t$ and normed by $\mu(T)=1$. Then for arbitrary $x_1,\dots,x_n\in A$ we have*

$$x_1\dots x_n=\frac{1}{n!}\cdot\int_T\dots\int_T\frac{(t_1x_1+\dots+t_nx_n)^n}{t_1\dots t_n}\,d\mu(t_1)\dots d\mu(t_n).$$

Proof. Calculating the power under the integral, we get on the right-hand side of the equations the terms:

$$\frac{1}{n!}\cdot\binom{n}{i_1,\dots,i_n}\cdot x_1^{i_1}\dots x_n^{i_n}\cdot\int_T\dots\int_T t_1^{i_1-1}\dots t_n^{i_n-1}d\mu(t_1)\dots d\mu(t_n),$$

which are equal to zero if at least one of the numbers i_j is zero. This implies the formula stated above.

EXAMPLES. (1) If $\boldsymbol{K}=\boldsymbol{R}$, then the unique measure μ has the form $\mu(\{1\})=\mu(\{-1\})=1/2$, and we get the "discrete" formula

$$x_1\dots x_n=\frac{1}{2^n\cdot n!}\cdot\sum_{\varepsilon_1,\dots,\varepsilon_n=\pm 1}\frac{(\varepsilon_1x_1+\dots+\varepsilon_nx_n)^n}{\varepsilon_1\dots\varepsilon_n}.$$

(2) If $\boldsymbol{K}=\boldsymbol{C}$ and μ is invariant under rotations, we get

$$x_1\dots x_n=\frac{1}{(2\pi)^n n!}\cdot\int_T\dots\int_T\frac{(t_1x_1+\dots+t_nx_n)^n}{t_1\dots t_n}|dt_1|\dots|dt_n|.$$

(3) The following formula for the case $\boldsymbol{K}=\boldsymbol{C}$ is a simple consequence of Cauchy's formula

$$x_1\dots x_n=\frac{1}{(2\pi i)^n n!}\cdot\oint_T\dots\oint_T\frac{(t_1x_1+\dots+t_nx_n)^n}{t_1^2\dots t_n^2}dt_1\dots dt_n.$$

PROPOSITION 4. *Let A be a commutative and associative algebra over $\boldsymbol{R}$ or $\boldsymbol{C}$ and let U,V denote absolutely convex neighborhoods of zero such that $x^n\in V$ for every $x\in U$. Then $U^n\subseteq e^n\cdot V$.*

Proof. For $x_1, \ldots, x_n \in U$ we get by a polarization formula

$$x_1 \ldots x_n = \frac{n^n}{n!} \cdot \int_T \ldots \int_T \frac{\big((t_1/n)x_1 + \ldots + (t_n/n)x_n\big)^n}{t_1 \ldots t_n} d\mu(t_1) \ldots d\mu(t_n).$$

If p denotes the seminorm corresponding to V, this formula implies the estimate

$$p(x_1 \ldots x_n) \leqslant \frac{n^n}{n!} < e^n.$$

4. A partial converse theorem. We shall prove the following theorem generalizing the cited result of Mitjagin, Rolewicz and Żelazko [2].

THEOREM. *Let A be a commutative and associative locally convex algebra, which is complete and metrizable. If every power series with positive radius of convergence is convergent in a certain neighborhood of zero in A, then A is an AE-algebra.*

To prove this, we only use the fact that the geometric series $\sum x^n$ is convergent in a certain neighborhood W_0 of zero in A. We assume that W_0 is open and absolutely convex and the geometric series is converging on $\overline{W}_0$, too.

At first, we prove three lemmas, each of which being formulated under the same assumptions as in our theorem.

LEMMA 4. *If p denotes a seminorm on A, then the geometric series converges absolutely with respect to p on W_0. Furthermore, there exists a neighborhood W of zero in A such that the geometric series converges absolutely and uniformly with respect to p on W.*

Proof. The first part of the proof is classical, the second part uses ideas from [2]. If $x \in W_0$, there is a real number $t > 1$ such that $t \cdot x \in W_0$. Hence $p\big((tx)^n\big) \leqslant 1$, this implies $p(x^n) \leqslant t^{-n}$ if n is sufficiently large, and the convergence is absolute with respect to p. Let q denote a seminorm on A satisfying $p(xy) = q(x) \cdot q(y)$ for all $x, y \in A$ and

$$W_m = \{x \in \overline{W}_0;\ q(x^n) \leqslant m \text{ for all } n\} \quad (m = 1, 2, \ldots).$$

Then the sets W_m are closed and $\bigcup_{m=1}^{\infty} W_m = \overline{W}_0$. Therefore there exists a set W_m with non-void interior and, using $W_m \subseteq m \cdot W_1$, we conclude that the interior of W_1 is not void. Now, a simple calculation shows that for $x \in \frac{1}{2} \cdot W_1 - \frac{1}{2} \cdot W_1$ holds $p(x^n) \leqslant 1$ for all n. But this set contains a neighborhood W of zero in A, and it is easy to see that the geometric series converges absolutely and uniformly with respect to p on $t \cdot W$ if $0 < t < 1$.

COROLLARY. *A is m-convex.*

Indeed, if V is an arbitrary absolutely convex neighborhood of zero in A, then by Lemma 4 there exists an absolutely convex neighborhood W of zero such that $x^n \in V$ for all n if $x \in W$. By Proposition 4 we have

$$W^n \subseteq e^n \cdot V \quad \text{or} \quad (e^{-1} \cdot W)^n \subseteq V \quad \text{for all } n.$$

The proof is complete by the same arguments as in the proof of Proposition 1.

LEMMA 5. *Let p be an arbitrary seminorm on A. Then the convergence of the geometric series is absolute and locally uniform with respect to p on W_0.*

Proof. By the last corollary we may assume that p is submultiplicative. Let $x_0 \in W_0$ be fixed. Then there exist real numbers $C \geqslant 1$, $t > 1$ such that $p(x_0^n) \leqslant C \cdot t^{-n}$ for all n (see the proof of Lemma 4). Choosing $\varepsilon > 0$ such that $t^{-1} + \varepsilon < 1$, we get for every $y \in A$ satisfying $p(y) < \varepsilon$ and for every n

$$p\big((x_0 + y)^n\big) \leqslant \sum_{k=0}^{n} \binom{n}{k} \cdot p(x_0^k) \cdot p(y^{n-k})$$

$$\leqslant \sum_{k=0}^{n} \binom{n}{k} \cdot C \cdot t^{-k} \varepsilon^{n-k} = C \cdot (t^{-1} + \varepsilon)^n.$$

This implies our statement, and moreover, the following

COROLLARY. *If M is a compact subset of W_0, then the geometric series converges absolutely and uniformly on M with respect to every seminorm.*

LEMMA 6. *Let M denote an absolutely convex compact subset of W_0 satisfying $e \cdot M \subseteq W_0$. If V is an arbitrary neighborhood of zero in A, then there exists a number m such that $M^n \subseteq V$ for all $n \geqslant m$.*

Proof. Since A is m-convex, we may assume $V^2 \subseteq V$. By the last corollary there is a number m such that $x^n \in V$ if $x \in e \cdot M$ and $n \geqslant m$. By Proposition 4, $(e \cdot M)^n \subseteq (e \cdot V)^n$, that means $M^n \subseteq V^n \subseteq V$ for $n \geqslant m$.

Proof of the theorem. Let $U_0 = (2e)^{-1} \cdot W_0$ be fixed. If V is an arbitrary absolutely convex neighborhood of zero in A with $V^2 \subseteq V$, and M is an absolutely convex compact subset of U_0, we define

$$V_M = (M + \tfrac{1}{2} \cdot V) \cap U_0.$$

Clearly, the system $\{V_M\}$, where V is fixed and M runs over all absolutely convex compact subsets of U_0, is a covering of U_0 and is directed upwards. To finish the proof in terms of Lemma 1 we have to show that $(V_M)^n \subseteq V$ for $n \geqslant m$, where m is a number depending on M. At first we remark that by Lemma 6 there exist numbers $C \geqslant 1$, $t > 2$ such that

$$M^n \subseteq C \cdot t^{-n} \cdot V \quad \text{for all } n.$$

This implies

$$(V_M)^n \subseteq (M+\tfrac{1}{2}\cdot V)^n \subseteq \sum_{k=0}^{n}\binom{n}{k}\cdot M^k\cdot\frac{1}{2^{n-k}}\cdot V$$

$$\subseteq \sum_{k=0}^{n}\binom{n}{k}\cdot C\cdot t^{-k}\cdot\frac{1}{2^{n-k}}\cdot V \subseteq C\cdot(t^{-1}+\tfrac{1}{2})^n\cdot V.$$

Since $(t^{-1}+\frac{1}{2})^n$ tends to zero if $n\to\infty$, it follows that $(V_M)^n \subseteq V$ if n is sufficiently large. By Lemma 1, A is an AE-algebra and the theorem is proved.

Remark. The metrizability of A is only used to prove m-convexity. We regard as an example the algebra $\mathcal{D}(\boldsymbol{R})$ of test functions, which is m-convex. Then our theorem shows that $\mathcal{D}(\boldsymbol{R})$ is an AE-algebra.

References

[1] H. Boseck, G. Czichowski, K.-P. Rudolph, *Analysis on Topological Groups — General Lie Theory*, Teubner Verlag Leipzig (to print).
[2] B. Mitjagin, S. Rolewicz, W. Żelazko, *Entire functions in B_0-algebras*, Studia Math. 21 (1962), 291–306.
[3] L. Waelbroeck, *Topological Vector Spaces and Algebras*, Lecture Notes in Mathematics 230 (1971).
[4] W. Żelazko, *Selected Topics in Topological Algebras*, Lecture Notes Series Aarhus University No. 31 (1971).

Received October 6, 1981 (1712)

STUDIA MATHEMATICA, T. LXXVI. (1983)

Spline bases in classical function spaces on compact C^∞ manifolds

Part II

by

Z. CIESIELSKI and T. FIGIEL (Sopot)

Abstract. Using spline functions the desired Schauder bases in Sobolev and Besov spaces on cubes with boundary conditions are constructed. The combination of these results and of the decomposition of function spaces established in Part I permits to complete in Section 11 the proofs of the main results formulated in Part I. Section 11 contains also applications (e.g., improved Sobolev type embedding theorems, estimates for the eigenvalues of integral operators and asymptotic estimates for the Kolmogorov diameters in the class of Besov spaces).

In this part we complete the proofs of Theorems A and B formulated in the Introduction to Part I of this paper. In order to read this part it is necessary to know some definitions and results given in Section 2.

In Section 4 of Part I the proofs of Theorems A and B are reduced to constructing suitable bases in the spaces $W_p^k(Q)_Z$ and $B_{p,q}^s(Q)_Z$, where $k\geqslant 0$, $s>0$, $1\leqslant p,q\leqslant\infty$, introduced in Section 2.

The boundary conditions induced by a set Z of the form (2.37) have a "tensor product" nature. This allows us to reduce our problem essentially to good approximation of vector-valued (L_p-valued) functions on some intervals and to constructing special bases in L_p spaces on the interval $\langle 0,1\rangle$. All this is carried out in Sections 7–10 by means of vector-valued splines and spline bases.

Section 7 contains the basics on vector-valued splines.

In Section 8 we consider families of orthogonal projections onto increasing subspaces of splines corresponding to the various boundary conditions. We also study associated families of projections relevant for the Sobolev spaces. The most important results are here the exponential estimates for the kernels of these projections and for the basic functions (cf. Proposition 8.10, Lemmas 8.13 and 8.27).

In Section 9 special spline bases on the unit interval are defined. Their tensor products (in the rectangular ordering) are bases in the spaces $W_p^0(Q)_Z$ and $W_p^m(Q)_Z$ for $1\leqslant p\leqslant\infty$, and they are unconditional bases if