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Associative and Lie subalgebras of finite codimension

by

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Abstract. It is shown that the algebra of compact operators on a Hilbert space of infinite dimension has no proper subalgebra of finite codimension, and no proper closed Lie subalgebra of finite codimension. Related results are established for more general C^*-algebras, and for the Schatten p-ideals ($p \geqslant 2$).

Introduction. It was shown by P. de la Harpe [3] that the Lie algebra $\mathscr{K}$ of compact operators on an infinite dimensional Hilbert space has no proper Lie ideal of finite codimension, and he conjectured that in fact $\mathscr{K}$ has no proper Lie subalgebra of finite codimension. We show below that the conjecture is true if the subalgebra is assumed to be closed. We also show that $\mathscr{K}$ has no proper associative subalgebra of finite codimension. The methods for this case are general and apply in a wider context, so that we prove some results about subalgebras of finite codimension in general C^*-algebras.

The term ideal (without qualification) will always mean a two-sided ideal.

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1. Finite codimensional ideals in C^*-algebras.

LEMMA 1.1. *Every ideal I of finite codimension in a C^*-algebra B is closed.*

Proof. Suppose first that I is a maximal ideal. If B/I is a radical algebra, then since it is finite-dimensional, there exists an integer $n \geqslant 1$ such that $(b_1+I)\ldots(b_n+I) = 0$ in B/I for all $b_1, \ldots, b_n \in B$. Hence $b_1 \ldots b_n \in I$, and since B is a C^*-algebra, B is linearly spanned by such products. Hence $B = I$, and I is certainly closed.

If B/I is not a radical algebra, it contains a primitive ideal, and by the maximality of I in B, this primitive ideal can only be the zero ideal. Thus I is a primitive ideal, and hence closed.

We prove the general result by induction on n, the codimension of I in B. Clearly every ideal of codimension 0 is closed. Suppose we know the lemma is true for $n = 0, 1, 2, \ldots, k-1$, and let us prove it for $n = k$.

If I is an ideal of codimension $n = k$, then either I is maximal, in which case it is closed by our argument above, or there is an ideal J of B such that $I \subsetneq J \subsetneq B$. Hence J has codimension $< k$, and so J is closed by the induction hypothesis. Thus J is a C^*-algebra, and I has codimension in J which is strictly less than k. Hence I is closed, again by the induction hypothesis. This completes the proof.

The following theorem is a special case of a result in [4], and its proof is quite elementary.

THEOREM 1.2 (Laffey [4]). *If B is a Banach algebra, and A is a subalgebra of finite codimension, then A contains an ideal I of B such that I has finite codimension in B.*

THEOREM 1.3. *A subalgebra of finite codimension in a C^*-algebra is necessarily closed.*

Proof. If A is a subalgebra of finite codimension of the C^*-algebra B, then there is an ideal I of B contained in A such that $\dim(B/I) < \infty$. By Lemma 1, I is closed. Also since $\dim(A/I) < \infty$, A/I is complete. Hence A is a Banach algebra, and so A is closed in B.

THEOREM 1.4. *Let B be a C^*-algebra, C a finite-dimensional Banach algebra, and $\varphi\colon B \to C$ an algebra homomorphism. Then φ is continuous.*

Proof. The ideal

$$I = \ker\varphi = \{x \in B\colon \varphi(x) = 0\}$$

is finite-codimensional in B, and hence closed by Lemma 1.1. Thus φ is the composition of the continuous maps

$$B \to B/\ker\varphi \qquad x \mapsto x + \ker\varphi$$

and

$$B/\ker\varphi \to C \qquad x + \ker\varphi \mapsto \varphi(x).$$

The second map is continuous because it is a linear map between finite-dimensional spaces.

Remark. Theorem 1.3 is not true for more general Banach algebras: Let B be the Banach algebra which is the completion of the algebra of all finite sums

$$\lambda r + \sum_{i=1}^{n} \lambda_i e_i,$$

where $\lambda, \lambda_1, \ldots, \lambda_n$ are complex numbers, the e_i are mutually orthogonal idempotents, $r^2 = 0$, and $re_i = e_i r = 0$. The norm is given by

$$\Big\|\lambda_r + \sum_{i=1}^{n} \lambda_i e_i\Big\| = \max\Big(\Big(\sum_{i=1}^{n} |\lambda_i|^1\Big)^{1/2}, \Big|\lambda - \sum_{i=1}^{n} \lambda_i\Big|\Big).$$

It is shown in Bade and Curtis [1] that B is a commutative algebra with one-dimensional radical, $\operatorname{rad}B$, linearly spanned by r, and that there is a subalgebra A of B, which is not closed, such that $A \oplus \operatorname{rad}B = B$. In fact, since $rB = 0$, it is easily seen A is an ideal.

Recall that a simple C^*-algebra is one having no proper closed ideals.

THEOREM 1.5. *If B is an infinite-dimensional simple C^*-algebra, then B has no proper subalgebra of finite codimension.*

Proof. Let A be a subalgebra of finite codimension. By Theorem 1.2 and Lemma 1.1 there is a closed ideal I of finite codimension such that $I \subseteq A$. Since B is simple and $\dim B = \infty$, $I = B$. Thus $A = B$.

THEOREM 1.6. *Let H be a Hilbert space of infinite dimension, and let $\mathscr{K}(H)$, $\mathscr{B}(H)$ denote the C^*-algebras of compact and bounded operators, respectively. Then $\mathscr{K}(H)$, $\mathscr{B}(H)$ and $\mathscr{B}(H)/\mathscr{K}(H)$ have no proper subalgebras of finite codimension.*

Proof. $\mathscr{K}(H)$ is simple.

As for $\mathscr{B}(H)$, its closed ideals are well known, and of them only $\mathscr{B}(H)$ itself is finite codimensional. (When H is separable, $\mathscr{K}(H)$ is the only proper closed ideal of $\mathscr{B}(H)$.)

The result for the Calkin algebra $\mathscr{B}(H)/\mathscr{K}(H)$ now follows easily.

Remark. In Theorem 1.5 one can replace the condition that B is a C^*-algebra by the weaker condition that B is a Banach algebra spanned by $B^2 = \{xy\colon x, y \in B\}$. Any Banach algebra with a bounded approximate identity is such an algebra, by the Cohen Factorization Theorem ([5], p. 26). Thus B could be taken to be $L^1(G)$ for G any locally compact group. The proofs are only a little more technical than those given here.

2. Lie subalgebras of finite codimension. In this section H denotes a Hilbert space of infinite dimension, and for $p > 0$, $\mathscr{C}_p$ denotes the Schatten p-ideal on H. This is a Banach algebra in its own right with the usual p-norm, and in particular $\mathscr{C}_2$ is a Hilbert space. $\mathscr{K}$ denotes the Banach algebra of compact operators on H. Thus $\mathscr{K}$ and the Schatten p-ideals are all Lie algebras.

THEOREM 2.1. *If $\mathscr{L}$ is a Lie subalgebra of $\mathscr{C}_2$ of finite codimension, then $\mathscr{L}$ is dense in $\mathscr{C}_2$.*

Proof. We assume $\mathscr{L}$ is closed and show that $\mathscr{L} = \mathscr{C}_2$. Now we can write $\mathscr{C}^2 = \mathscr{L} \oplus \mathscr{L}^\perp$, an orthogonal direct sum, with the inner product $(S, T) = \operatorname{tr}(ST^*)$ for $S, T \in \mathscr{C}_2$. Also, replacing $\mathscr{L}$ by $\mathscr{L} \cap \mathscr{L}^*$ if necessary, we can assume $\mathscr{L} = \mathscr{L}^*$ and $\mathscr{L}^\perp = (\mathscr{L}^\perp)^*$. Suppose the operators $T_1, \ldots, T_n$ are a basis for the finite dimensional space $\mathscr{L}^\perp$.

First we show that $T_1, \ldots, T_n$ are finite rank operators: Fix a unit vector $e_0 \in H$ and let M be the span of $e_0, T_1e_0, \ldots, T_ne_0$. Suppose $f \in M^\perp$.

Then the rank one operator $f\otimes e_0$ is in $\mathscr{L}$ for if $T\in\mathscr{L}^\perp$, we have

$$\operatorname{tr}((f\otimes e_0)T^*) = \operatorname{tr}(f\otimes Te_0) = (f, Te_0) = 0$$

since $Te_0 \in M$.

Since $\mathscr{L}$ is self-adjoint, we can also conclude that $(f\otimes e_0)^* = e_0\otimes f\in\mathscr{L}$. Now since $\mathscr{L}$ is a Lie algebra, we have for $f, g\in M^\perp$,

$$(f\otimes e_0)(e_0\otimes g)-(e_0\otimes g)(f\otimes e_0) = f\otimes g-(f,g)e_0\otimes e_0\in\mathscr{L}.$$

By taking arbitrary orthogonal pairs f, g in $M^\perp$ we obtain $f\otimes g\in\mathscr{L}$.

Let P be the orthogonal projection of H onto $M^\perp$. Then for $T\in\mathscr{L}^\perp$ and f, g as above, $0 = \operatorname{tr}((f\otimes g)T) = \operatorname{tr}((Pf\otimes Pg)T) = \operatorname{tr}(Pf\otimes T^*Pg) = (PTPf, g)$. It follows that PTP is a scalar on $M^\perp$, and since it is a Hilbert–Schmidt operator, $PTP = 0$.

Thus we have shown every operator in $\mathscr{L}^\perp$ has finite rank (and its rank is no greater than $2\dim M$).

Now we let N be the span of the ranges of $T_1, \ldots, T_n, T_1^*, \ldots, T_n^*$, and Q be the orthogonal projection of H onto this finite dimensional space N. Then $T = QTQ$ $(T\in\mathscr{L}^\perp)$. It follows that for $T\in\mathscr{C}_2$, $T(1-Q)$ and $(1-Q)TQ$ are in $\mathscr{L}$, so to show $T\in\mathscr{L}$ we need only show $QTQ\in\mathscr{L}$.

Let V be the partial isometry with initial space N and range in $N^\perp$. Then $V = VQ = (1-Q)V = (1-Q)VQ\in\mathscr{L}$. Also $QTV^* = QTV^*(1-Q)\in\mathscr{L}$, and $VTV^* = (1-Q)VTV^*(1-Q)\in\mathscr{L}$. Finally, $QTQ = QTV^*V = [QTV^*, V]+VQTV^* = [QTV^*, V]+VTV^*\in\mathscr{L}$.

THEOREM 2.2. *Every Lie subalgebra of $\mathscr{K}$ of finite codimension is dense in $\mathscr{K}$.*

Proof. This follows from considering the Lie subalgebra $L\cap\mathscr{C}_2$ of $\mathscr{C}_2$, and from the fact that $\mathscr{C}_2$ is dense in $\mathscr{K}$.

Remarks. (i) Theorem 2.1 is not true for $\mathscr{C}_1$ since the closed Lie subalgebra $\mathscr{L} = \{T\in\mathscr{C}_1: \operatorname{tr}(T) = 0\}$ is of codimension 1. The same conclusion holds for any $\mathscr{C}_p$ with $0<p<1$.

(ii) For $p>0$ there exist dense Lie subalgebras (even associative subalgebras) of $\mathscr{C}_p$ of every given finite codimension. To see this observe that for T, S in $\mathscr{C}_p$ we have $TS\in\mathscr{C}_{p/2}$ [2], Lemma XI.9.9, p. 1093. It follows that if $\mathscr{L}$ is any linear submanifold of $\mathscr{C}_p$ containing $\mathscr{C}_{p/2}$, then $\mathscr{L}$ is automatically a subalgebra of $\mathscr{C}_p$. Now $\mathscr{C}_{p/2}$ has infinite codimension in $\mathscr{C}_p$; thus $\mathscr{L}$ could be chosen to have any finite codimension. The density of $\mathscr{L}$ follows from that of $\mathscr{C}_{p/2}$.

(iii) The analogue of Theorem 2.2 holds (with a similar proof) for $\mathscr{C}_p$ with $p\geqslant 2$ in place of $\mathscr{K}$.

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ERRATA

Page 127 line 13 from below:

for $\bigoplus\limits_{\emptyset\neq e\subseteq D}\otimes$ read $\bigoplus\limits_{\emptyset\neq e\subseteq D}$