Finally, since \( v \in L_w^+(x) \) is finite a.e., then taking \( w(x) = \tilde{v}(x) + (1 + |x|)^a \),
with \( a > n(p-1) \), (2), (3) and (4) imply (i). We observe that for \( a < 2n(p-1) \)
the weight \( w \) is smaller than that in Wo-Sang Young’s paper.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY
New Brunswick, New Jersey 08903

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Equivalent Cauchy sequences and contractive fixed points
in metric spaces

by

SOLOMON LEADER (New Brunswick, N.J.)

Abstract. The sequences \([x_i], [y_j]\) in a metric space \((X, d)\) are equivalent Cauchy
sequences if and only if given \( \varepsilon > 0 \) there exist \( \delta \) in \((0, \infty)\) and a positive integer \( r \)
such that \( d(x_i, y_j) < \delta \) for all \( i, j \) with \( d(x_i, y_j) < \varepsilon + \delta \). As a typical application
let \( f: X \to X \) with complete graph such that given \( \varepsilon > 0 \) there exist \( \delta \) in \((0, \infty)\) and an
integer \( r \) with \( d(f^n x, f^n y) < \delta \) for all \( n \), \( y \) with \( d(x, y) < \varepsilon + \delta \). Then \( f \) has a unique
fixed point \( u \) and \( f^n u \to u \) as \( n \to \infty \) for all \( u \).

1. Introduction. Let \((X, d)\) be a metric space, \( f: X \to X \), and \( N \) be the
natural numbers. We call \( w \) in \( X \) a contractive fixed point of \( f \) if \( f = w \)
and \( f^n x \to w \) as \( n \to \infty \) in \( N \) for all \( x \) in \( X \). For the existence of a contractive
fixed point it is necessary (and under certain mildly restrictive conditions,
sufficient) that all orbits \([f^n x]\) be equivalent Cauchy sequences. Sequences
\([x_i]\) and \([y_j]\) in \( X \) are called equivalent if \( d(x_i, y_j) \to 0 \) as \( i \to \infty \). Equivalent
Cauchy sequences converge to a common point in the completion of \( X \).

Our basic contribution here (Theorem 1) is a characterization (EC)
of equivalent Cauchy sequences. Application of (EC) to two identical
sequences yields a refinement of the Cauchy convergence criterion (Corol-
ary 1) with correspondingly refined estimates for \( d(x_i, w) \) as \( x_i \to w \)
(Theorem 2). (EC) is applied to orbits for single and multivalued mappings
to yield fixed points. Theorem 3 subsumes a body of fixed point theorems.
In particular, it easily yields the theorems in [1], [2], [4] and Theorem 1.2
in [3].

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2. Sequences in metric spaces.

Theorem 1. Two sequences \([x_i]\) and \([y_j]\) in a metric space \((X, d)\) are
equivalent-Cauchy if and only if

\[ d(x_i + r, y_j + r) < \varepsilon \quad \text{for all } i, j \text{ with } d(x_i, y_j) < \varepsilon + \delta. \]

(1)
Proof. Given (EO) define

\[ d_k(n) = \max\{d(x_i, y_j) : n \leq i, j \leq n + k\}. \]

**Lemma 1.** For all \( k \) in \( N \), \( \inf[d_k(n) : n \in N] = 0. \)

Suppose \( \inf[d_k(n) : n \in N] = e > 0 \) for some \( k \). Apply (EO) to get \( e \), so that (1) holds. Then choose \( e \) so that \( d_k(n) < e + \varepsilon \). By (1) and (2) \( d_k(n + r) < e \), contradicting the definition of \( e \). So Lemma 1 holds.

**Lemma 2.** Let \( e, \varepsilon, \delta, \tau \) satisfy (1). In terms of (3) let \( n \) satisfy

\[ d(n) < \min\{e, \delta, \tau\}. \]

Then

\[ d(x_i, y_j) < 3e \quad \text{for all} \quad i, j \geq n. \]

We contend that the assumption

\[ d(x_{i+r}, y_{j+r}) \geq e \quad \text{for some} \quad j \geq n \]

yields a contradiction. Take the smallest \( j \) satisfying (5). Then

\[ d(x_{i+r}, y_{j+r}) < e \quad \text{for} \quad n \leq i < j. \]

By (2) and (3), \( j > n + r \). So \( n < j - r < j \). Hence (6) with \( i = j - r \) implies

\[ d(x_{i+r}, y_{j-r}) < e. \]

So \( d(x_{i+r}, y_{j-r}) < d(x_i, y_j) + d(y_j, y_{i+r}) + d(x_{i+r}, y_{j-r}) < 2d(x_i, y_j) + e < 2e + \varepsilon \)

by (2), (3), (7). That is, \( d(x_{i+r}, y_{j-r}) < \delta + e \), which implies \( d(x_{i+r}, y_{j-r}) < e \) by (1), contradicting (5). So (5) is false. That is,

\[ d(x_{i+r}, y_{j-r}) < e \quad \text{for all} \quad i \geq n. \]

Similarly,

\[ d(x_i, y_j) < e \quad \text{for all} \quad i \geq n. \]

For \( i, j \geq n \) we have \( d(x_i, y_j) \leq d(x_i, x_{i+r}) + d(x_{i+r}, y_{j+r}) + d(y_{j+r}, y_j) < 3e \) by (8), (9), which gives (4). So Lemma 2 holds.

Given \( e > 0 \) apply (EO) to get \( e, \varepsilon, \delta, \tau \) so that (1) holds. Lemma 1 gives \( n \) such that (3) holds. So Lemma 2 implies \( [x_i] \) and \( [y_j] \) are equivalent Cauchy sequences.

The converse, that equivalent Cauchy sequences satisfy (EO), is trivial with \( \delta = \infty \). Indeed, in all the results of this section the case \( \delta = \infty \) is the corresponding standard result.

**Corollary 1 (Cauchy Sequences).** A sequence \([x_i]\) in \((X, d)\) is Cauchy if and only if given \( e > 0 \) there exist \( \delta \) in \((0, \infty)\) and \( r \) in \( N \) such that

\[ d(x_{i+r}, x_{j+r}) < e \quad \text{for all} \quad i, j \text{ with} \quad d(x_i, x_j) < e + \delta. \]

**Proof.** Apply Theorem 1 with \( y_i = x_i. \)

**Corollary 2 (Convergent Sequences).** \( x_i \to w \) in \((X, d)\) as \( i \to \infty \) if and only if given \( e > 0 \) there exist \( \delta \) in \((0, \infty)\) and \( r \) in \( N \) such that

\[ d(x_{i+r}, w) < e \quad \text{for all} \quad i \text{ with} \quad d(x_i, w) < e + \delta. \]

**Proof.** Apply Theorem 1 with \( y_i = w. \)

**Corollary 3 (Equivalent Sequences).** \( d(x_i, y_i) \to 0 \) as \( i \to \infty \) if and only if given \( e > 0 \) there exist \( \delta \) in \((0, \infty)\) and \( r \) in \( N \) such that

\[ d(x_{i+r}, y_{i+r}) < e \quad \text{for all} \quad i \text{ with} \quad d(x_i, y_i) < e + \delta. \]

**Proof.** Apply Corollary 2 to the real sequence \([d(x_i, y_i)]\) converging to 0.

**Theorem 2.** Let \( x_i \to w \) in \((X, d)\) and \( e, \delta, \tau \) satisfy (10). Then

\[ d(x_{i+r}, w) < e \quad \text{for all} \quad i \text{ with} \quad d(x_i, w) < e + \delta. \]

**Proof.** We get (13) by letting \( k \to \infty \) in the following lemma.

**Lemma 3.** Under (10) if \( d(x_i, x_{i+r}) < e \), then

\[ d(x_{i+r}, x_{i+2r}) < e \]

for all \( k \) in \( N \).

To prove the lemma we use induction on \( k \). (14) is trivial for \( k = 1 \).

Given (14) for \( k = m \), let \( j = i + mr \), so \( d(x_i, x_j) < e \).

Thus \( d(x_i, x_j) < d(x_i, x_{i+r}) + d(x_{i+r}, x_{i+2r}) < e + e \)

by (6), (10), (11). Applying (10) we get (14) with \( k = m + 1 \).

3. Contractive fixed points.

**Theorem 3.** Let \((X, d)\) be a metric space and \( f: X \to X \) with complete graph (i.e. closed in \( X^2 \), where \( X \) is the completion of \( X \)). Then

(i) \( f \) has a contractive fixed point if and only if given \( x, y \) in \( X \) and \( e \) in \( X \) there exist \( \delta \) in \((0, \infty)\) and \( r \) in \( N \) with \( d(f^{i+r}x, f^{i+r}y) < e \) for all \( i, j \) with \( d(f^i x, f^j y) < e + \delta \).

(ii) \( f \) has a fixed point if and only if there exists \( x \) in \( X \) such that given \( e > 0 \) there exist \( \delta \) in \((0, \infty)\) and \( r \) in \( N \) with

\[ d(f^{i+r}x, f^{i+r}x) < e \quad \text{for all} \quad i, j \text{ with} \quad d(f^i x, f^j x) < e + \delta. \]

Moreover, if \( f^i x \to w \) as \( i \to \infty \) and \( e, \delta, \tau \) satisfy (10), then \( d(f^{i+r}x, w) \leq e \) for all \( i \) with \( d(f^i x, w) \leq e + \delta. \)

**Proof.** The contraction condition in (i) is just (EO) in Theorem 1 applied to the \( f \)-orbits of \( x \) and \( y \). The contraction condition in (ii) is the convergence criterion in Corollary 1 applied to the orbit of \( x \). Now \( (x_i, y_i) \) is in the graph of \( f \) for any \( f \)-orbit \([x_i]\). Hence, since the graph is
complete, a Cauchy orbit converges to a fixed point of f. The final state-
ment in Theorem 3 follows from Theorem 2.

The next result is a trivial consequence of Theorem 3.

**Corollary 4.** Let \((X, \delta)\) be a metric space and \(f: X \to X\) with complete graph. Assume that given \(\varepsilon > 0\) there exist \(\delta > 0\) and \(\tau > 0\) in \(X\) such that

\[
\delta(f^n x, f^n y) < \varepsilon \quad \text{for all} \quad x, y \in X \text{ with } d(x, y) < \varepsilon + \delta.
\]

Then \(f\) has a contractive fixed point \(w\) in \(X\). Moreover, if \(\varsigma, \delta, \tau\) satisfy

\[
\delta(f^n w, w) < \varepsilon \quad \text{for all} \quad x \in X \text{ with } d(x, f^n x) < \delta.
\]

The special case of Corollary 4 with \(\tau = 1\) gives the Meir-Keeler
contraction theorem [4]. The essential novelty of Corollary 4 is that \(\tau\)
may vary with \(\varepsilon\). Indeed, the case with \(\tau\) constant follows from the case
with \(\tau = 1\) since a contractive fixed point of an iterate \(f^n\) is a contractive
fixed point of \(f\). (See Lemma 3 in [5].)

**4. Fixed points for multifunctions.** Theorem 1 can also be used to get
fixed points for multifunctions. Our final result is an extension of Corollary 4 to multifunctions.

A multifunction \(F\) in \(X\) is a subset of \(X^2\). Let \(F^0\) be the set of all \(y\) with
\((x, y)\) in \(F\). For \(x\) in \(X\) define \(F^0\) by

\[
(x, y) \in F \text{ if there exist } s_0, a_1, \ldots, a_r \text{ with } s_0 = x \text{ and } a_r = y \\
\text{such that}
\]

\[
(a_{i-1}, a_i) \in F
\]

for \(i = 1, \ldots, r\). A sequence \([x_0, x_1, \ldots]\) is an \(F\)-orbit of \(x\) if \(x_0 = x\) and

\[
(x_{i-1}, x_i) \in F
\]

holds for all \(i \in N\).

**Theorem 4.** Let \((X, \delta)\) be a metric space. Let \(F\) be a multifunction in
\(X\) with complete graph such that \(F\) is nonempty for all \(x\) in \(X\) and given
\(\varepsilon > 0\) there exist \(\delta > 0\) and \(\tau > 0\) in \(X\) so that for all \(x, y, u, v \in X\)

\[
\delta(x, y) < \varepsilon + \delta, \quad u \in F^0 x, v \in F^0 y \quad \text{implies} \quad \delta(u, v) < \varepsilon.
\]

Then there exists a unique \(w\) in \(X\) to which all \(F\)-orbits converge. Moreover, \(Fw = w\) and if \(\varepsilon, \delta, \tau\) satisfy (18), then

\[
\delta(x, w) < \delta, \quad u \in F^0 x \quad \text{implies} \quad \delta(u, w) < \varepsilon.
\]

**Proof.** Since \(Fw\) is nonempty, every point \(x\) is the initial point of some
\(F\)-orbit. (18) gives (1) of Theorem 1 for all \(F\)-orbits \([x_0], [y_0]\). So all \(F\)-orbits
are equivalent Cauchy sequences by Theorem 1. Hence, by (17) and the
completeness of \(F\), all orbits converge to a common point \(w\) with \((w, w)\)
in \(F\). So \(w \in Fw\). We need only to show that \(Fw = w\).

Now (18) applies with \(y = x\) implies \(\delta(x, w) < \varepsilon\) for all \(x\). Since
\(w\) is in \(Fw\), \(Fw\) is contained in \(F^n w\) for all \(n \in N\). Hence, \(Fw = w\).

Finally, (19) follows from Theorem 2.