Finally, since $v \in L^1_{loc}\bar{\bar{v}}(x)$ is finite a.e., then taking $w(x) = \bar{\bar{v}}(x) + (1+|x|)^a$, with $a > n(p-1)$, (2), (3) and (4) imply (i). We observe that for $a < 2n(p-1)$ the weight w is smaller than that in Wo-Sang Young's paper.

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Equivalent Cauchy sequences and contractive fixed points in metric spaces

by

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Abstract. The sequences $[x_i]$, $[y_i]$ in a metric space (X, d) are equivalent Cauchy sequences if and only if given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and a positive integer r such that $d(x_{i+r}, y_{j+r}) < \varepsilon$ for all i, j with $d(x_i, y_j) < \varepsilon + \delta$. As a typical application let $f\colon X \to X$ with complete graph such that given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and an integer r with $d(f^r x, f^r y) < \varepsilon$ for all x, y with $d(x, y) < \varepsilon + \delta$. Then f has a unique fixed point w and $f^i x \to w$ as $i \to \infty$ for all x.

1. Introduction. Let (X, d) be a metric space, $f\colon X \to X$, and N be the natural numbers. We call w in X a *contractive fixed point* of f if $fw = w$ and $f^i x \to w$ as $i \to \infty$ in N for all x in X. For the existence of a contractive fixed point it is necessary (and under certain mildly restrictive conditions, sufficient) that all orbits $[f^i x]$ be equivalent Cauchy sequences. Sequences $[x_i]$ and $[y_i]$ in X are called *equivalent* if $d(x_i, y_i) \to 0$ as $i \to \infty$. Equivalent Cauchy sequences converge to a common point in the completion of X.

Our basic contribution here (Theorem 1) is a characterization (EC) of equivalent Cauchy sequences. Application of (EC) to two identical sequences yields a refinement of the Cauchy convergence criterion (Corollary 1) with correspondingly refined estimates for $d(x_i, w)$ as $x_i \to w$ (Theorem 2). (EC) is applied to orbits for single and multivalued mappings to yield fixed points. Theorem 3 subsumes a body of fixed point theorems. In particular, it easily yields the theorems in [1], [2], [4] and Theorem 1.2 in [3].

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2. Sequences in metric spaces.

THEOREM 1. *Two sequences $[x_i]$ and $[y_i]$ in a metric space (X, d) are equivalent-Cauchy if and only if*

(EC) *given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N such that*

$$d(x_{i+r}, y_{j+r}) < \varepsilon \quad \text{for all } i, j \text{ with } d(x_i, y_j) < \varepsilon + \delta. \tag{1}$$

Proof. Given (EC) define

(2) $$d_k(n) = \mathrm{Max}[d(x_i, y_j)\colon n \leqslant i, j \leqslant n+k].$$

LEMMA 1. *For all k in N,* $\mathrm{Inf}[d_k(n)\colon n \in N] = 0$.

Suppose $\mathrm{Inf}[d_k(n)\colon n \in N] = \varepsilon > 0$ for some k. Apply (EC) to get δ, r so that (1) holds. Then choose n so that $d_k(n) < \varepsilon+\delta$. By (1) and (2) $d_k(n+r) < \varepsilon$ contradicting the definition of ε. So Lemma 1 holds.

LEMMA 2. *Let ε, δ, r satisfy* (1). *In terms of* (2) *let n satisfy*

(3) $$d_r(n) < \mathrm{Min}\{\varepsilon, \delta/2\}.$$

Then

(4) $$d(x_i, y_j) < 3\varepsilon \quad \textit{for all } i, j \geqslant n.$$

We contend that the assumption

(5) $$d(x_{n+r}, y_j) \geqslant \varepsilon \quad \text{for some } j \geqslant n$$

yields a contradiction. Take the smallest j satisfying (5). Then

(6) $$d(x_{n+r}, y_i) < \varepsilon \quad \text{for} \quad n \leqslant i < j.$$

By (2) and (3), $j > n+r$. So $n < j-r < j$. Hence (6) with $i = j-r$ implies

(7) $$d(x_{n+r}, y_{j-r}) < \varepsilon.$$

So $d(x_n, y_{j-r}) \leqslant d(x_n, y_n) + d(y_n, x_{n+r}) + d(x_{n+r}, y_{j-r}) < 2d_r(n) + \varepsilon < \delta+\varepsilon$ by (2), (3), (7). That is, $d(x_n, y_{j-r}) < \delta+\varepsilon$ which implies $d(x_{n+r}, y_j) < \varepsilon$ by (1), contradicting (5). So (5) is false. That is,

(8) $$d(x_{n+r}, y_j) < \varepsilon \quad \text{for all } j \geqslant n.$$

Similarly,

(9) $$d(x_i, y_{n+r}) < \varepsilon \quad \text{for all } i \geqslant n.$$

For $i, j \geqslant n$ we have $d(x_i, y_j) \leqslant d(x_i, y_{n+r}) + d(y_{n+r}, x_{n+r}) + d(x_{n+r}, y_j) < 3\varepsilon$ by (8), (9), which gives (4). So Lemma 2 holds.

Given $\varepsilon > 0$ apply (EC) to get δ, r so that (1) holds. Lemma 1 gives n such that (3) holds. So Lemma 2 implies $[x_i]$ and $[y_i]$ are equivalent Cauchy sequences.

The converse, that equivalent Cauchy sequences satisfy (EC), is trivial with $\delta = \infty$. Indeed, in all the results of this section the case $\delta = \infty$ is the corresponding standard result.

COROLLARY 1 (Cauchy Sequences). *A sequence $[x_i]$ in (X, d) is Cauchy if and only if given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N such that*

(10) $$d(x_{i+r}, x_{j+r}) < \varepsilon \quad \textit{for all } i, j \textit{ with } d(x_i, x_j) < \varepsilon+\delta.$$

Proof. Apply Theorem 1 with $y_i = x_i$.

COROLLARY 2 (Convergent Sequences). *$x_i \to w$ in (X, d) as $i \to \infty$ if and only if given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N such that*

(11) $$d(x_{i+r}, w) < \varepsilon \quad \textit{for all } i \textit{ with } d(x_i, w) < \varepsilon+\delta.$$

Proof. Apply Theorem 1 with $y_i = w$.

COROLLARY 3 (Equivalent Sequences). *$d(x_i, y_i) \to 0$ as $i \to \infty$ if and only if given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N such that*

(12) $$d(x_{i+r}, y_{i+r}) < \varepsilon \quad \textit{for all } i \textit{ with } d(x_i, y_i) < \varepsilon+\delta.$$

Proof. Apply Corollary 2 to the real sequence $[d(x_i, y_i)]$ converging to 0.

THEOREM 2. *Let $x_i \to w$ in (X, d) and ε, δ, r satisfy* (10). *Then*

(13) $$d(x_{i+r}, w) \leqslant \varepsilon \quad \textit{for all } i \textit{ with } d(x_i, x_{i+r}) \leqslant \delta.$$

Proof. We get (13) by letting $k \to \infty$ in the following lemma.

LEMMA 3. *Under* (10) *if $d(x_i, x_{i+r}) \leqslant \delta$, then*

(14) $$d(x_{i+r}, x_{i+kr}) < \varepsilon$$

for all k in N.

To prove the lemma we use induction on k. (14) is trivial for $k = 1$. Given (14) for $k = m$ let $j = i+mr$, so $d(x_{i+r}, x_j) < \varepsilon$. Thus $d(x_i, x_j) \leqslant d(x_i, x_{i+r}) + d(x_{i+r}, x_j) < \delta+\varepsilon$. Applying (10) we get (14) with $k = m+1$.

3. Contractive fixed points.

THEOREM 3. *Let (X, d) be a metric space and $f\colon X \to X$ with complete graph (i.e. closed in Y^2 where Y is the completion of X). Then*

(i) *f has a contractive fixed point if and only if given x, y in X and $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N with $d(f^{i+r}x, f^{j+r}y) < \varepsilon$ for all i, j with $d(f^i x, f^j y) < \varepsilon+\delta$.*

(ii) *f has a fixed point if and only if there exists x in X such that given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N with*

(15) $$d(f^{i+r}x, f^{j+r}x) < \varepsilon \quad \textit{for all } i, j \textit{ with } d(f^i x, f^j x) < \varepsilon+\delta.$$

Moreover, if $f^i x \to w$ as $i \to \infty$ and ε, δ, r satisfy (15), *then $d(f^{i+r}x, w) \leqslant \varepsilon$ for all i with $d(f^i x, f^{i+r}x) \leqslant \delta$.*

Proof. The contraction condition in (i) is just (EC) in Theorem 1 applied to the f-orbits of x and y. The contraction condition in (ii) is the convergence criterion in Corollary 1 applied to the orbit of x. Now (x_i, x_{i+1}) is in the graph of f for any f-orbit $[x_i]$. Hence, since the graph is complete, a Cauchy orbit converges to a fixed point of f. The final statement in Theorem 3 follows from Theorem 2.

The next result is a trivial consequence of Theorem 3.

COROLLARY 4. *Let (X, d) be a metric space and $f\colon X \to X$ with complete graph. Assume that given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N such that*

$$d(f^r x, f^r y) < \varepsilon \quad \text{for all } x, y \text{ with } d(x, y) < \varepsilon + \delta. \tag{16}$$

Then f has a contractive fixed point w in X. Moreover, if ε, δ, r satisfy (16), then $d(f^r x, w) \leqslant \varepsilon$ for all x with $d(x, f^r x) \leqslant \delta$.

The special case of Corollary 4 with $r = 1$ gives the Meir–Keeler contraction theorem [4]. The essential novelty of Corollary 4 is that r may vary with ε. Indeed, the case with r constant follows from the case with $r = 1$ since a contractive fixed point of an iterate f^r is a contractive fixed point of f. (See Lemma 3 in [5].)

4. Fixed points for multifunctions. Theorem 1 can also be used to get fixed points for multivalued mappings. Our final result is an extension of Corollary 4 to multifunctions.

A *multifunction* F in X is a subset of X^2. Let Fx be the set of all y with (x, y) in F. For r in N define F^r as follows: $(x, y) \in F^r$ if there exist $x_0, x_1, \ldots, x_r$ with $x_0 = x$ and $x_r = y$ such that

$$(x_{i-1}, x_i) \in F \tag{17}$$

for $i = 1, \ldots, r$. A sequence $[x_0, x_1, \ldots]$ is an *F-orbit* of x if $x_0 = x$ and (17) holds for all i in N.

THEOREM 4. *Let (X, d) be a metric space. Let F be a multifunction in X with complete graph such that Fx is nonempty for all x in X and given $\varepsilon > 0$ there exist δ in $(0, \infty]$ and r in N so that for all x, y, u, v in X*

$$d(x, y) < \varepsilon + \delta,\ u \in F^r x,\ v \in F^r y \quad \text{imply} \quad d(u, v) < \varepsilon. \tag{18}$$

Then there exists a unique w in X to which all F-orbits converge. Moreover, $Fw = w$ and if ε, δ, r satisfy (18), then

$$d(x, u) \leqslant \delta,\ u \in F^r x \quad \text{imply} \quad d(u, w) \leqslant \varepsilon. \tag{19}$$

Proof. Since Fx is nonempty, every point x is the initial point of some F-orbit. (18) gives (1) of Theorem 1 for all F-orbits $[x_i]$, $[y_i]$. So all F-orbits are equivalent Cauchy sequences by Theorem 1. Hence, by (17) and the completeness of F, all orbits converge to a common point w with (w, w) in F. So $w \in Fw$. We need only to show $\operatorname{diam} Fw = 0$.

Now (18) applied with $y = x$ implies $\operatorname{diam} F^r x \leqslant \varepsilon$ for all x. Since w is in Fw, Fw is contained in $F^n w$ for all n in N. Hence, $\operatorname{diam} Fw \leqslant \varepsilon$ for all $\varepsilon > 0$. So $\operatorname{diam} Fw = 0$.

Finally, (19) follows from Theorem 2.

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