[30] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, Paris 1968.
[32] S. M. Nikol'skij, *Approximation of functions of several variables and embedding theorems*, Nauka, Moskva 1977.
[33] S. Schonefeld, *Schauder bases in spaces of differentiable functions*, Bull. Amer. Math. Soc. 75 (1969), 586–590.
[34] V. A. Solonnikov, *A priori estimates for second order differential equations of parabolic type*, Trudy Mat. Inst. AN SSSR 70 (1964), 132–212.
[35] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton 1970.

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# On weighted norm inequalities for the maximal function

by

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**Abstract.** We give a refinement of a lemma of C. Fefferman and E. Stein, and we show an application to weighted norm inequalities.

The lemma of C. Fefferman and E. Stein given in [3], p. 111, states that

$$\int_{R^n} Mf(x)^p g(x)\,dx \leqslant C \int_{R^n} f(x)^p Mg(x)\,dx, \tag{1}$$

where $1 < p < \infty$, $M$ is the Hardy–Littlewood maximal function, and $f$ and $g$ are positive measurable functions.

In this note we show that by restricting the radius in the definition of the maximal function a similar inequality holds. This inequality can be used as a substitute for (1) in weighted norm inequalities when the assumption $Mg < \infty$ cannot be made.

LEMMA. *Let $f$ be a measurable function and define*

$$\tilde{f}(x) = \sup_{r<(|x|+1)/2} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(t)|\,dt,$$

*where $B_r(x)$ is the ball of center $x$ and radius $r$, and*

$$\tilde{\tilde{f}}(x) = \sup_{\substack{x\in B_r\\ r<|x|+1}} \frac{1}{|B_r|} \int_{B_r} |f(t)|\,dt,$$

*where $B_r$ is a ball of radius $r$. If $g > 0$ almost everywhere, then*

$$\int_{R^n} \tilde{f}(x)^p g(x)\,dx \leqslant C \int_{R^n} |f(x)|^p \tilde{\tilde{g}}(x)\,dx.$$

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**Proof.** This proof is inspired by [1]. We shall first show that the operator $Tf = \bar{f}$ is of weak type $(1, 1)$ with respect to the measures $\bar{\bar{g}}(x)\,dx$, $g(x)\,dx$. Let $E_\lambda$ denote the set $\{x: \bar{f}(x) > \lambda\}$, and $E_\lambda^m = E_\lambda \cap \{|x| \leqslant m\}$, $m > 0$. Then for each $x \in E_\lambda^m$ there exists $B_r(x)$, $r < (|x|+1)/2$, such that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(t)|\,dt > \lambda .$$

By Besicovitch's covering lemma we can extract a subfamily $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ whose union covers $E_\lambda^m$ and

$$\sum_i \chi_{B_{r_i}(x_i)} \leqslant C_n ,$$

where $\chi_A$ denotes the characteristic function of the set $A$. Then

$$|E_\lambda^m|_{g\,dx} = \int_{E_\lambda^m} g(x)\,dx \leqslant \sum_{i=1}^{\infty} \int_{B_{r_i}(x_i)} g(x)\,dx$$

$$\leqslant \sum_{i=1}^{\infty} \left(\frac{1}{|B_{r_i}(x_i)|}\int_{B_{r_i}(x_i)} g(x)\,dx\right) \frac{1}{\lambda}\int_{B_{r_i}(x_i)} |f(y)|\,dy .$$

Observe that for all $y \in B_{r_i}(x_i)$ we have

$$r_i < (|x_i|+1)/2 = (1+|x_i|) - (1+|x_i|)/2 < 1 + |x_i| - r_i < 1 + |y| ,$$

and consequently

$$\frac{1}{|B_{r_i}(x_i)|}\int_{B_{r_i}(x_i)} g(y)\,dy \leqslant \bar{\bar{g}}(y) .$$

Therefore

$$|E_\lambda^m|_{g\,dx} \leqslant \frac{1}{\lambda}\sum_{i=1}^{\infty}\int_{B_{r_i}(x_i)} |f(y)|\bar{\bar{g}}(y)\,dy \leqslant \frac{C_n}{\lambda}\int_{R^n} |f(y)|\bar{\bar{g}}(y)\,dy .$$

Letting $m \to \infty$, the weak type $(1, 1)$ follows. On the other hand, since $g > 0$,

$$\|\bar{f}\|_{\infty, g\,dx} = \|\bar{f}\|_{\infty, dx} \leqslant \|f\|_{\infty, dx} = \|f\|_{\infty, \bar{\bar{g}}\,dx} .$$

Now the lemma follows from Marcinkiewicz's interpolation theorem.



We shall prove now the following theorem that has also been obtained independently by Wo-Sang Young.

THEOREM. *Let $v$ be a non-negative function, $Mf$ be the Hardy–Littlewood maximal function, and $1 < p < \infty$. The following conditions are equivalent:*

(i) *There exists $w(x) \geqslant 0$ and finite a.e. such that*

$$\int_{R^n} Mf(x)^p v(x)\,dx \leqslant C \int_{R^n} |f(x)|^p w(x)\,dx .$$

(ii) $$\int_{R^n} \frac{v(x)}{(1+|x|)^{np}}\,dx < \infty .$$

**Proof.** (i) $\Rightarrow$ (ii): We take a function $f \neq 0$ and a ball $B$ such that $\int |f(x)|^p w(x)\,dx < \infty$ and $0 < \int_B |f(x)|\,dx < \infty$. Then there exists $C > 0$ such that

$$Mf(x) \geqslant \frac{C}{(1+|x|)^n}\int_B |f(y)|\,dy .$$

Therefore

$$\int_{R^n} \frac{v(x)}{(1+|x|)^{np}}\,dx \leqslant C \int_{R^n} Mf(x)^p v(x)\,dx \leqslant C \int_{R^n} |f(x)|^p w(x)\,dx < \infty .$$

(ii) $\Rightarrow$ (i): Since $v+1$ satisfies (ii), we may assume $v > 0$. Observe that

(2) $$Mf(x) \leqslant \sup_{r<(|x|+1)/2} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,dy + \sup_{r\geqslant(|x|+1)/2} \frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y)|\,dy = \bar{f}(x) + \overset{*}{f}(x) .$$

For $\bar{f}(x)$, applying the lemma we have

(3) $$\int_{R^n} \bar{f}(x)^p v(x)\,dx \leqslant C \int_{R^n} |f(x)|^p \bar{\bar{v}}(x)\,dx .$$

On the other hand, taking $\beta > n/q$, $1/p + 1/q = 1$, and using Hölder's inequality, we get

$$\overset{*}{f}(x) \leqslant \frac{C}{(1+|x|)^n}\int_{R^n} |f(y)|\,dy \leqslant \frac{C'}{(1+|x|)^n}\left(\int_{R^n} |f(y)|^p (1+|y|)^{\beta p}\,dy\right)^{1/p} ,$$

where $C' = C\left(\int (1+|y|)^{-\beta q}\,dy\right)^{1/q}$. Then

(4) $$\int_{R^n} \overset{*}{f}(x)^p v(x)\,dx \leqslant C\left(\int_{R^n} \frac{v(x)}{(1+|x|)^{np}}\,dx\right)\left(\int_{R^n} |f(y)|^p (1+|y|)^{\beta p}\,dy\right).$$

Finally, since $v \in L^1_{\mathrm{loc}} \bar{\bar{v}}(x)$ is finite a.e., then taking $w(x) = \bar{\bar{v}}(x) + (1+|x|)^a$, with $a > n(p-1)$, (2), (3) and (4) imply (i). We observe that for $a < 2n(p-1)$ the weight $w$ is smaller than that in Wo-Sang Young's paper.

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### References

[1] A. P. Calderón, *Notes of the course "Análisis Real Avanzado" given at the University of Buenos Aires*, 1979.
[2] L. Carleson and P. Jones, *Weighted norm inequalities and a theorem of Koosis*, preprint.
[3] C. Fefferman and E. Stein, *Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115.
[4] B. Muckenhoupt, *Weighted norm inequalities for classical operators*, Proc. of Symposia in Pure Math., vol. XXXV, part 1, 68–83.
[5] J. L. Rubio de Francia, *Boundedness of maximal functions and singular integrals in weighted $L^p$ spaces*, preprint.

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# Equivalent Cauchy sequences and contractive fixed points in metric spaces

by

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**Abstract.** The sequences $[x_i]$, $[y_i]$ in a metric space $(X, d)$ are equivalent Cauchy sequences if and only if given $\varepsilon > 0$ there exist $\delta$ in $(0, \infty]$ and a positive integer $r$ such that $d(x_{i+r}, y_{j+r}) < \varepsilon$ for all $i, j$ with $d(x_i, y_j) < \varepsilon + \delta$. As a typical application let $f: X \to X$ with complete graph such that given $\varepsilon > 0$ there exist $\delta$ in $(0, \infty]$ and an integer $r$ with $d(f^r x, f^r y) < \varepsilon$ for all $x, y$ with $d(x, y) < \varepsilon + \delta$. Then $f$ has a unique fixed point $w$ and $f^i x \to w$ as $i \to \infty$ for all $x$.

**1. Introduction.** Let $(X, d)$ be a metric space, $f: X \to X$, and $N$ be the natural numbers. We call $w$ in $X$ a *contractive fixed point* of $f$ if $fw = w$ and $f^i x \to w$ as $i \to \infty$ in $N$ for all $x$ in $X$. For the existence of a contractive fixed point it is necessary (and under certain mildly restrictive conditions, sufficient) that all orbits $[f^i x]$ be equivalent Cauchy sequences. Sequences $[x_i]$ and $[y_i]$ in $X$ are called *equivalent* if $d(x_i, y_i) \to 0$ as $i \to \infty$. Equivalent Cauchy sequences converge to a common point in the completion of $X$.

Our basic contribution here (Theorem 1) is a characterization (EC) of equivalent Cauchy sequences. Application of (EC) to two identical sequences yields a refinement of the Cauchy convergence criterion (Corollary 1) with correspondingly refined estimates for $d(x_i, w)$ as $x_i \to w$ (Theorem 2). (EC) is applied to orbits for single and multivalued mappings to yield fixed points. Theorem 3 subsumes a body of fixed point theorems. In particular, it easily yields the theorems in [1], [2], [4] and Theorem 1.2 in [3].

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**2. Sequences in metric spaces.**

THEOREM 1. *Two sequences $[x_i]$ and $[y_i]$ in a metric space $(X, d)$ are equivalent-Cauchy if and only if*

(EC) *given $\varepsilon > 0$ there exist $\delta$ in $(0, \infty]$ and $r$ in $N$ such that*

$$d(x_{i+r}, y_{j+r}) < \varepsilon \quad \textit{for all } i, j \textit{ with } d(x_i, y_j) < \varepsilon + \delta. \tag{1}$$