$F = \mathrm{Rad}_{(n)}(E)$ for some $E \subset X$ and a positive integer n. Then the factorization of i_E given in (iii) yields a factorization of i_F, $i_F = B \circ A$ such that $\|A\| \cdot \|B\| \operatorname{unc}\{r_i V_j\}_{\substack{i \leq n \\ j \leq N}} \leq C^2$.

(ii) ⇒ (i). This implication is obvious. ■

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References

[1] P. Enflo, J. Lindenstrauss and G. Pisier, *On the "three space problem"* Math. Scand. 36 (1975), 199–210.
[2] T. Figiel and W. B. Johnson, *Large subspaces of l_∞^n and estimates of the Gordon–Lewis constant*, Israel J. Math. 37 (1980), 92–112.
[3] T. Figiel, W. B. Johnson and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure with applications to Lorentz function spaces*, J. Approximation Theory 13 (1975), 395–412.
[4] Y. Gordon, *Unconditional Schauder decompositions of normed ideals of operators between some l_p-spaces*, Pacific J. Math. 60 (1975), 71–82.
[5] Y. Gordon and D. R. Lewis, *Absolutely summing operators and local unconditional structures*, Acta Math. 133 (1974), 27–48.
[6] W. B. Johnson, J. Lindenstrauss and G. Schechtman, *On the relations between several notions of unconditional structure*, Israel J. Math. 37 (1980), 120–129.
[7] W. B. Johnson and L. Tzafriri, *Some more Banach spaces which do not have local unconditional structure*, Houston J. Math. 3 (1977), 55–60.
[8] M. I. Kadec and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the space L_p*, Studia Math. 21 (1962), 161–176.
[9] T. Ketonen, *On unconditionality in L_p spaces*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 35.
[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence spaces*, Springer-Verlag, 1977.
[11] —, — *Clasical Banach Spaces II, Function spaces*, Springer-Verlag, 1979.
[12] A. Pełczyński and H. P. Rosenthal, *Localization techniques in L_p spaces*, Studia Math. 52 (1975), 263–289.
[13] G. Pisier, *Some results on Banach spaces without local unconditional structure*, Compositio Math. 37 (1978), 3–19.
[14] H. P. Rosenthal, *On the subspaces of L_p ($p > 2$) spanned by sequences of independent random variables*, Israel J. Math. 8 (1970), 273–303.

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On impulsive control with long run average cost criterion

by

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Abstract. Discrete and continuous time impulsive control problems with long run average cost criterion are considered. The paper generalizes the results obtained by M. Robin in [9]. The methods of the proofs are different from those of [9].

Introduction. Impulsive control, introduced first by Bensoussan and Lions in [1], is the one of the most applicable types of stochastic control. This control consists in shifting current states of a Markov process (x_t) to new random states ξ_i, $i = 1, 2, \ldots$, at moment τ_i, respectively. With each strategy $V = (\tau_i, \xi_i)_{i \in N}$ is associated the long run average cost functional $J(V)$ consisting of the "holding cost" $f(x)$ and "replacement cost" $k(x_{\tau_i}, \xi_i) = c(x_{\tau_i}) + d(\xi_i)$ per unit time

$$J_x(V) = \liminf_{t \uparrow \infty} (1/t) E_x^V \Big\{ \int_0^t f(y_s)\, ds + \sum_{i=1}^{\infty} \chi_{\tau_i \leq t} [c(x_{\tau_i}) + d(\xi_i)] \Big\}.$$

The studies of impulsive control problems with such functional were originated by M. Robin in [9] for Markov processes having nice ergodic properties. This paper generalizes his results. We complete and extend his results to Fellerian Markov processes with general state space E. In particular, we show that the value function is constant and find optimal or ε-optimal strategies. We also prove that the use of general stopping times τ_i, instead of those of the form $\tau_i = \tau_{i-1} + \sigma_i \cdot \Theta_{\tau_{i-1}}$ as in the paper [9], does not change the optimal value of the functional.

We start with the discrete time impulsive control. Due to the special form of the controlled system we obtain results more general and complete than those which follow from the existing theory of the long run average cost, see [3]. Next we consider continuous time impulsive control. Methods of some of the proofs are similar to the martingale ones introduced by P. Mandl in the context of adaptive control ([4], [5]).

1. Discrete time case. Let $\Omega = E^N$ be the space of all sequences with values in E, where $(E, \mathcal{B})$ denotes a measurable state space. Suppose that for any $\omega \in \Omega$, $x_n(\omega) = \omega(n)$ and $F_n = \sigma\{x_m, m \leq n\}$, $F = F_\infty$.

Let $X = (\Omega, F_n, \Theta, x_n, P_x)$ be the discrete time homogeneous Markov process with the state space E and the transition function $P(x, \Gamma)$, $x \in E$, $\Gamma \in B$.

Let us assume that we have opportunity to control the process X with the help of impulses. This means that at a Markov time τ we can shift the process to the random point ξ. The choice of τ and ξ is done with a view to our knowledge about the controlled process X till time τ. The impulsive strategy $V = (\tau_i, \xi_i)_{i \in N}$ consists of pairs of Markov times τ_i and F_{τ_i} measurable random variables ξ_i. Without loss of generality we can assume that at each time τ only one shift can be done. A strategy V can be equivalently written as a sequence

$$\Pi = (u_0, u_1, u_2, \ldots),$$

where u_i are measurable transformations from E^{n+1} into E of the form

$$u_k(x_0, \ldots, x_k) = x_k \quad \text{if} \quad \tau_i(\omega) \neq k \text{ for } i = 1, 2, \ldots,$$
$$u_k(x_0, \ldots, x_k) = \xi_i \quad \text{if} \quad \tau_i(\omega) = k.$$

Then the corresponding probability measure P^Π (which there exists by I. Tulcea theorem [7]) has the property

$$P^\Pi(x_{n+1} \in \Gamma | F_n) = P\big(u_n(x_0, \ldots, x_n); \Gamma\big) \tag{1.1}$$

for $n \in N$, $\Gamma \in B$.

On the other hand, with the controlled Markov process with the transition probabilities defined in (1.1) we can associate the impulsive strategy V in the following way:

$$\begin{aligned} \tau_1 &= \inf\{k \geqslant 0 \colon u_k(x_0, \ldots, x_k) \neq x_k\}, \\ \tau_{i+1} &= \inf\{k > \tau_i \colon u_k(x_0, \ldots, x_k) \neq x_k\}, \\ \xi_i &= u_{\tau_i}(x_0, \ldots, x_{\tau_i}) \quad \text{for} \quad i \in N. \end{aligned} \tag{1.2}$$

Let us introduce the following long run average cost functional

$$J_x(\Pi) = \liminf_{t \to \infty} (1/t) E_x^\Pi g_t(\Pi), \tag{1.3}$$

where

$$g_t(\Pi) = \sum_{i=0}^{t-1} f(u_i) + \sum_{i=0}^{t-1} \chi_{u_i \neq x_i} [c(x_i) + d(u_i)] \tag{1.4}$$

and functions f, c, d are nonnegative, bounded, measurable, c is a strictly positive function (i.e. $\exists\, a > 0$ $c(x) \geqslant a$ for $x \in E$).

Suppose that y_t,

$$\begin{aligned} y_t(\omega) &= x_t(\omega) \quad \text{for} \quad t \in]\tau_{n-1}, \tau_n[\cap N, \; \tau_0 = 0, \\ y_{t_n}(\omega) &= \xi_n(\omega), \end{aligned} \tag{1.5}$$

denotes the impulsive control trajectory. Subsequently the measure P^Π will be denoted by P^V to underline the impulsive strategy V.

Then the functional (1.3) has the equivalent form

$$J_x(V) = \liminf_{t \uparrow \infty} (1/t) E_x^V g_t(V), \tag{1.6}$$

where

$$g_t(V) = \sum_{i=0}^{t-1} f(y_i) + \sum_{i=1}^{\infty} \chi_{\tau_i < t} [c(x_{\tau_i}) + d(y_{\tau_i})]. \tag{1.7}$$

We want to minimize the functional $J_x(V)$, find the value $u(x)$,

$$u(x) = \inf_V J(V), \tag{1.8}$$

and optimal or ε-optimal strategies.

Let

$$\lambda = \inf_x \inf_\tau \frac{E_x\Big\{\sum_{i=0}^{\tau-1} f(x_i) + c(x_\tau)\Big\} + d(x)}{E_x\{\tau\}}. \tag{1.9}$$

We will assume henceforth that for τ, such that $E_x \tau = \infty$, the notation

$$E_x\Big\{\sum_{i=0}^{\tau-1} f(x_i) + c(x_\tau)\Big\}$$

or

$$\frac{E_x\Big\{\sum_{i=0}^{\tau-1} f(x_i) + c(x_\tau)\Big\}}{E_x\{\tau\}}$$

means

$$\liminf_{t \uparrow \infty} E_x\Big\{\sum_{i=0}^{\tau \wedge t-1} f(x_i) + c(x_{\tau \wedge t})\Big\}$$

or

$$\liminf_{t \uparrow \infty} \frac{E_x\Big\{\sum_{i=0}^{\tau \wedge t-1} f(x_i) + c(x_{\tau \wedge t})\Big\}}{E_x\{\tau \wedge t\}},$$

respectively. Suppose that

$$w(x) \overset{\text{def}}{=} \inf_\tau E_x\Big\{\sum_{i=0}^{\tau-1} \big(f(x_i) - \lambda\big) + c(x_\tau)\Big\}, \tag{1.10}$$

$$\tau^* = \inf\{s \geqslant 0 \colon w(x_s) \geqslant c(x_s)\}. \tag{1.11}$$

Let $V(\tau, y)$ denotes impulsive strategy consisting of the times τ_i of the form

$$\begin{aligned} \tau_1 &= \tau, \\ \tau_2 &= \tau_1 + \tau \circ \Theta_{\tau_1}, \\ &\cdots\cdots\cdots \\ \tau_n &= \tau_{n-1} + \tau \circ \Theta_{\tau_{n-1}} \end{aligned} \tag{1.12}$$

and deterministic fix point y.

The following theorem is the main result of this section.

THEOREM 1. *The optimal value $u(\cdot)$ is constant and equals λ. Moreover,*

(a) *if there exists $y \in E$ such that $E_y\tau^* = \infty$, then the strategy $V(\tau^*, y)$ is optimal one,*

(b) *if for each $y \in E$, $E_y\tau^* < \infty$, and*

$$\inf_y [w(y) + d(y)] = 0, \tag{1.13}$$

then the strategy $V(\tau^*, x^\varepsilon)$, where x^ε is such that $w(x^\varepsilon) + d(x^\varepsilon) \leqslant \varepsilon$, is $\varepsilon/E_{x^\varepsilon}\{\tau^*\}$ optimal; moreover, τ^* is optimal Markov time in the definition of w,

(c) *if*

$$\sup_x \liminf_{t\uparrow\infty} E_x\Big\{\sum_{i=0}^{t-1}\big(f(x_i) - \lambda\big) + c(x_t)\Big\} < \infty, \tag{1.14}$$

then the strategy "not to do impulses" $V(\infty, y)$ is optimal one.

Proof. Let us first define the following impulsive strategy $\overline{V} = (\tau_i, y)$, where

$$\begin{aligned} \tau_1 &= 1, \\ \tau_2 &= 1 + \tau \circ \Theta_1, \\ \tau_{i+1} &= \tau_i + \tau \circ \Theta_{\tau_i}, \quad i \in N. \end{aligned} \tag{1.15}$$

τ is an arbitrary Markov time such that $0 < E_y\tau < \infty$. Then trajectory y for $t < \tau_i$, together with the random variable x_{τ_i}, is independent of the trajectory y_{τ_i+t} for $t \geqslant 0$ with respect to the probability measure $P^{\overline{V}}$. Similarly, for each $i \in N$, the stopping times τ_i and $\tau \circ \Theta_{\tau_i}$ are $P^{\overline{V}}$ independent. Thus

$$J_x(\overline{V}) \leqslant \liminf_{t\uparrow\infty} (1/t) E_x^{\overline{V}}\Big\{\sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t}\Big\} E_y\Big\{\sum_{i=0}^{\tau-1} f(x_i) + c(x_\tau) + d(x)\Big\}$$

and using the Blackwell renewal theorem we obtain

$$J_x(\overline{V}) \leqslant \frac{E_y\Big\{\sum_{i=0}^{\tau-1} f(x_i) + c(x_\tau)\Big\} + d(y)}{E_y\{\tau\}}. \tag{1.16}$$

Since the last inequality is also satisfied for $\tau = 0$, so recalling the remark after definition (1.9), we have

$$u(x) \leqslant \lambda. \tag{1.17}$$

We will prove the reverse inequality. We start with the following

LEMMA 1. *For each bounded measurable function z*

$$M_t = z(y_t) - \sum_{i=0}^{t-1}(P - I)z(y_i) - \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t}\big(z(\xi_i) - z(x_{\tau_i})\big) - z(x) \tag{1.18}$$

is P^V martingale.

The proof of Lemma 1 is an easy consequence of the well-known result stating that

$$z(x_t) - \sum_{i=0}^{t-1}\big(P^{u_i(x_0,\ldots,x_i)} - I\big) z(x_i)$$

is P^V martingale.

Next we define the function

$$S_t(V)(x) = E_x^V\{g_t(V) - t\lambda + z(y_t)\}, \tag{1.19}$$

where z is a bounded measurable function. Then by Lemma 1

$$\begin{aligned} S_t(V)(x) &= E_x^V\Big\{g_t(V) - t\lambda + M_t + \sum_{i=0}^{t-1}(P - I)z(y_i) + \\ &\quad + \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t}\big(z(\xi_i) - z(x_{\tau_i})\big)\Big\} + z(x) \\ &= E_x^V\Big\{\sum_{i=0}^{t-1}[(P - I)z(y_i) + f(y_i) - \lambda] + \\ &\quad + \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t}\big(z(\xi_i) + d(\xi_i) - z(x_{\tau_i}) + c(x_{\tau_i})\big)\Big\} + z(x). \end{aligned} \tag{1.20}$$

So if we find the function z for which the following inequalities are satisfied:

$$\begin{aligned} z(x) + d(x) &\geqslant 0, \\ c(x) - z(x) &\geqslant 0, \qquad x \in E, \\ (P - I)z(x) + f(x) - \lambda &\geqslant 0, \end{aligned} \tag{1.21}$$

then we obtain

$$\liminf_{t\uparrow\infty}(1/t)S_t(V)(x) \geqslant \liminf_{t\uparrow\infty}(1/t)z(x) = 0. \tag{1.22}$$

Since

$$\liminf_{t\uparrow\infty}(1/t)S_t(V)(x) = \liminf_{t\uparrow\infty}(1/t)E_x^V\{g_t(V)\}-\lambda \tag{1.23}$$

and the strategy V can be chosen arbitrarily, we have from (1.22)

$$u(x) \geqslant \lambda. \tag{1.24}$$

It remains to find the desired function z.

LEMMA 2. *For function $z(x) = w(x)$, where w is defined by* (1.10), *the inequalities* (1.21) *are satisfied.*

Proof. Let τ be such that $0 < E_x\tau < \infty$. Then

$$\lambda \leqslant \frac{E_x\{\sum_{i=0}^{\tau-1} f(x_i)+c(x_\tau)\}+d(x)}{E_x\{\tau\}}$$

and

$$E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x)_i-\lambda\big)+c(x_\tau)\Big\}+d(x) \geqslant 0. \tag{1.25}$$

Since for $\tau = 0$ the inequality (1.25) is also satisfied, we have $w(x)+d(x) \geqslant 0$.

The second inequality (1.21) is trivial by the definition of the function w. We will prove the third one.

We have

$$w(x) \leqslant E_x\Big\{\sum_{i=0}^{(1+\sigma\cdot\Theta_1)-1}\big(f(x_i)-\lambda\big)+c(x_{(1+\sigma\cdot\Theta_1)})\Big\} \tag{1.26}$$

$$= f(x)-\lambda+E_x\Big\{E_{x_1}\Big\{\sum_{i=0}^{\sigma-1}\big(f(x_i)-\lambda\big)+c(x_\sigma)\Big\}\Big\}$$

for arbitrary Markov time σ, $E_x\sigma < \infty$. Moreover,

$$\operatorname*{ess\,inf}_{\sigma} E_{x_1}\Big\{\sum_{i=0}^{\sigma-1}\big(f(x_i)-\lambda\big)+c(x_\sigma)\Big\} \leqslant \|c\| \tag{1.27}$$

and there exists a sequence σ_n, for which we have

$$E_{x_1}\Big\{\sum_{i=0}^{\sigma_n-1}\big(f(x_i)-\lambda\big)+c(x_{\sigma_n})\Big\}\downarrow \operatorname*{ess\,inf}_{\sigma} E_{x_1}\Big\{\sum_{i=0}^{\sigma-1}\big(f(x_i)-\lambda\big)+c(x_\sigma)\Big\}. \tag{1.28}$$

So summarizing (1.26)–(1.28), we obtain

$$w(x) \leqslant f(x)-\lambda+Pw(x). \tag{1.29}$$

Finally, from (1.17) and (1.24), $u(x) = \lambda$.

Now we deal with finding out the optimal strategies characterizing the value u. To do this we have to prove some lemmas associated with the function w.

LEMMA 3. *The following identity holds:*

$$w(x) = \min\{c(x), f(x)-\lambda+Pw(x)\}. \tag{1.30}$$

Proof. Suppose for some $x \in E$, and $\varepsilon > 0$

$$w(x) \leqslant c(x)-\varepsilon, \qquad w(x) \leqslant f(x)-\lambda+Pw(x)-\varepsilon.$$

Since $w(x_t)-\sum_{i=0}^{t-1}(P-I)w(x_i)$ is P_x-martingale, for a bounded Markov time τ we have

$$w(x) = E_x\Big\{w(x_\tau)-\sum_{i=0}^{\tau-1}(P-I)w(x_i)\Big\} \leqslant E_x\{\chi_{\tau=0}w(x)\}+$$

$$+E_x\Big\{\chi_{\tau>0}\Big(c(x_\tau)-\varepsilon+\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)\Big)\Big\}$$

$$\leqslant E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)+c(x_\tau)\Big\}-\varepsilon.$$

But a bounded Markov time τ can be chosen arbitrarily, so the last inequality contradicts the definition of function w.

LEMMA 4. *The function w satisfies the identity*

$$w(x) = \inf_\tau E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)+w(x_\tau)\Big\}. \tag{1.31}$$

Proof. Let us write

$$\overline{w}(x) = \inf_\tau E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)+w(x_\tau)\Big\}.$$

Immediately, we have $\overline{w}(x) \leqslant w(x)$. On the other hand, for each Markov time τ, $E_x\tau < \infty$, using the same consideration as in (1.26)–(1.29), we obtain

$$w(x) \leqslant \inf_{\sigma} E_x\Big\{ \sum_{i=0}^{(\tau+\sigma\cdot\Theta_\tau)-1} \big(f(x_i)-\lambda\big) + c(x_{\tau+\sigma\cdot\Theta_\tau})\Big\}$$

$$= E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)\Big\} + \inf_{\sigma} E_x\Big\{E_{x_\tau}\Big\{\sum_{i=0}^{\sigma-1}\big(f(x_i)-\lambda\big)+c(x_\sigma)\Big\}\Big\}$$

$$\leqslant E_x\Big\{\sum_{i=0}^{\tau-1}\big(f(x_i)-\lambda\big)+w(x_\tau)\Big\}.$$

Thus $w(x) \leqslant \overline{w}(x)$, and (1.31) is satisfied.

LEMMA 5. *Let for* $\varepsilon \geqslant 0$

(1.32) $$\tau_\varepsilon = \inf\{s \geqslant 0:\ w(x_s) \geqslant c(x_s)-\varepsilon\}.$$

If for some ε, $E_x\tau_\varepsilon < \infty$, *then*

(1.33) $$w(x) = E_x\Big\{\sum_{i=0}^{\tau_\varepsilon-1}\big(f(x_i)-\lambda\big)+w(x_{\tau_\varepsilon})\Big\}$$

and τ_ε *is* ε*-optimal stopping time in the definition* (1.10) *of function* $w(x)$.

Proof. Suppose $\varepsilon > 0$, and $E_x\tau_\varepsilon < \infty$. Let, for $\delta > 0$, $\tau(\delta)$ denote a bounded δ-optimal stopping time from the definition of function w. Then

(1.34) $$\lim_{\delta\downarrow 0} E_x\Big\{\sum_{i=0}^{\tau(\delta)-1}\big(f(x_i)-\lambda\big)+c(x_{\tau(\delta)})\Big\} = w(x)$$

and

(1.35) $$w(x) \leqslant E_x\Big\{\sum_{i=0}^{\tau(\delta)-1}\big(f(x_i)-\lambda\big)+w(x_{\tau(\delta)})\Big\}$$

$$\leqslant E_x\Big\{\chi_{\tau(\delta)<\tau_\varepsilon}\Big(\sum_{i=0}^{\tau(\delta)-1}\big(f(x_i)-\lambda\big)+c(x_{\tau(\delta)})-\varepsilon\Big)\Big\}+$$

$$+E_x\Big\{\chi_{\tau(\delta)\geqslant\tau_\varepsilon}\Big(\sum_{i=0}^{\tau(\delta)-1}\big(f(x_i)-\lambda\big)+c(x_{\tau(\delta)})\Big)\Big\}$$

$$= -\varepsilon E_x\{\chi_{\tau(\delta)<\tau_\varepsilon}\}+E_x\Big\{\sum_{i=0}^{\tau(\delta)-1}\big(f(x_i)-\lambda\big)+c(x_{\tau(\delta)})\Big\}.$$

Letting $\delta\to 0$, we obtain

(1.36) $$\lim_{\delta\downarrow 0} E_x\{\chi_{\tau(\delta)<\tau_\varepsilon}\} = 0.$$

Thus $\chi_{\tau(\delta)<\tau_\varepsilon}\to 0$ in probability as $\delta\downarrow 0$, and there exists a sequence $\delta_n\to 0$ such that $\chi_{\tau(\delta_n)<\tau_\varepsilon}\to 0$ a.s. This implies $\tau(\delta_n)\wedge\tau_\varepsilon\uparrow\tau_\varepsilon$ a.s. as $n\to\infty$.

Since without any trouble one can check that

(1.37) $$\Psi(t) = w(x_t)+\sum_{i=0}^{t-1}\big(f(x_i)-\lambda\big)$$

is submartingale, we have

(1.38) $$w(x) \leqslant E_x\Psi\big(\tau_\varepsilon\wedge\tau(\delta_n)\big) \leqslant E_x\Psi\big(\tau(\delta_n)\big).$$

Letting $n\to\infty$ and using (1.34), we obtain

(1.39) $$w(x) = E_x\Psi(\tau_\varepsilon)$$

so (1.33) is satisfied, and from definition τ_ε is really ε optimal stopping time.

Assume now $E_x\tau_0 < \infty$. Then $E_x\tau_\varepsilon < \infty$ for $\varepsilon > 0$, since $\tau_\varepsilon\uparrow\tau_0$ as $\varepsilon\downarrow 0$. Putting in (1.39) $\varepsilon\downarrow 0$, we obtain

(1.40) $$w(x) = E_x\Psi(\tau_0)$$

which implies that τ_0 is optimal stopping time in the definition of function w. This completes the proof of Lemma 5.

Using Lemmas 3, 4, 5, we can finish the proof of Theorem 1. Let us have a look at (1.20). For the strategy $V(\tau^*, y)$, where $E_y\tau^* = \infty$, using Lemma 3 we obtain

(1.41) $$S_t\big(V(\tau^*, y)\big)(x) = E_x^V\Big\{\sum_{i=1}^{\infty}\chi_{\tau_i\leqslant t}\Big\}\big(w(y)+d(y)\big)+w(x).$$

Let us fix real number $T > 0$. Then

(1.42) $$S_t\big(V(\tau^*, y)\big)(x) \leqslant S_t\big(V(\tau^*\wedge T, y)\big)(x)$$

$$\leqslant E_x^{V(\tau^*\wedge T, y)}\Big\{\sum_{i=1}^{\infty}\chi_{\tau_i\wedge T\leqslant t}\Big\}\big(2\|w\|+\|d\|+\|c\|\big)+w(x)$$

and, with the help of classical renewal theorem, we have the estimation

(1.43) $$J_x\big(V(\tau^*, y)\big)(x) = \liminf_{t\uparrow\infty}(1/t)S_t\big(V(\tau^*, y)\big)(x)+\lambda$$

$$\leqslant \frac{2\|w\|+\|d\|+\|c\|}{E_y\{\tau^*\wedge T\}}+\lambda.$$

Letting $T\to\infty$, we obtain $J_x\big(V(\tau^*, y)\big)(x) = \lambda$. Thus $V(\tau^*, y)$ is optimal.

The proof of (b) is similar, so can be omitted.

Suppose now that (1.14) is satisfied. Then, we take as a function z in (1.20) the following

(1.44) $$z(x) \overset{\text{def}}{=} \liminf_{t\uparrow\infty} E_x\Big\{\sum_{i=0}^{t-1}\big(f(x_i)-\lambda\big)+c(x_t)\Big\}.$$

Using Fatou's lemma, one shows without difficulty that z satisfies the inequality

$$(P-I)w(x)+f(x)-\lambda \leqslant 0. \tag{1.45}$$

Hence

$$S_t(V(\infty, y))(x) \leqslant w(x)$$

and $u(x) = J_x(V(\infty, y)) = \lambda$. The proof of the theorem is thus complete.

As an example, we consider a special class of Markov processes having "nice ergodic properties". Let (E, B) be a compact state space, and let there exist invariant measure m such that

$$|P_x(x_n \in \Gamma)-m(\Gamma)| \leqslant K\beta^n \quad \text{for each } \Gamma \in B, \tag{1.46}$$

where $0 \leqslant \beta < 1$, and β, K are independent of Γ. Let us write $\bar{f} = \int_E f(x)m(dx)$.

Theorem 2. *Under assumption* (1.46) *one of conditions* (1.13) *or* (1.14) *is satisfied. If $\lambda < \bar{f}$, then* (1.13) *and $E_y\tau^* < \infty$ for $y \in E$; if $\lambda = \bar{f}$, then* (1.14) *holds.*

Proof. From (1.46)

$$v(x) \overset{\text{def}}{=} \sum_{i=0}^{\infty} \left(\bar{f}-P^i f(x)\right) \tag{1.47}$$

is the bounded function (see Lemma A.2, [9], for the continuous time case). Hence for any Markov time τ, $E_x\tau < \infty$,

$$v(x) = E_x\Big\{\sum_{i=0}^{\tau-1}\left(\bar{f}-f(x_i)\right)+v(x_\tau)\Big\} \tag{1.48}$$

and

$$\lambda = \inf_x \inf_\tau \frac{E_x\{\bar{f}\tau+v(x_\tau)+c(x_\tau)\}+d(x)-v(x)}{E_x\{\tau\}}. \tag{1.49}$$

From the definition of inf, there exists a sequence (x_n, τ^n), $E_{x_n}\{\tau^n\} < \infty$, such that

$$\lambda = \lim_{n\to\infty} \frac{E_{x_n}\Big\{\sum_{i=0}^{\tau^n-1} f(x_i)+c(x_{\tau^n})\Big\}+d(x_n)}{E_{x_n}\{\tau^n\}} \tag{1.50}$$

and we have two situations:

(1) $E_{x_n}\tau^n$ is bounded as $n\to\infty$; then

$$0 = \lim_{n\to\infty} E_{x_n}\Big\{\sum_{i=0}^{\tau^n-1}\left(f(x_i)-\lambda\right)+c(x_{\tau^n})+d(x_n)\Big\}$$

and

$$\inf_{x\in E}\left(w(x)+d(x)\right) = 0$$

or

(2) $E_{x_n}\tau^n$ is unbounded as $n\to\infty$, and then from (1.49) $\lambda = \bar{f}$ and since v is bounded, (1.14) is satisfied.

Finally, let us note that if $\lambda < \bar{f}$ then we have situation (1) and $E_y\tau^* < \infty$ for $y \in E$.

We close Section 1 with the remark associated with assumption (1.46).

Remark 1. If the Markov kernel $P(x, dy)$ is of the form

$$P(x, dy) = r(x, y)\mu(dy), \tag{1.51}$$

where μ is a probability measure on (E, B), $r(x, y)$ is nonnegative and continuous, and there exists ball U such that

$$\mu(U) > 0 \quad \text{and} \quad r(x, y) > 0 \quad \text{for} \quad x \in E, y \in U \tag{1.52}$$

then assumption (1.46) is satisfied.

The proof of this fact can be found in Doob [2].

2. Continuous time case. This section presents the completion of the result due to M. Robin [9]. Methodologically, it is very similar to the previous one.

Let $\Omega = D(R^+, E)$ be the space of right continuous left limited functions from R^+ into E, a locally compact with countable base state space.

Let $x_t(\omega) = \omega(t)$ for any $\omega \in \Omega$, $F_t^0 = \sigma\{x_s, s \leqslant t\}$, $F^0 = F_\infty^0$, and F_t, F be universally completed σ-fields of F_t^0, F^0, respectively. We will denote by $C(C_0)$ the Banach space of continuous bounded (vanishing at infinity in addition) functions on E. Suppose $X = (\Omega, F_t, \Theta_t, x_t, P_x)$ is a homogeneous Markov process with the semigroup $(\Phi(t))_{t\geqslant 0}$. We assume

$$\Phi(t)C \subset C \quad \text{for} \quad t \geqslant 0 \quad \text{(Feller property)}, \tag{2.1}$$

$$\Phi(t)C_0 \subset C_0 \quad \text{for} \quad t \geqslant 0. \tag{2.2}$$

To describe the evolution of the controlled process we have to recall the construction of the new probability space due to M. Robin [8].

Let $\tilde{\Omega} = \Omega^N$. The impulsive strategy $V = (\tau_i, \xi_i)_{i\in N}$ consists of pairs of Markov times τ_i and random variables ξ_i such that

τ_1 is $F_t \otimes \{\emptyset, \Omega\} \otimes \ldots \otimes \{\emptyset, \Omega\} \otimes \ldots$ Markov time,

ξ_1 is $F_{\tau_1} \otimes \{\emptyset, \Omega\} \otimes \ldots \otimes \{\emptyset, \Omega\} \otimes \ldots$ measurable random variable,

. . . .

τ_n is $F_t^n \otimes \{\emptyset, \Omega\} \otimes \ldots \otimes \{\emptyset, \Omega\} \otimes \ldots$ Markov time,

ξ_n is $F_{\tau_n}^n \otimes \{\emptyset, \Omega\} \otimes \ldots \otimes \{\emptyset, \Omega\} \otimes \ldots$ measurable random variable, where $F_t^n = F_t^{n-1} \otimes F_t$ and $F_t^1 = F_t$. So for $\omega \in \tilde{\Omega}$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$ we have

$$\begin{aligned} \tau_1(\omega) &= \tau_1(\omega_1), & \xi_1(\omega) &= \xi_1(\omega_1), \\ \tau_2(\omega) &= \tau_2(\omega_1, \omega_2), & \xi_2(\omega) &= \xi_2(\omega_1, \omega_2), \\ &\ldots\ldots\ldots & &\ldots\ldots\ldots \end{aligned}$$

and the trajectory of the controlled process X is of the form

$$(2.3)\qquad \begin{aligned} y_t(\omega) &= x_t^{n-1}(\omega_n) \quad \text{for} \quad t \in [\tau_{n-1}, \tau_n[,\ \tau_0 = 0, \\ y_{\tau_n}(\omega) &= \xi_n(\omega_1, \ldots, \omega_n). \end{aligned}$$

The impulsive control V generates a probability measure P^V on the space $\tilde{\Omega}$. Let us denote by $G_{\tau_1}, G_{\tau_2}, \ldots, G_{\tau_n}$, the following σ-fields on the spaces $\Omega^n = \mathop{\mathsf{X}}_{i=1}^{n+1} \Omega$, $n = 1, 2, \ldots$,

$$\begin{aligned} G_{\tau_1} &= \sigma\{F_{\tau_1}^2, F_{\tau_1}^1 \otimes \{\emptyset, \Omega\}\}, \\ &\ldots\ldots\ldots\ldots\ldots \\ G_{\tau_n} &= \sigma\{F_{\tau_n}^{n+1}, F_{\tau_n}^n \otimes \{\emptyset, \Omega\}\}. \end{aligned}$$

The projections P^n of the measure P^V on the spaces Ω^n, $n = 0, 1, \ldots$, have the following properties:

$$\begin{aligned} &P_x^0 = P_x, \\ &\ldots\ldots \\ &P_x^n = P_x^{n-1} \otimes \varepsilon_{\varphi_y} \quad \text{on} \quad G_{\tau_n}, \\ &P_x^n(\Theta_{n,\tau_n}^{-1} B | G_{\tau_n}) = \varepsilon_{\varphi_{\omega_1(\tau_1)}}(\omega_1) \otimes \ldots \otimes \varepsilon_{\varphi_{\omega_{n-1}(\tau_{n-1})}}(\omega_{n-1}) \otimes P_{\xi(\omega_1, \ldots, \omega_n)}(B), \end{aligned}$$

where $\varphi_y(t) = y$ for $t \geqslant 0$ denotes the constant trajectory, and $B \in F_\infty^{n+1}$.

The continuous time analog of the long run average cost functional is of the form

$$(2.4)\qquad J_x(V) = \liminf_{t \uparrow \infty} \frac{E_x^V g_t(V)}{t},$$

where

$$(2.5)\qquad g_t(V) = \int_0^t f(y_s)\, ds + \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t} [c(x_{\tau_i}^{i-1}) + d(y_{\tau_i})],$$

$f, c, d \in C$, $f \geqslant 0$, $d \geqslant 0$, c is strictly positive, $c(x) \geqslant a > 0$. Our aim is a minimization of $J_X(V)$, and characterization of $u(x) = \inf_V J_x(V)$.

In this section we will consider first impulsive control consisting of the special form stopping times. Namely, similarly as in [9], the times τ_i will be of the form

$$(2.6)\qquad \tau_i(\omega) = \tau_{i-1}(\omega_1, \ldots, \omega_{i-1}) + \sigma_i(\omega_i)\, \Theta_{\tau_{i-1}(\omega_1, \ldots, \omega_{i-1})}, \qquad i \geqslant 2,$$

where σ_i is an arbitrary F_t Markov time. Let

$$(2.7)\qquad \lambda = \inf_x \inf_\tau \frac{E_x\{\int_0^\tau f(x_s)\, ds + c(x_\tau)\} + d(x)}{E_x\{\tau\}},$$

$$(2.8)\qquad w(x) = \inf_\tau E_x\Big\{\int_0^\tau \big(f(x_s) - \lambda\big)\, ds + c(x_\tau)\Big\}$$

and

$$(2.9)\qquad \tau_\varepsilon = \inf\{s \geqslant 0:\ w(x_s) \geqslant c(x_s) - \varepsilon\} \quad \text{for} \quad \varepsilon > 0.$$

Suppose $V(\tau, y)$ denotes similarly as in Section 1 impulsive strategy consisting of the time τ_i, where $\tau_1 = \tau = \sigma_i$ for $i \in N$, and deterministic fix point y.

The following theorem holds:

THEOREM 1. *The optimal value $u(x)$ with respect to strategies with Markov times satisfying (2.6) is constant and equals λ. Moreover,*

(a) *if, for some $y \in E$, $E_y\{\tau_\varepsilon\} \uparrow \infty$ as $\varepsilon \to 0$, then the strategy $V(\tau^*, y)$, with the stopping time $\tau^* = \lim_{\varepsilon \downarrow 0} \tau_\varepsilon$, is optimal;*

(b) *if, for each $y \in E$, $E_y \tau^* < \infty$, and*

$$(2.10)\qquad \inf_y [w(y) + d(y)] = 0$$

then the strategy $V(\tau_\varepsilon, x^\varepsilon)$, where x^ε is such that $w(x^\varepsilon) + d(x^\varepsilon) \leqslant \varepsilon$, is $2\varepsilon / E_{x^\varepsilon}\{\tau_\varepsilon\}$ optimal;

(c) *if*

$$(2.11)\qquad \sup_x \liminf_{t \uparrow \infty} E_x \Big\{\int_0^t \big(f(x_s) - \lambda\big)\, ds + c(x_t)\Big\} < \infty$$

then the strategy "do not interfere in the run of process" $V(\infty, y)$ is optimal.

Proof. We will follow the proof of Theorem 1.1. The identical consideration as in (1.15)–(1.17) leads to the inequality $u(x) \leqslant \lambda$.

To prove the reverse inequality we have to introduce the function $S_t(V)(x)$:

$$(2.12)\qquad \begin{aligned} S_t(V)(x) &= E_x^V \Big\{\int_0^t f(y_s)\, ds + \sum_{i=1}^{\infty} \chi_{\tau_i \geqslant t} [c(x_{\tau_i}^{i-1}) + d(\xi_i)] - t\lambda + w(y_t)\Big\} \\ &= E_x^V \Big\{\Big\{\int_0^t \big(f(y_s) - \lambda\big)\, ds + \end{aligned}$$

$$+\sum_{i=1}^{\infty}\chi_{\tau_i\leqslant t}[w(x_{\tau_i}^{i-1})-w(\xi_i)]+w(y_t)-w(x)\}+$$

$$+\Big\{\sum_{i=1}^{\infty}\chi_{\tau_i\leqslant t}[w(\xi_i)-w(x_{\tau_i}^{i-1})+c(x_{\tau_i}^{i-1})+d(\xi_i)]+w(x)\Big\}\Big\}.$$

Let us denote by A_t the first expression in the brackets:

$$A_t(V)(x)\overset{\text{def}}{=}E_x^V\Big\{\int_0^t\big(f(y_s)-\lambda\big)ds+\tag{2.13}$$

$$+\sum_{i=1}^{\infty}\chi_{\tau_i\leqslant t}[w(x_{\tau_i}^{i-1})-w(\xi_i)]+w(y_t)-w(x)\Big\}$$

and by $N(0,t)$ the number of impulses in the time interval $[0, t]$. Since $c(x)\geqslant a>0$ and $u(x)\leqslant\|f\|$, we can restrict in future to such impulsive strategies V for which

$$\liminf_{t\uparrow\infty}(1/t)E_x^VN(0,t)\cdot a\leqslant 2\|f\|.\tag{2.14}$$

This means that $\tau_i\to\infty$ a.s. as $i\to\infty$. Then

$$S_t(V)=\lim_{n\uparrow\infty}S_{t\wedge\tau_n}(V)\tag{2.15}$$

and

$$A_{t\wedge\tau_n}(V)(x)=E_x^V\Big\{\sum_{i=0}^{n-1}\chi_{\tau_i\leqslant t}\Big[\int_{\tau_i}^{\tau_{i+1}\wedge t}\big(f(x_s^i)-\lambda\big)ds+w(x_{\tau_{i+1}\wedge t}^i)-w(x_{\tau_i}^i)\Big]\Big\}.\tag{2.16}$$

Further on we will study the properties of the function w defined in (2.8).

PROPOSITION 1. *The function w is $\mathscr{C}_0$ continuous, that is for each $x\in E$*

$$P_x\big(\lim_{t\downarrow 0}w(x_t)=w(x)\big)=1.\tag{2.17}$$

Moreover,

$$\Psi(t)=\int_0^t\big(f(x_s)-\lambda\big)ds+w(x_t)\tag{2.18}$$

is the right continuous submartingale.

Proof. The proof will follow from some lemmas. We begin from the discounted, finite time optimal stopping problem. The following lemma is proved in [8]:

LEMMA 1. *If X is a right continuous, homogeneous Markov process on the state space E, and assumptions (2.1), (2.2) are satisfied, then*

$$w_0^{T,\alpha}(t,x)\overset{\text{def}}{=}\inf_\tau E_x\Big\{\int_0^{\tau\wedge(T-t)}e^{-\alpha s}\big(f(x_s)-\lambda\big)ds+\chi_{\tau\leqslant T-t}e^{-\alpha\tau}c(x_\tau)\Big\}\tag{2.19}$$

is the continuous function on $[0,T]\times E$.

Then

LEMMA 2.

$$w_0^T(t,x)\overset{\text{def}}{=}\inf_\tau E_x\Big\{\int_0^{\tau\wedge(T-t)}\big(f(x_s)-\lambda\big)ds+\chi_{\tau\leqslant T-t}c(x_\tau)\Big\}\tag{2.20}$$

is the continuous function on $[0,T)\times E$.

Proof. We have $w_0^{T,\alpha}\to w_0^T$ uniformly as $\alpha\to 0$ since

$$|w_0^{T,\alpha}(t,x)-w_0^T(t,x)|\leqslant\sup_\tau E_x\Big\{\|f\|\int_0^{\tau\wedge(T-t)}(1-e^{-\alpha s})ds+\chi_{\tau\leqslant T-t}(1-e^{-\alpha\tau})\|c\|\Big\}$$

$$\leqslant\|f\|\int_0^t(1-e^{-\alpha s})ds+\|c\|(1-e^{-\alpha T})\to 0\quad\text{as}\quad\alpha\to 0.$$

Hence

LEMMA 3.

$$w^T(t,x)\overset{\text{def}}{=}\inf_\tau E_x\Big\{\int_0^{\tau\wedge(T-t)}\big(f(x_s)-\lambda\big)ds+c(x_{\tau\wedge(T-t)})\Big\}\tag{2.21}$$

is the continuous function on $[0,T]\times E$.

Proof. This follows from Lemma 2 if we note that

$$\inf_\tau E_x\Big\{\int_0^{\tau\wedge(T-t)}\big(f(x_s)-\lambda\big)ds+c(x_{\tau\wedge(T-t)})-\|c\|\Big\}$$

$$=\inf_\tau E_x\Big\{\int_0^{\tau\wedge(T-t)}\big(f(x_s)-\lambda\big)ds+\chi_{\tau\leqslant T-t}\big(c(x_\tau)-\|c\|\big)\Big\}.$$

Taking into account the previous result, we easy assure that

$$\Psi^T(t)=\int_0^t\big(f(x_s)-\lambda\big)ds+w^T(t,x_t)\tag{2.22}$$

is for $t\in[0,T]$ the right continuous submartingale.

On the other hand,

$$w^T(t,x) = \inf_{\tau \leqslant T-t} E_x\Big\{\int_0^\tau \big(f(x_s)-\lambda\big)\,ds + c(x_\tau)\Big\} \tag{2.23}$$

so $w^T(t,x)\downarrow w(x)$ as $T\to\infty$, and

$$w(x) = \inf_\tau E_x\Big\{\int_0^\tau \big(f(x_s)-\lambda\big)\,ds + c(x_\tau)\Big\}. \tag{2.24}$$

This imply that $(\Psi^T(t))_{t\leqslant T}$ presents a family of decreasing (on each finite interval $[0, T_1]$, for $T\geqslant T_1$) right continuous submartingales. Then using Theorem VI 16, [6] we finally obtain that $(\Psi(t))_{t\geqslant 0}$ is almost surely the right continuous submartingale. Hence $w(x)$ is $\mathscr{C}_0$ continuous. In order to characterize the function u we need the next three lemmas. Since proofs of Lemmas 4 and 5 are very similar to those from Section 1, they will not be given here.

LEMMA 4. *The function w satisfies the inequalities*

$$\begin{aligned} &w(x)\leqslant c(x),\\ &w(x)+d(x)\geqslant 0. \end{aligned} \tag{2.25}$$

LEMMA 5. *The function w is a solution of the equation*

$$w(x) = \inf_\tau E_x\Big\{\int_0^\tau \big(f(x_s)-\lambda\big)\,ds + w(x_\tau)\Big\}. \tag{2.26}$$

The next lemma is a continuous-time analog of Lemma 1.5.

LEMMA 6. *If*

$$\tau_\varepsilon = \inf\{s\geqslant 0\colon\ w(x_s)\geqslant c(x_s)-\varepsilon\} \quad \text{for} \quad \varepsilon>0 \tag{2.27}$$

and $E_x\tau_\varepsilon<\infty$, then τ_ε is ε optimal stopping time in the definition of function w. Moreover,

$$w(x) = E_x\Big\{\int_0^{\tau_\varepsilon} \big(f(x_s)-\lambda\big)\,ds + w(x_{\tau_\varepsilon})\Big\}. \tag{2.28}$$

Proof. We follow the proof of Lemma 1.5. From (1.38), (1.36), using the fact that $\chi_{\tau(\delta_n)<\tau_\varepsilon}\to 0$ a.s. as $n\to\infty$, we obtain

$$\begin{aligned} w(x) &= \lim_{n\uparrow\infty} E_x\Psi\big(\tau(\delta_n)\wedge\tau_\varepsilon\big)\\ &= \lim_{n\uparrow\infty} E_x\Big\{\int_0^{\tau(\delta_n)\wedge\tau_\varepsilon}\big(f(x_s)-\lambda\big)\,ds +\\ &\qquad + \chi_{\tau(\delta_n)<\tau_\varepsilon} w(x_{\tau(\delta_n)}) + \chi_{\tau_\varepsilon\leqslant\tau(\delta_n)} w(x_{\tau_\varepsilon})\Big\}\\ &= E_x\Big\{\int_0^{\tau_\varepsilon}\big(f(x_s)-\lambda\big)\,ds + w(x_{\tau_\varepsilon})\Big\}. \end{aligned} \tag{2.29}$$

Hence τ_ε is ε-optimal stopping time.

Now we can return to the proof of Theorem 1. We will use the same method as in the proof of Theorem 1.1. Taking into account the submartingale property (2.18) and the inequalities (2.25) with (2.12), (2.13), (2.15), (2.16), we obtain $u(x)\geqslant\lambda$. Thus the first part of the proof is established.

The proof of points (a), (b), (c) is similar as in Section 1. The only nontrivial one is the fact that the time $\tau_\varepsilon(x^\varepsilon)$, when the process starts from x^ε, is a.e. positive for sufficiently small $\varepsilon>0$. In fact, suppose $w(x^\varepsilon)\geqslant c(x^\varepsilon)-\varepsilon$. Then since $w(x^\varepsilon)+d(x^\varepsilon)\leqslant\varepsilon$, we have $c(x^\varepsilon)+d(x^\varepsilon)\leqslant 2\varepsilon$, which contradicts the strict positiveness of c. So $\tau_\varepsilon(x^\varepsilon)>0$ a.e. because the functions w and c are $\mathscr{C}_0$ continuous.

The next theorem completes Robin's result.

THEOREM 2. *Suppose that E is compact and there exists invariant probability measure m, constants K, $\gamma>0$ such that*

$$|P_x(x_t\in\Gamma)-m(\Gamma)|\leqslant Ke^{-\gamma t} \tag{2.30}$$

for any Borel set Γ.

Let $\bar f = \int_E f(x)m(dx)$. Then one of conditions (2.10) or (2.11) is satisfied. Moreover, if $\lambda<\bar f$ then (2.10) and $E_y\{\tau^*\}<\infty$, $y\in E$; if $\lambda=\bar f$ then (2.11) holds.

Proof. See Theorem 1.2.

So far we considered impulsive control with the Markov times of the form (2.6). Now we will be interested in the general case; the times τ_i will be arbitrary Markov times.

THEOREM 3. *The use of the general impulsive control does not change the optimal value $u(x)$ of the functional (2.4), $u(x)=\lambda$.*

Proof. Let us define first the finite time functional

$$J_x(V,t) = (1/t)E_x^V\Big\{\int_0^t f(y_s)\,ds + \sum_{i=1}^\infty \chi_{\tau_i\leqslant t}[c(x_{\tau_i}^{i-1})+d(y_{\tau_i})]\Big\}. \tag{2.31}$$

Suppose $u(x)$ denotes now the optimal value of the functional (2.4) with respect to arbitrary impulsive controls. Then there exists an ε-optimal strategy V_ε,

$$J_x(V_\varepsilon) \leqslant u(x) + \varepsilon. \tag{2.32}$$

Moreover, from the definition of lim inf, there exists a sequence t_n, $t_n \to \infty$ as $n \to \infty$ such that

$$J_x(V_\varepsilon, t_n) \to J_x(V_\varepsilon). \tag{2.33}$$

The following result will play an important role in the proof of this theorem.

Lemma 7. *For each t there exists the impulsive strategy V_t consisting of the times of the form* (2.6) *such that*

$$J_x(V_t, t) - J_x(V_\varepsilon, t) \leqslant \tfrac{1}{3}\varepsilon. \tag{2.34}$$

Proof. We have to recall the finite time discounted impulsive control problem. The following proposition is a consequence of the results from [8]:

Proposition 2. *Let*

$$u_a(x) = \inf_V J_x^a(V, t), \tag{2.35}$$

where

$$J_x^a(V, t) = (1/t) E_x^V \Big\{ \int_0^t f(y_s) e^{-as} ds + \sum_{i=1}^{\infty} \chi_{\tau_i \leqslant t} e^{-a\tau_i} [c(x_{\tau_i}^{i-1}) + d(y_{\tau_i})] \Big\}; \tag{2.36}$$

then there exists the optimal impulsive strategy of the form (2.6).

Since $J_x^a(V, t) \to J_x(V, t)$ as $a \to 0$, using Proposition 2 we assure that (2.34) is satisfied for a strategy V_t. Taking into account these facts we will construct the strategy $\overline{V}_\varepsilon$, consisting of the stopping times of the form (2.6), such that

$$J_x(\overline{V}_\varepsilon) \leqslant \varepsilon + J_x(V_\varepsilon). \tag{2.37}$$

We choose the subsequence $(t_{n_k})_{k \in N}$ from the sequence $(t_n)_{n \in N}$ successively in the following way

$$t_{n_1} = t_1,$$

(2.38) for given t_{n_i} we take $t_{n_{i+1}}$ such that

$$t_{n_i} \leqslant \tfrac{1}{3} t_{n_{i+1}},$$

$$2\|f\| t_{n_i} + [\|c\| + \|d\|] N_{V_\varepsilon}(0, t_{n_i}) \leqslant \tfrac{1}{3}\varepsilon t_{n_{i+1}},$$

where $N_{V_\varepsilon}(0, t)$ denotes the number of shifts in the time interval $[0, t]$.

Let us denote by $V^{[t_1, t_2[}$ the impulsive strategy restricted to the time interval $[t_1, t_2[$. Next we define the strategy $\overline{V}_\varepsilon$:

$$\overline{V}_\varepsilon = (V_{t_{n_1}}^{[0, t_{n_1}[}, V_{t_{n_2}}^{[t_{n_1}, t_{n_2}[}, \ldots, V_{t_{n_k}}^{[t_{n_{k-1}}, t_{n_k}[}, \ldots). \tag{2.39}$$

This means that in the time interval $[t_{n_i}, t_{n_{i+1}}[$ the impulsive strategy $V_{t_{n_{i+1}}}$ defined by (2.34) is adopted.

It remains to check that $\overline{V}_\varepsilon$ is sufficiently good, indeed. We have

$$\begin{aligned} J_x(\overline{V}_\varepsilon, t_{n_{i+1}}) = {} & \frac{t_{n_i}}{t_{n_{i+1}}} \big(J_x(\overline{V}_\varepsilon, t_{n_i}) - J_x(V_\varepsilon, t_{n_i})\big) + \\ & + \big(J_x(V_{t_{n_{i+1}}}, t_{n_{i+1}}) - J_x(V_\varepsilon, t_{n_{i+1}})\big) + \\ & + \frac{t_{n_i}}{t_{n_{i+1}}} \big(J_x(V_\varepsilon, t_{n_i}) - J_x(V_{t_{n_{i+1}}}, t_{n_i})\big) + J_x(V_\varepsilon, t_{n_{i+1}}). \end{aligned} \tag{2.40}$$

Recalling the definition of V_t and (2.38), we easily obtain

$$J_x(\overline{V}_\varepsilon, t_{n_{i+1}}) \leqslant \varepsilon + J_x(V_\varepsilon, t_{n_{i+1}}). \tag{2.41}$$

Using (2.33) we see that (2.37) is satisfied. Since ε is arbitrary, the optimal value $u(x)$ can be approximated with the use of strategies $\overline{V}_\varepsilon$ of the form (2.6). The proof of theorem is complete.

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References

[1] A. Bensoussan, J. L. Lions, *Nouvelles Methodes en Contrôle Impulsionnel*, Applied Math. and Optimiz. 1 (1975), 289–312.
[2] J. L. Doob, *Stochastic Processes*, Wiley, New York 1967.
[3] E. B. Dynkin, A. Yuskevich, *Controlled Markov processes and their applications*, Nauka, Moskva 1975.
[4] P. Mandl, *Impulsive control with average cost criterion*, Abstracts of 14th European Meeting of Statisticians, Wrocław 1981.
[5] — *On self optimizing control of Markov processes*, to appear in Banach Center Publications, Vol. 13.
[6] P. A. Meyer, *Probabilités et potentiel*, Hermann, Paris 1966.
[7] J. Neveu, *Bases mathématiques du calcul des probabilités*, Masson, Paris 1964.

[8] M. Robin, *Contrôle impusionnel des processus de Markov, thesis,* Université de Paris IX, 1978.

[9] — *On some impulsive control problems with long run average cost,* SIAM J. Control 19 (1981), 333–358.

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