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On a theorem of Lebow and Mlak for several commuting operators

by

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Abstract. A result of Mlak concerning the spectral radius of an operator in a Hilbert space is extended to several commuting operators.

Let H be a complex Hilbert space. Denote by $L(H)$ the Banach algebra of all bounded linear operators in H. For an n-tuple of pairwise commuting operators $T_1, \ldots, T_n$ with the Taylor joint spectrum $\sigma(T_1, \ldots, T_n)$ contained in the open unit ball $B \subset C^n$ ($B = \{z \in C^n, |z| < 1\}$) we denote by

$$M(\xi, T) = I - \sum_{i=1}^{n} \xi_i T_i, \qquad \xi \in \partial B$$

the topological boundary of B.

Note that $M(\xi, T)$ is invertible for every $\xi \in \partial B$ (by the spectral mapping theorem for $\sigma(T_1, \ldots, T_n)$).

The operator-valued function $M(\xi, T)^{-n}$ plays the role of the Fredholm resolvent for the above system $T_1, \ldots, T_n$. In fact, it is easy to prove that for every function f holomorphic in B and continuous in $\bar{B}$ (the closure) we have the equality

$$f(T_1, \ldots, T_n) = \int_{\partial B} M(\xi, T)^{-n} f(\xi)\, \Omega(\xi),$$

where $\Omega(\xi)$ is the $(n-1, n)$ differential form given explicitly by Henkin; see [6] for the definition.

Let us recall some definitions and notations. Denote by $U = \{z \in C, |z| < 1\}$ the open unit disc. For $p \geqslant 1$ and $\alpha \geqslant 0$ let

$$A^{p,\alpha} = \Big\{f,\ f\colon U \to C \text{ is holomorphic and } \int_U |f|^p (1-|z|^2)^\alpha\, dx\, dy < +\infty\Big\}.$$

For $f \in A^{p,\alpha}$ let $\|f\|_{p,\alpha}^p = \int_U |f|^p (1-|z|^2)^\alpha\, dx\, dy$. The space $A^{p,\alpha}$ is called the *Bergman space* and has been investigated in detail by Horowitz [2], [3] Let σ be the surface Lebesgue measure on ∂B. We define the *Hardy space*

$$H^p(B) = \Big\{f,\ f\colon B\to C \text{ is holomorphic and } \sup_{0<r<1} \int_{\partial B} |f(rz)|^p d\sigma(z) < \infty\Big\}.$$

By using the polar coordinates it is easy to check the following

PROPOSITION 1. *We have $g \in H^p(B)$ iff for every $\lambda \in \partial B$ the function $g_\lambda(w) = g(\lambda_1 w, \ldots, \lambda_n w)$, $w \in U$, belongs to $A^{p,n-2}$, $n \geqslant 2$.*

Now let us recall the Hardy inequality. Let $h \in H^1(U)$ and $h(z) = \sum_{k\geqslant 0} c_k z^k$; then $\sum_k |c_k| k^{-1} < +\infty$, see [1]. Now we shall give an analogous inequality for $g \in A^{1,n-2}$.

PROPOSITION 2. *If $g \in A^{1,n-2}$ $(n \geqslant 2)$ and $g(z) = \sum_k a_k z^k$, then $\sum_k |a_k| k^{-n} < +\infty$.*

Proof. Denote $n-2 = a$. Following the proof of the Hardy inequality given in [1] we first assume that $a_k \geqslant 0$, $\forall k$. Now

$$\operatorname{Im} g(re^{i\theta}) = \sum_{m=1}^{\infty} a_m r^m \sin m\theta, \quad 0 \leqslant r < 1.$$

Hence

$$\sum_{m=1}^{\infty} m^{-1} a_m r^m \leqslant (1/2) \int_0^{2\pi} |g(re^{i\theta})|\, d\theta.$$

Multiply both sides of the last inequality by $(1-r^2)^a r$ and integrate over $(0,1)$. Then we have

$$\sum_{m=1}^{\infty} m^{-1} a_m \int_0^1 r^m (1-r^2)^a r\,dr \leqslant (1/2)\|g\|_{1,a}.$$

But

$$\int_0^1 r^m (1-r^2)^a r\,dr \geqslant C_n m^{1-n} \quad \text{for a certain } C_n \text{ and } m = 1, 2, \ldots$$

Thus

$$(*) \qquad C_n \sum_{m=1}^{\infty} a_m m^{-n} \leqslant (1/2)\|g\|_{1,a}.$$

If $g \in A^{1,a}$ is arbitrary, then by Theorem 1 of [3] there exist $g_1, g_2 \in A^{2,a}$ such that $g = g_1 g_2$ and

$$\|g_1\|_{2,a} \|g_2\|_{2,a} \leqslant D_a \|g\|_{1,a},$$

where D_a does not depend on g. Let

$$g_1(z) = \sum_k b_k z^k, \quad g_2(z) = \sum_k c_k z^k.$$

Then

$$G_1 = \sum_k |b_k| z^k, \quad G_2 = \sum_k |c_k| z^k.$$

also belong to $A^{2,a}$ and $\|g_s\|_{2,a} = \|G_s\|_{2,a}$, $s = 1, 2$. Put $H = G_1 G_2$. Then $H \in A^{1,a}$. But

$$H = \sum_k \tilde{a}_k z^k, \quad \tilde{a}_k \geqslant 0 \quad \text{and} \quad |a_k| \leqslant \tilde{a}_k, \quad \forall k.$$

Applying $(*)$ to H we have

$$\sum_k |a_k| k^{-n} \leqslant \sum_k \tilde{a}_k k^{-n} \leqslant C_n \|H\|_{1,a}.$$

Since

$$\|H\|_{1,a} \leqslant \|G_1\|_{2,a} \|G_2\|_{2,a} = \|g_1\|_{2,a} \|g_2\|_{2,a} \leqslant D_a \|g\|_{1,a},$$

and so the proof is complete.

Before we proceed further let us recall the above-mentioned result of Mlak [5]. Let T be an operator in a complex Hilbert space H. Assume that $\big((I-zT)^{-1}x, y\big)$, $z \in U$, belongs to $H^1(U)$ for every $x, y \in H$. Then $r(T) < 1$, where $r(T)$ is the spectral radius of T. Mlak's result extends an earlier theorem of Lebow [4]. Here is a generalization of Mlak's theorem to higher dimensions. Let $T_1, \ldots, T_n$ be a system of commuting operators in H. First of all note that the following extension of the above result is obvious.

If $\big(\prod_{i=1}^{n} (I - z_i T_i)^{-1} x, y\big) \in H^1(U^n)$, then $r(T_i) < 1$, $i = 1, \ldots, n$.

This is immediate by Mlak's theorem. But we also have the following generalization:

PROPOSITION 3. *Let $T_1, \ldots, T_n$ be a commuting system of operators in a complex Hilbert space H, $n \geqslant 2$. Assume that $\big(M(\bar{z}, T)^{-n} x, y\big) \in H^1(B)$ for every $x, y \in H$. Then*

$$\sigma(T_1, \ldots, T_n) \subset B,$$

where $\sigma(T_1, \ldots, T_n)$ denotes the Taylor joint spectrum of $T_1, \ldots, T_n$.

Proof. By Proposition 1 we have

$$\big(M(\bar{z}, T)^{-n}x, y\big) \in H^1(B) \Leftrightarrow \forall \xi \in \partial B,\ \big(M(\xi, wT)^{-n} x, y\big) \in A^{1,n-2}.$$

But it is clear that

$$\sigma(T_1, \ldots, T_n) \subset B \Leftrightarrow \forall \xi \in \partial B,\ r\Big(\sum_i \xi_i T_i\Big) < 1.$$

Fix $\xi \in \sigma B$ and let $T_\xi = \sum_i \xi_i T_i$. We know that $((I-wT_\xi)^{-n}x, y) \in A^{1,n-2}$. Suppose that there exists a $\lambda \in \partial U$ and $\bar{\lambda} \in \sigma(T_\xi)$. Put $S_\xi = \bar{\lambda} T_\xi$. We have $1 \in \sigma(S_\xi)$ and $((I-wS_\xi)^{-n}x, y) \in A^{1,n-2}$. Since

$$((I-wS_\xi)^{-n}x, y) = \Big(\sum_{k=1}^{\infty} \binom{n+k-1}{n-1} (wS_\xi)^k x, y\Big).$$

applying Proposition 2 we can write

$$\Big|\Big(\sum_{k=1}^{s} k^{-n} \binom{n+k-1}{n-1} S_\xi^k x, y\Big)\Big| \leqslant \sum_{k=1}^{\infty} \Big|\Big(k^{-n} \binom{n+k-1}{n-1} S_\xi^k x, y\Big)\Big| \leqslant M_{x,y} < +\infty.$$

Hence the sequence of operators

$$R_s = \sum_{k=1}^{s} k^{-n} \binom{n+k-1}{n-1} S_\xi^k$$

is bounded in norm, i.e., $\|R_s\| < M < +\infty$, $s = 1, 2, \ldots$

Denote by $\mathscr{P}$ the Banach algebra generated by S_ξ and I. Then there exists a $\eta \in \operatorname{Sp}\mathscr{P}$ such that $\eta(S_\xi) = 1$. Hence $|\eta(R_s)| \leqslant \|R_s\| < M$, $s = 1, 2, \ldots$

But on the other hand

$$\eta(R_s) = \sum_{k=1}^{s} k^{-n} \binom{n+k-1}{n-1} \eta(S_\xi)^k = \sum_{k=1}^{s} k^{-n} \binom{n+k-1}{n-1} > \sum_{k=1}^{s} k^{-n} k^{n-1} = \sum_{k=1}^{s} k^{-1}, \quad s = 1, 2, \ldots$$

This contradiction completes the proof.

Remark 1. It is clear that the same result also holds for an n-tuple of commuting operators in a complex Banach space.

Remark 2. Proposition 3 can be extended to more general domains in C^n, as a result of certain integral formulas, but this will be shown in another paper.

Note added in proof. Applying a method of Nikolski we have extended the above result to a more general context (to appear in Bull. Acad. Polon. Sci.).

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