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Note on the strong maximal operator

by

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Abstract. The following conjecture has been stated: Let $f \in L(\log^+ L)^{n-1}$, then f^* is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n$ (f^* denotes the strong maximal function).

We give here a partial answer. See Corollary 6 below.

Introduction and statement of results. Let $f(y_1, y_2)$ be an integrable function with support in the unit cube of R^2 defined by the inequalities $0 \leqslant y_i \leqslant 1$ $(i = 1,2)$. We consider the *partial maximal operators* M_i defined by

$$M_1 f(y_1, y_2) = \sup_{a<y_1<b} \frac{1}{b-a} \int_a^b f(\theta, y_2)\, d\theta,$$

$$M_2 f(y_1, y_2) = \sup_{a<y_2<b} \frac{1}{b-a} \int_a^b f(y_1, \eta)\, d\eta$$

at each point (y_1, y_2) in R^2. We also consider the *strong maximal operator* $f \to f^*$ defined by

$$f^*(y_1, y_2) = \sup_{(y_1, y_2) \in I} \frac{1}{|I|} \int_I f(\theta, \eta)\, d\theta\, d\eta,$$

where the supremum is taken over the set of all intervals I (cells with sides parallel to the axes) containing the point (y_1, y_2). We denote by $L(\log^+ L)^k$ the class of all functions f such that the integral

$$\int |f(\theta, \eta)| \big(\log^+ |f(\theta, \eta)|\big)^k\, d\theta\, d\eta$$

is finite.

The purpose of this work is to show some properties concerning the strong maximal operator and the partial maximal operators.

We start with some definitions of geometric nature related to the strong maximal operator.

Then we prove an inequality involving the strong maximal operator and the partial maximal operators. (See Theorem 1 below.)

Further on we show that the strong maximal operator can be characterized as an average. (See Theorem 3 below.)

We then prove that the local integrability in R^2 of the strong maximal operator cannot be characterized by means of $L(\log^+ L)^2$.

The following conjecture has been stated: Let $f \in L(\log^+ L)^{n-1}$; then f^* is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n$. (See [1]; [2]).

We give here a partial answer. See Corollary 6 below.

Finally, we state a rarity theorem concerning Baire's category, related to the strong maximal operator.

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Section I. All functions considered will be non-negative and supported in the unit cube of R^2 which we shall denote by S.

DEFINITION 1. Let $f \in L^1(R^2)$. A point $(y_1, y_2) \in R^2$ is of *partial K-eccentricity* for this function if there exist intervals $(I_n \times Q_n)_{n \geqslant 1}$ which satisfy

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n \times Q_n} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n$; $y_2 \in Q_n$ $(n \geqslant 1)$ and $1/K \leqslant |I_n|/|Q_n| \leqslant K$ for all $n \geqslant n_0$.

DEFINITION 2. Let $f \in L^1(R^2)$. A point $(y_1, y_2) \in R^2$ is of *semi-absolute eccentricity* for this function if there exist intervals $(I_n \times Q_n)_{n \geqslant 1}$ which satisfy

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n \times Q_n} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n$; $y_2 \in Q_n$ $(n \geqslant 1)$ and

$$|I_n|/|Q_n| \to +\infty \quad \text{or} \quad |Q_n|/|I_n| \to +\infty \quad (n \to \infty).$$

DEFINITION 3. Let $f \in L^1(R^2)$. A point $(y_1, y_2) \in R^2$ is of *absolute eccentricity* for this function if for every sequence of intervals $(I_n \times Q_n)_{n \geqslant 1}$ which satisfies

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n \times Q_n} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n$; $y_2 \in Q_n$ $(n \geqslant 1)$, we have either

$$|I_n|/|Q_n| \to +\infty \quad \text{or} \quad |Q_n|/|I_n| \to +\infty \quad (n \to \infty).$$

DEFINITION 4. Let $f \in L^1(R^2)$. A point $(y_1, y_2) \in R^2$ is of *K-eccentricity* for this function if for every sequence of intervals $(I_n \times Q_n)_{n \geqslant 1}$, which satisfies

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n \times Q_n} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n$; $y_2 \in Q_n$ $(n \geqslant 1)$, we have

$$1/K \leqslant |I_n|/|Q_n| \leqslant K \quad (n \geqslant n_0).$$

Remarks. (i) A point of *absolute eccentricity* is of *semi-absolute eccentricity*, but the converse is obviously not true.

(ii) A point of *K-eccentricity* is of *partial K-eccentricity*.

(iii) A point of *semi-absolute eccentricity* may be of partial K-eccentricity, and conversely. For example: If we define

$$f(y_1, y_2) = \begin{cases} \text{Constant} & \text{if} \quad (y_1, y_2) \in S, \\ 0 & \text{if} \quad (y_1, y_2) \notin S, \end{cases}$$

then every point of S is of *semi-absolute eccentricity* and of partial K-eccentricity for every $K \geqslant 1$.

PROPOSITION 1. *If a point (y_1, y_2) is not of partial K-eccentricity for any $K \geqslant 1$, then it is of absolute eccentricity, and conversely.*

Proof. Let us assume that (y_1, y_2) is not of *absolute eccentricity*; then there exist intervals $(I_n \times Q_n)_{n \geqslant 1}$ which satisfy

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n \times Q_n} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n$; $y_2 \in Q_n$ $(n \geqslant 1)$, and such that

$$\alpha = \lim_n \frac{|I_n|}{|Q_n|} < +\infty; \qquad \beta = \lim_n \frac{|Q_n|}{|I_n|} < +\infty;$$

then $1/\gamma \leqslant |I_n|/|Q_n| \leqslant \gamma$ if $n \geqslant n_0$, where $\gamma > \max(\alpha, \beta) > 0$ $(\gamma \in N)$. Hence (y_1, y_2) is of partial γ-eccentricity. The converse is obvious.

PROPOSITION 2. *If a point (y_1, y_2) is not of K-eccentricity for any $K \geqslant 1$, then it is of semi-absolute eccentricity, and conversely.*

Proof. Let us assume that (y_1, y_2) is not of K-eccentricity for any $K \geqslant 1$. Then, given $K \in N$, there exist intervals $(I_n^k \times Q_n^k)_{n \geqslant 1}$, which satisfy

$$(1) \qquad f^*(y_1, y_2) = \lim_n \frac{1}{|I_n^k|\,|Q_n^k|} \int_{I_n^k \times Q_n^k} f(\theta, \eta)\, d\theta\, d\eta,$$

where $y_1 \in I_n^k$; $y_2 \in Q_n^k$ $(n \geqslant 1)$ and

$$|I_n^k|/|Q_n^k| > K \quad \text{for all } n \geqslant 1 \quad \text{or} \quad |Q_n^k|/|I_n^k| > K \quad \text{for all } n \geqslant 1.$$

Next we choose a subsequence of $(I_n^k \times Q_n^k)_{n\geqslant 1, k\geqslant 1}$, which we call again $(I_n^k \times Q_n^k)_{n\geqslant 1, k\geqslant 1}$ for the sake of clarity. Then (1) holds for the subsequence as $n\to\infty$ for any fixed $K \geqslant 1$, and

$$\frac{|I_n^k|}{|Q_n^k|} \underset{k\to\infty}{\to} +\infty \quad \text{or} \quad \frac{|Q_n^k|}{|I_n^k|} \underset{k\to\infty}{\to} +\infty \quad \text{if} \quad n \geqslant 1.$$

Clearly

$$f^*(y_1, y_2) = \lim_k \lim_n \frac{1}{|I_n^k|\,|Q_n^k|} \int_{I_n^k \times Q_n^k} f(\theta, \eta)\, d\theta\, d\eta.$$

We can therefore conclude that

$$f^*(y_1, y_2) = \lim_J \frac{1}{|I_{n_J}^{k_J}|\,|Q_{n_J}^{k_J}|} \int_{I_{n_J}^{k_J} \times Q_{n_J}^{k_J}} f(\theta, \eta)\, d\theta\, d\eta,$$

and

$$\frac{|I_{n_J}^{k_J}|}{|Q_{n_J}^{k_J}|} \underset{J\to\infty}{\to} +\infty \quad \text{or} \quad \frac{|Q_{n_J}^{k_J}|}{|I_{n_J}^{k_J}|} \underset{J\to\infty}{\to} +\infty$$

for a subsequence $(I_{n_J}^{k_J} \times Q_{n_J}^{k_J})_{J\geqslant 1}$.

This is based upon the fact that if $(a_{k,n})_{k,n\geqslant 1}$ is a double sequence such that $\lim_k \lim_n a_{k,n} = a$, then we can select a subsequence $(a_{k_J,n_J})_{J\geqslant 1}$, verifying $\lim_J a_{k_J,n_J} = a$.

THEOREM 1. *Let $f \in L\log^+ L$. Then $M_i f(y_1, y_2) \leqslant f^*(y_1, y_2)$ $(i = 1, 2)$ for almost any point $(y_1, y_1) \in R^2$.*

Proof. We define

$$T = \Big\{(y_1, y_2)\,|\; M_i f(y_1, y_2) < +\infty \ (i = 1, 2),\ f^*(y_1, y_2) < +\infty,$$

$$f(y_1, y_2) = \lim_{\substack{y_1 \in I \\ \delta I \to 0}} \frac{1}{|I|} \int_I f(\theta, y_2)\, d\theta = \lim_{\substack{y_2 \in Q \\ \delta Q \to 0}} \frac{1}{|Q|} \int_Q f(y_1, \eta)\, d\eta,$$

$$\int_B M_i f(y_1, \eta)\, d\eta < +\infty; \ \int_B M_i f(\theta, y_2)\, d\theta < +\infty \ (i = 1, 2)$$

for every interval B, $\int f(\theta, y_2)\, d\theta < +\infty$; $\int f(y_1, \eta)\, d\eta < +\infty\Big\}$.

(a) *T is measurable*: We take the *upper partial derivatives* of $(\int f)$ defined at each (y_1, y_2) by

$$\overline{D_1}\Big(\int f, (y_1, y_2)\Big) = \sup\Big\{ \overline{\lim_{\substack{k \\ \delta I_k \to 0}}} \frac{1}{|I_k|} \int_{I_k} f(\theta, y_2)\, d\theta \,\Big|\, y_1 \in I_k \Big\},$$

$$\overline{D_2}\Big(\int f, (y_1, y_2)\Big) = \sup\Big\{ \overline{\lim_{\substack{k \\ \delta I_k \to 0}}} \frac{1}{|I_k|} \int_{I_k} f(y_1, \eta)\, d\eta \,\Big|\, y_2 \in I_k \Big\},$$

and the *lower partial derivatives* of $(\int f)$

$$\underline{D_1}\Big(\int f, (y_1, y_2)\Big) = \inf\Big\{ \underline{\lim_{\substack{k \\ \delta I_k \to 0}}} \frac{1}{|I_k|} \int_{I_k} f(\theta, y_2)\, d\theta \,\Big|\, y_1 \in I_k \Big\},$$

$$\underline{D_2}\Big(\int f, (y_1, y_2)\Big) = \inf\Big\{ \underline{\lim_{\substack{k \\ \delta I_k \to 0}}} \frac{1}{|I_k|} \int_{I_k} f(y_1, \eta)\, d\eta \,\Big|\, y_2 \in I_k \Big\},$$

with $(I_k)_{k\geqslant 1}$ any sequence of intervals in R. $\overline{D_i}(\int f)$ and $\underline{D_i}(\int f)$ are measurable $(i = 1, 2)$; then

(1) $$L_i = \Big\{(y_1, y_2)\,\Big|\, \underline{D_i}\Big(\int f, (y_1, y_2)\Big) = \overline{D_i}\Big(\int f, (y_1, y_2)\Big) = f(y_1, y_2)\Big\} \quad (i = 1, 2)$$

are measurable. Let

$$H_n = \Big\{(y_1, y_2)\Big|\, \int_{-n}^{n} M_i f(y_1, \eta)\, d\eta < +\infty,\ \int_{-n}^{n} M_i f(\theta, y_2)\, d\theta < +\infty \ (i = 1, 2)\Big\}.$$

Since $f \in L\log^+ L$, $M_i f$ is locally integrable $(i = 1, 2)$, so that given $n \geqslant 1$, almost all (y_1, y_2) belong to H_n. Then almost all (y_1, y_2) belong to $\bigcap_{n\geqslant 1} H_n$. Since $f \in L\log^+ L$, we also have $f^*(y_1, y_2) < +\infty$ a.e. Hence T is measurable.

(b) *Almost all points $(y_1, y_2) \in T$*: We have only to demonstrate (by (1)) that

$$|S \cap L_i| = |S| \quad (i = 1, 2),$$

for if $(y_1, y_2) \notin S$, then $f(y_1, y_2) = 0$ by hypothesis, and

$$\frac{1}{|I|} \int_I f(\theta, y_2)\, d\theta = 0 = f(y_1, y_2),$$

$$\frac{1}{|Q|} \int_Q f(y_1, \eta)\, d\eta = 0 = f(y_1, y_2)$$

if $\delta(I) < \varepsilon$; $\delta(Q) < \varepsilon$; for $\varepsilon > 0$ sufficiently small. $S \cap L_i$ is measurable; then

$$|L_i \cap S| = \int_0^1 |(L_i \cap S)_{y_1}| dy_1 = \int_0^1 |(L_i \cap S)_{y_2}| dy_2 \quad (i=1,2).$$

For almost all (y_1, y_2)

(2) $$\int f(y_1, \eta) d\eta < +\infty \quad \text{and} \quad \int f(\theta, y_2) d\theta < +\infty.$$

Then if we take (y_1, y_2) for which (2) holds, we get

$$\lim_{\substack{\theta_1 \in I \\ \delta I \to 0}} \frac{1}{|I|} \int_I f(\theta, y_2) d\theta = f(\theta_1, y_2) \text{ a.e.}$$

These are the differentiation points of $f(\ , y_2)$: $R \to R$, so that $|(L_1 \cap S)_{y_2}| = 1$ for nearly all y_2, with $0 \leqslant y_2 \leqslant 1$. Hence $|L_1 \cap S| = |S| = 1$. Similarly,

$$|L_2 \cap S| = |S| = 1.$$

Now let

$$\Omega = \{(y_1, y_2) \in T |\ |(T \cap S)_{y_1}| = |(T \cap S)_{y_2}| = 1 \text{ if } 0 \leqslant y_1, y_2 \leqslant 1\}$$

(c) *Almost all points* $(y_1, y_2) \in \Omega$: Clearly almost all points $(y_1, y_2) \notin S$ belong to Ω. We shall see that $|\Omega \cap S| = 1$. Since

$$\Omega = T \cap \{(y_1, y_2) |\ |(T \cap S)_{y_1}| = |(T \cap S)_{y_2}| = 1 \text{ if } 0 \leqslant y_1, y_2 \leqslant 1\},$$

then

$$\Omega \cap S = (T \cap S) \cap \{(y_1, y_2) |\ |(T \cap S)_{y_1}| = |(T \cap S)_{y_2}| = 1\}$$
$$= (T \cap S) \cap (\{y_1 |\ |(T \cap S)_{y_1}| = 1\} \times \{y_2 |\ |(T \cap S)_{y_2}| = 1\}),$$

and

$$\int_0^1 |(T \cap S)_{y_1}| dy_1 = |T \cap S| = 1.$$

Since $|(T \cap S)_{y_1}| \leqslant 1$ for all y_1, then $|(T \cap S)_{y_1}| = 1$ a.e. Consequently,

$$\left|\{y_1 |\ |(T \cap S)_{y_1}| = 1\}\right| = 1.$$

Similarly, $\left|\{y_2 |\ |(T \cap S)_{y_2}| = 1\}\right| = 1$, hence $|\Omega \cap S| = 1$, as wanted. Let $(y_1, y_2) \in \Omega$. Then $M_2 f(y_1, y_2) < +\infty$. Given $\varepsilon > 0$, there exists (a, b) with $a < y_2 < b$ such that

(3) $$M_2 f(y_1, y_2) < \frac{1}{b-a} \int_a^b f(y_1, \eta) d\eta + \varepsilon,$$

(a, b) depending on (y_1, y_2) and ε. We have

$$f(y_1, \eta) = \lim_{\substack{\delta I \to 0 \\ y_1 \in I}} \frac{1}{|I|} \int_I f(\theta, \eta) d\theta$$

for almost all η since $(y_1, y_2) \in \Omega$. Now given

$$g_n(y_1, \eta) = \frac{1}{|I_n|} \int_{I_n} f(\theta, \eta) d\theta, \quad y_1 \in I_n;\ n \geqslant 1,$$

where $\delta I_n \to 0$, obviously $g_n(y_1, \eta) \leqslant M_1 f(y_1, \eta)$. Since $(y_1, y_2) \in \Omega$, $M_1 f(y_1, \cdot)$ is locally integrable, then by dominated convergence we have

$$\frac{1}{b-a} \int_a^b f(y_1, \eta) d\eta = \frac{1}{b-a} \int_a^b \lim_n g_n(y_1, \eta) d\eta$$
$$= \lim_n \frac{1}{b-a} \int_a^b g_n(y_1, \eta) d\eta$$
$$= \lim_n \frac{1}{b-a} \frac{1}{|I_n|} \int_a^b \Big(\int_{I_n} f(\theta, \eta) d\theta \Big) d\eta \leqslant f^*(y_1, y_2).$$

Hence, by (3),

$$M_2 f(y_1, y_2) \leqslant f^*(y_1, y_2) + \varepsilon \quad (\varepsilon > 0 \text{ arbitrary})$$

for every $(y_1, y_2) \in \Omega$. Similarly,

$$M_1 f(y_1, y_2) \leqslant f^*(y_1, y_2)$$

for every $(y_1, y_2) \in \Omega$.

COROLLARY 1. *Let* $f \in L^1(R^2)$; *then*

$$M_i f(y_1, y_2) \leqslant f^*(y_1, y_2) \ \text{a.e.} \quad (i = 1, 2).$$

Proof. We take $f_n \xrightarrow{L^1} f$ such that $f_n \in L\log^+ L$ for all $n \geqslant 1$. Let $\varphi_n = \inf_{k \geqslant n} f_k$. Therefore $\varphi_n \nearrow f$ and $\varphi_n \in L\log^+ L$ $(n \geqslant 1)$. Hence,

$$M_i f(y_1, y_2) = \lim_n M_i \varphi_n(y_1, y_2) \leqslant \lim_n \varphi_n^*(y_1, y_2) \leqslant f^*(y_1, y_2) \text{ a.e.}$$

DEFINITION 5. We define the product operators $M_1 M_2$ and $M_2 M_1$ by

$$M_1 M_2 f(y_1, y_2) = \sup_{y_1 \in I} \frac{1}{|I|} \int_I \Big(\sup_{y_2 \in Q} \frac{1}{|Q|} \int_Q f(\theta, \eta) d\eta \Big) d\theta,$$

$$M_2 M_1 f(y_1; y_2) = \sup_{y_2 \in Q} \frac{1}{|Q|} \int_Q \Big(\sup_{y_1 \in I} \frac{1}{|I|} \int_I f(\theta, \eta) d\eta \Big) d\theta.$$

Remarks. (i) $f^* \leqslant M_1M_2f$ and $f^* \leqslant M_2M_1f$.

(ii) If $f \in L\log^+L$, then M_2M_1f and M_1M_2f are well defined and finite almost everywhere. See [2].

THEOREM 2. *Let $f \in L\log^+L$. Let (y_1, y_2) be a point of semi-absolute eccentricity for this function with $(y_1, y_2) \in \Omega$; and such that (y_1, y_2) is a differentiation point, then*

$$f^*(y_1, y_2) \leqslant M_1f(y_1, y_2)$$

or

$$f^*(y_1, y_2) \leqslant M_2f(y_1, y_2).$$

Proof. There exists $(I_n \times Q_n)_{n\geqslant 1}$ such that

$$f^*(y_1, y_2) = \lim_n \frac{1}{|I_n|\,|Q_n|} \int_{I_n\times Q_n} f(\theta, \eta)\,d\theta\,d\eta .$$

Without loss of generality let us assume

$$\frac{|I_n|}{|Q_n|} \to +\infty .$$

As $0 \leqslant |I_n|$; $|Q_n| \leqslant$ constant, there exist subsequences $|I_n|$; $|Q_n|$ $(n \geqslant 1)$ with $|I_n| \to c$ and $|Q_n| \to 0$. Given $\varepsilon > 0$ since $f^*(y_1, y_2) < +\infty$ if $(y_1, y_2) \in \Omega$ then

$$f^*(y_1, y_2) < \frac{1}{|I_n|\,|Q_n|} \int_{I_n\times Q_n} f(\theta, \eta)\,d\theta\,d\eta + \varepsilon/2 \tag{1}$$

if $n \geqslant n_0$. On the other hand,

$$\frac{1}{|Q_n|} \int_{Q_n} f(\theta,\eta)\,d\eta \to f(\theta, y_2)$$

for almost all θ since $|Q_n| \to 0$ and $(y_1, y_2) \in \Omega$. Then

$$f^*(y_1, y_2) \leqslant \lim \frac{1}{|I_n|} \int_{I_n} [f(\theta, y_2)\,\theta]\,d\theta + \varepsilon$$

since the inequality

$$\lim \frac{1}{|I_n|} \int_I [f(\theta, y_2)\,\theta]\,d\theta + \varepsilon < f^*(y_1, y_2),$$

together with (1), would give

$$0 = \lim \frac{1}{|I_n|} \int_{I_n} \left(\frac{1}{|Q_n|} \int_{Q_n} f(\theta, \eta)\,d\eta \right) d\theta - \lim \frac{1}{|I_n|} \int_{I_n} f(\theta, y_2)\,d\theta$$

$$> \big(\varepsilon - f^*(y_1, y_2)\big) + \big(f^*(y_1, y_2) - \varepsilon/2\big) = \varepsilon/2$$

if $|I_n| \to 0$ because (y_1, y_2) is a differentiation point which belongs to Ω. If $|I_n| \to |I| = c > 0$, we have

$$\varepsilon/2 < \lim \frac{1}{|I_n|} \int_{I_n} \left[\frac{1}{|Q_n|} \int_{Q_n} f(\theta, \eta)\,d\eta - f(\theta, y_2) \right] d\theta$$

$$= \frac{1}{|I|} \int_I \left[\lim \frac{1}{|Q_n|} \int_{Q_n} f(\theta, \eta)\,d\eta - f(\theta, y_2) \right] d\theta = 0,$$

In both cases we have an absurd. Then

$$f^*(y_1, y_2) \leqslant \lim \frac{1}{|I_n|} \int_{I_n} [f(\theta, y_2)]\,d\theta + \varepsilon \leqslant M_1f(y_1, y_2) + \varepsilon .$$

COROLLARY 2. *Let $f \in L\log^+L$. If almost all points $(y_1, y_2) \in S$ are of semi-absolute eccentricity for this function, then*

$$f^*(y_1, y_2) = \max\big(M_1f(y_1, y_2);\ M_2f(y_1, y_2)\big) \quad a.e. \text{ in } S.$$

THEOREM 3. *Let $f \in L\log^+L$. Let $(y_1, y_2) \in \Omega$ be a differentiation point for this function. Then one of the following statements is valid:*

$$f^*(y_1, y_2) = \frac{1}{|Q|} \int_Q f(\theta, \eta)\,d\theta\,d\eta; \quad (y_1, y_2) \in Q, \tag{i}$$

$$f^*(y_1, y_2) = \frac{1}{|I|} \int_I f(\theta, y_2)\,d\theta; \quad y_1 \in I, \tag{ii}$$

$$f^*(y_1, y_2) = \frac{1}{|H|} \int_H f(y_1, \eta)\,d\eta; \quad y_2 \in H, \tag{iii}$$

$$f^*(y_1, y_2) = f(y_1, y_2), \tag{iv}$$

where Q, I and H are intervals.

Let us consider $\Psi_{(y_1,y_2)}\colon F \to R$ defined in the following manner:

$$F = \{(a, b, c, d)|\ a \leqslant y_1 \leqslant b;\ c \leqslant y_2 \leqslant d\},$$

$$\Psi_{(y_1,y_2)}(a,b,c,d) = \begin{cases} \dfrac{1}{b-a}\displaystyle\int_a^b \left(\dfrac{1}{d-c}\int_c^d f(\theta,\eta)\,d\eta\right)d\theta & \text{if } a \neq b,\ c \neq d, \\ \dfrac{1}{b-a}\displaystyle\int_a^b f(\theta, y_2)\,d\theta & \text{if } a \neq b,\ c = d, \\ \dfrac{1}{d-c}\displaystyle\int_c^d f(y_1,\eta)\,d\eta & \text{if } a = b,\ c \neq d, \\ f(y_1,y_2) & \text{if } a = b,\ c = d. \end{cases}$$

$\Psi_{(y_1,y_2)}$ *is continuous*:

Case I. $a \neq b$; $c \neq d$. Let us consider $(a_n, b_n, c_n, d_n) \to (a, b, c, d)$; then $b_n - a_n > 0$; $d_n - c_n > 0$ if $n \geqslant n_0$ and

$$\frac{1}{b_n-a_n}\,\frac{1}{d_n-c_n}\int_{a_n}^{b_n}\left(\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta\right)d\theta \to \frac{1}{b-a}\,\frac{1}{d-c}\int_a^b\left(\int_c^d f(\theta,\eta)\,d\eta\right)d\theta.$$

Case II. $a \neq b$; $c = d$. We take $(a_n, b_n, c_n, d_n) \to (a, b, c, d)$, $b_n - a_n > 0$; $0 \leqslant d_n - c_n$ if $n \geqslant n_0$. Suppose $0 < d_n - c_n$ if $n \geqslant n_0$; then

$$(1) \qquad \lim_n \frac{1}{b_n-a_n}\int_{a_n}^{b_n}\left(\frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta\right)d\theta$$

$$= \lim_n \frac{1}{b_n-a_n}\cdot\lim_n \int_{a_n}^{b_n}\left(\frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta\right)d\theta.$$

Since

$$\frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta \leqslant M_2 f(\theta, y_2)$$

(which is locally integrable because $(y_1, y_2) \in \Omega$), (1) can be written as

$$\frac{1}{b-a}\int \lim_n \chi_{[a_n,b_n]}(\theta)\cdot\lim_n\left(\frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta\right)d\theta = \frac{1}{b-a}\int_a^b f(\theta,y_2)\,d\theta$$

because

$$\lim_n \frac{1}{d_n-c_n}\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta = f(\theta, y_2) \text{ a.e.}$$

(This being true because $(y_1, y_2) \in \Omega$.) If there exists a subsequence $(c_{n_k}, d_{n_k})_{k\geqslant 1}$ such that $c_{n_k} = d_{n_k}$, $k \geqslant 1$, obviously,

$$\frac{1}{b_{n_k}-a_{n_k}}\int_{a_{n_k}}^{b_{n_k}} f(\theta,y_2)\,d\theta \underset{k\to\infty}{\to} \frac{1}{b-a}\int_a^b f(\theta,y_2)\,d\theta.$$

Case III. $a = b$; $c \neq d$; analogous to *Case* II.

Case IV. $a = b$; $c = d$. Since (y_1, y_2) is a differentiation point,

$$\lim_n \frac{1}{b_n-a_n}\,\frac{1}{d_n-c_n}\int_{a_n}^{b_n}\left(\int_{c_n}^{d_n} f(\theta,\eta)\,d\eta\right)d\theta = f(y_1,y_2)$$

if $(a_n, b_n, c_n, d_n) \to (a, b, c, d)$, $b_n - a_n > 0$, $d_n - c_n > 0$ $(n \geqslant n_0)$. Moreover, if there exists a subsequence $(a_{n_k}, b_{n_k})_{k\geqslant 1}$ such that $a_{n_k} = b_{n_k}$, $d_{n_k} - c_{n_k} > 0$ $(k \geqslant 1)$, then

$$\frac{1}{d_{n_k}-c_{n_k}}\int_{c_{n_k}}^{d_{n_k}} f(y_1,\eta)\,d\eta \to f(y_1,y_2)$$

since $(y_1, y_2) \in \Omega$. Hence $\Psi_{(y_1,y_2)}$ is continuous.

Now suppose without loss of generality that $(y_1, y_2) \in S$; then

$$\sup_{(a,b,c,d)\in F} \Psi_{(y_1,y_2)}(a,b,c,d) = \sup_{(a,b,c,d)\in(S\times S)\cap F} \Psi_{(y_1,y_2)}(a,b,c,d),$$

where $S \times S = \{(a,b,c,d) \mid 0 \leqslant a, b, c, d \leqslant 1\}$. This is clear since f is supported in S, which means that the average

$$\frac{1}{b-a}\,\frac{1}{d-c}\int_a^b\left(\int_c^d f(\theta,\eta)\,d\eta\right)d\theta$$

is the greatest if we select $(a, b) \times (c, d) \subseteq S$, that is, $(a, b, c, d) \in S \times S$. Similarly, for any intervals (a, b), (c, d), the averages

$$\text{(A)} \qquad \frac{1}{b-a}\int_a^b f(\theta,y_2)\,d\theta;$$

$$\text{(B)} \qquad \frac{1}{d-c}\int_c^d f(y_1,\eta)\,d\eta$$

are the greatest if $(a, b), (c, d) \subseteq [0, 1]$, that is if we take in case (A) $0 \leqslant c = y_2 = d \leqslant 1$ and in case (B) $0 \leqslant a = y_1 = b \leqslant 1$. Hence there exists $(\bar a, \bar b, \bar c, \bar d) \in F$ such that

$$\sup_{(a,b,c,d)\in F} \Psi_{(y_1,y_2)}(a,b,c,d) = \Psi_{(y_1,y_2)}(\bar a, \bar b, \bar c, \bar d).$$

DEFINITION 6. Let $f \in L^1(R^2)$. We define the *wide partial* upper derivatives of $(\int f)$ in (y_1, y_2) by:

$$\overline{D^1_w}\left(\int f, (y_1, y_2)\right) = \sup\left\{\overline{\lim_{\substack{(y_1,y_2)\in I_n\\ |I^\theta_n|\to 0}}} \frac{1}{|I_n|}\int_{I_n} f(\theta, \eta)\,d\theta\,d\eta\right\},$$

$$\overline{D^2_w}\left(\int f, (y_1, y_2)\right) = \sup\left\{\overline{\lim_{\substack{(y_1,y_2)\in I_n\\ |I^\eta_n|\to 0}}} \frac{1}{|I_n|}\int_{I_n} f(\theta, \eta)\,d\theta\,d\eta\right\},$$

where

$$|I^\theta_n| = \text{height of the interval } I_n,$$

$$|I^\eta_n| = \text{measure of the base of } I_n.$$

COROLLARY 3. *Let $f \in L\log^+ L$; then*

$$\overline{D^1_w}\left(\int f, (y_1, y_2)\right) = M_1 f(y_1, y_2) \text{ a.e.}$$

$$\overline{D^2_w}\left(\int f, (y_1, y_2)\right) = M_2 f(y_1, y_2) \text{ a.e.}$$

Proof. For any sequence of intervals $(I_n)_{n\geqslant 1}$ such that $|I^\eta_n|\to 0$ with $(y_1, y_2) \in I_n$ $(n \geqslant 1)$, $(y_1, y_2)\in \Omega$ being a differentiation point, we have

$$(1)\quad \overline{\lim_{|I^\eta_n|\to 0}} \frac{1}{|I_n|}\int_{I_n} f(\theta,\eta)\,d\theta\,d\eta$$

$$= \lim_{|I^\eta_{n_k}|\to 0} \frac{1}{|I_{n_k}|}\int_{I_{n_k}} f(\theta,\eta)\,d\theta\,d\eta$$

$$= \lim_{|I^\eta_{n_k}|\to 0} \frac{1}{d_{n_k}-c_{n_k}}\int_{c_{n_k}}^{d_{n_k}} \left(\frac{1}{|I^\eta_{n_k}|}\int_{I^\eta_{n_k}} f(\theta,\eta)\,d\theta\right) d\eta$$

$$= \frac{1}{d-c}\int_c^d f(y_1,\eta)\,d\eta \quad \text{by Theorem 3; case II}$$

if $|I^\theta_{n_k}| = d_{n_k} - c_{n_k} \to d - c > 0$, and (1) is equal to $f(y_1, y_2)$ if $|I^\theta_{n_k}|\to 0$ (since (y_1, y_2) is a differentiation point). Hence

$$\overline{D^2_w}\left(\int f, (y_1, y_2)\right) \leqslant M_2 f(y_1, y_2) \text{ a.e.}$$

Now let

$$M_2 f(y_1, y_2) < \frac{1}{d-c}\int_c^d f(y_1,\eta)\,d\eta + \varepsilon,$$

where $c < y_2 < d$, $(M_2 f(y_1, y_2) < +\infty$ since $(y_1, y_2) \in \Omega)$ (c, d) depending on (y_1, y_2) and ε.

Now choose $I_n = H_n \times Q_n$ $(n \geqslant 1)$ such that

$$|Q_n| = |I^\theta_n| \to d - c \quad\text{and}\quad |H_n| = |I^\eta_n| \to 0.$$

Again by Theorem 3; Case II,

$$\overline{\lim_{|I^\eta_n|\to 0}} \frac{1}{|I_n|}\int_{I_n} f(\theta,\eta)\,d\theta\,d\eta = \frac{1}{d-c}\int_c^d f(y_1,\eta)\,d\eta,$$

hence $\overline{D^2_w}(\int f, (y_1, y_2)) + \varepsilon > M_2 f(y_1, y_2)$ $(\varepsilon > 0$ arbitrary) if $(y_1, y_2)\in\Omega$. This implies

$$\overline{D^2_w}\left(\int f, (y_1, y_2)\right) \geqslant M_2 f(y_1, y_2) \text{ a.e.}$$

PROPOSITION 3. *There exists a function $f \in L\log^+ L$ and some interval $I \subseteq S^0$, being S^0 the interior of S, such that $\chi_I f \notin L(\log^+ L)^2$ and $M_1 f(y_1, y_2) = M_1 M_2 f(y_1, y_2)$ for all $(y_1, y_2) \in S$.*

Proof. Let us select $g\colon R \to R$ such that

$$g \equiv 0 \text{ in } \mathscr{C}[0,1]; \qquad g \neq 0 \text{ in } [0,1]; \qquad g \in L\log^+ L$$

and $\chi_{(\alpha\beta)}\cdot g \notin L(\log^+ L)^2$ being $0 < \alpha < \beta < 1$.

Let $f\colon R^2 \to R$ be defined by

$$f(y_1, y_2) = \begin{cases} g(y_1) & \text{if } S_{y_2} \neq \emptyset,\\ 0 & \text{if } S_{y_2} = \emptyset;\end{cases}$$

then f is supported in the unit cube S. Let $(y_1, y_2)\in S^0$

$$M_1 M_2 f(y_1, y_2) = \sup_{a<y_1<b} \frac{1}{b-a}\int_a^b \left(\sup_{c<y_2<d}\int_c^d \frac{f(\theta,\eta)}{d-c}\,d\eta\right) d\theta$$

$$= \sup_{a<y_1<b}\frac{1}{b-a}\int_a^b f(\theta, y_2)\,d\theta = M_1 f(y_1, y_2),$$

because if $c \leqslant \eta \leqslant d$, then $f(\theta,\eta) = f(\theta, y_2) = g(\theta)$, and the average $\frac{1}{d-c}\int_c^d f(\theta,\eta)\,d\eta$ is the greatest if $(c, d) \subset [0, 1]$. Obviously, $f \in L\log^+ L$ and $\chi_I f \notin L(\log^+ L)^2$ for some interval I, $I \subseteq S^0$.

COROLLARY 4. *There exists $f \in L\log^+L$ such that $\chi_I f \notin L(\log^+L)^2$ ($I \subseteq S^0$ an interval) satisfying*

$$\int_S M_1 f = \int_S f^* = \int_S M_1 M_2 f < +\infty.$$

Proof. The result follows from Theorem 1 and Proposition 3. The last corollary shows us that the integrability over *every measurable set* is an essential condition for the characterization of the product operator M_1M_2 by means of $L(\log^+L)^2$. See [1].

THEOREM 4. *Let $f \in L^1(R^2)$ such that $\int_S f^* < +\infty$; then $(\chi_I f)^*$ is integrable over every bounded set for any interval $I \subseteq S^0$.*

Proof. Let

$$A = \Big\{(y_1, y_2) \in S \mid \int_0^1 f^*(y_1, \eta)\,d\eta < +\infty,$$
$$\int_0^1 f^*(\theta, y_2)\,d\theta < +\infty \text{ and } f^*(y_1, y_2) < +\infty\Big\};$$

then $|A| = |S| = 1$.

Now we define

$$B = \{(y_1, y_2) \in A \mid |A_{y_1}| = |A_{y_2}| = 1\}$$

and

$$C = \{(y_1, y_2) \in B \mid |B_{y_1}| = |B_{y_2}| = 1\},$$

so that $|B| = |C| = |S| = 1$.

Given $\delta_0 > 0$, we choose $(p_1, q_1) \in C$ such that

$$\|(p_1, q_1) - (0, 0)\| = \|(p_1, q_1)\| < \delta_0.$$

By definition, $|B_{q_1}| = |B_{p_1}| = 1$. Since $|B_{q_1}| = 1$, we can select (p_2, q_1), satisfying

$$\|(p_2, q_1) - (1, 0)\| < \delta_0 \quad \text{and} \quad (p_2, q_1) \in B.$$

Since $|B_{p_1}| = |A_{p_2}| = 1$, we take $q_2 \in B_{p_1} \cap A_{p_2}$ such that

$$\|(p_1, q_2) - (0, 1)\| < \delta_0; \quad \|(p_2, q_2) - (1, 1)\| < \delta_0.$$

Then $(p_1, q_1) \in C$; (p_2, q_1) and $(p_1, q_2) \in B$; $(p_2, q_2) \in A$. Consequently, there exists an interval $I \subseteq S^0$ with vertex (p_i, q_J) ($i = 1, 2$; $J = 1, 2$) arbitrarily near from the vertex of the unit cube S such that $(p_i, q_J) \in A$ ($i = 1, 2$; $J = 1, 2$).

Next let us consider the sets $(Q_i)_{1 \le i \le 8}$ that are shown in Fig. 2, $R^2 = \bigcup_{i=1}^{8} Q_i \cup S$.

Let $g = \chi_I f$, let I be the interval we have constructed, f the function of the hypothesis.

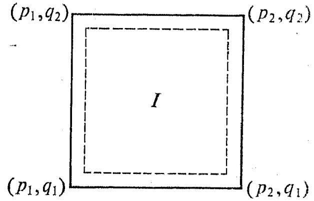

Fig. 1

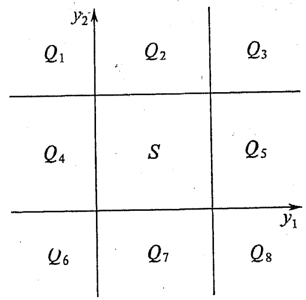

Fig. 2

Given $\varepsilon > 0$, if $g^*(y_1, y_2) < +\infty$ ($g^* < +\infty$ a.e.) and $(y_1, y_2) \in Q_8$, there exists an interval H for which $(y_1, y_2) \in H$ and

$$g^*(y_1, y_2) < \frac{1}{|H|} \int_H g(\theta, \eta)\,d\theta\,d\eta + \varepsilon$$
$$= \frac{1}{|H|} \int_{H \cap I} g(\theta, \eta)\,d\theta\,d\eta + \varepsilon \le \frac{1}{|H \cap S|} \int_{H \cap I} g(\theta, \eta)\,d\theta\,d\eta + \varepsilon$$
$$\le \frac{1}{|H \cap S|} \int_{H \cap S} f(\theta, \eta)\,d\theta\,d\eta + \varepsilon \le f^*(p_2, q_1) + \varepsilon,$$

for if H is an interval such that $|H\cap I|>0$ and $(y_1, y_2)\in H$, with $(y_1, y_2)\in Q_8$, then

$$(p_2, q_1)\in H\cap S.$$

(See Fig. 1; Fig. 2.) Hence $g^*(y_1, y_2)\leqslant f^*(p_2, q_1)$ a.e. in Q_8. Now let D be a bounded set.

$$\int_{D\cap Q_8} g^*\leqslant f^*(p_2, q_1)|D\cap Q_8|<+\infty$$

since $(p_2, q_1)\in A$. The procedure is analogous with $D\cap Q_6$; $D\cap Q_1$; $D\cap Q_3$. Let $(y_1, y_2)\in Q_5$ and $g^*(y_1, y_2)<+\infty$. Given $\varepsilon>0$, there exists H for which $(y_1, y_2)\in H$ and

$$g^*(y_1, y_2)<\frac{1}{|H|}\int_H g(\theta, \eta)\,d\theta\, d\eta+\varepsilon$$

$$\leqslant\frac{1}{|H\cap S|}\int_{H\cap S} f(\theta, \eta)\,d\theta\, d\eta+\varepsilon\leqslant f^*(p_2, y_2)+\varepsilon.$$

Since $(p_2, y_2)\in H\cap S$, $g^*(y_1, y_2)\leqslant f^*(p_2, y_2)$. Hence

$$\int_{D\cap Q_5} g^*(y_1, y_2)\,dy_1dy_2=\int_0^1\Big(\int_{(D\cap Q_5)y_2} g^*(y_1, y_2)\,dy_1\Big)\,dy_2$$

$$\leqslant\int_0^1\Big(\int_{(D\cap Q_5)y_2} f^*(p_2, y_2)\,dy_1\Big)\,dy_2$$

$$=\int_0^1 |(D\cap Q_5)_{y_2}|f^*(p_2, y_2)\,dy_2,$$

$$\leqslant M\int_0^1 f^*(p_2, y_2)\,dy_2<+\infty$$

since (p_2, q_1), $(p_2, q_2)\in A$ and since D is bounded, $|(D\cap Q_5)_{y_2}|\leqslant M$.

The procedure is analogous for $Q_4\cap D$; $Q_7\cap D$; $Q_2\cap D$. Hence $\int_D f^*<+\infty$.

COROLLARY 5. *There exists $f\in L\log^+L$ such that $f\notin L(\log^+L)^2$, and such that f^* is locally integrable.*

Proof. Follows immediately from the previous theorem.

THEOREM 5. *Let $f\in L\log^+L$, and let f^* be integrable over $\{f^*>1\}$ if $\varphi(x)=x(\log^+x)^2$; then*

$$\overline{\lim_N}\frac{1}{N^2}\int_{\{g_N^i\geqslant f-K\}\cap S}\varphi(f/2)\,d\theta\, d\eta<+\infty\qquad (i=1,2),$$

where

$$g_N^1(\theta, \eta)=\sum_{k=1}^N\chi_{I_k}(\eta)\frac{1}{|I_k|}\int_{I_k} f(\theta, \eta)\,d\eta,$$

$$g_N^2(\theta, \eta)=\sum_{k=1}^N\chi_{I_k}(\theta)\frac{1}{|I_k|}\int_{I_k} f(\theta, \eta)\,d\theta,$$

I_k being disjoint intervals in R such that $\bigcup_{k=1}^N I_k=[0,1]$; $|I_k|=1/N$, and $K\colon R\to R$ a measurable function such that $\int_0^1\varphi(K)\,dx<+\infty$ and $K\geqslant 0$.

Proof. Let $A=\{(y_1, y_2)\mid f^*(y_1, y_2)>a\}$. A_{y_1} is open and $A_{y_1}\neq R$ for almost all y_1 since A is a set of finite measure. Then $A_{y_1}=\bigcup_{J\geqslant 1} Q_J$, where the sets Q_J are half-open cubic intervals, disjoint and such that for each J one has

$$1\leqslant\frac{d(Q_j, \partial A_{y_1})}{\delta(Q_J)}<3$$

(Whitney's covering theorem). See [3], p. 10. Then there exists an expansion Q_J^* of Q_J with center at the center of Q_J such that $|Q_J^*|=C_2|Q_J|$. It is clear that, choosing conveniently the constant C_2 that depends only on the dimension, we have

$$Q_J^*\cap A_{y_1}^c\neq\emptyset.$$

So if $a<y_1<b$, we have

$$\frac{1}{|Q_J^*|}\int_{Q_J^*}\Big(\frac{1}{b-a}\int_a^b f(\theta, \eta)\,d\theta\Big)\,d\eta\leqslant a.$$

Hence

$$|A_{y_1}|=\sum_{J\geqslant 1}|Q_J|\geqslant\frac{1}{a\cdot c_2}\sum_{J\geqslant 1}\int_{Q_J^*}\frac{1}{b-a}\int_a^b f$$

$$\geqslant\frac{1}{a\cdot c_2}\int_{\bigcup_{J\geqslant 1} Q_J}\frac{1}{b-a}\int_a^b f=\frac{1}{a\cdot c_2}\int_{A_{y_1}}\frac{1}{b-a}\int_a^b f$$

so that

$$(1)\qquad |\{y_2\mid f^*(y_1, y_2)>a\}|\geqslant(c/a)\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\int_{(f^*>a)_{y_1}} f,$$

c being a constant that depends only on the dimension. By (1) and the hypothesis we have

$$+\infty > \int_{\{f^*>1\}} f^* > \int_1^\infty |f^*>a|\,da = \int_R \Big(\int_1^\infty |(f^*>a)_{y_1}|\,da\Big)\, dy_1$$

$$\geqslant C\int_R \Big(\int_1^\infty \Big(\frac{1}{a}\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\Big(\int_{(f^*>a)_{y_1}} f(\theta,\eta)\,d\eta\Big)d\theta\Big)\Big)\,da\Big)\,dy_1$$

$$\geqslant C\int_R\Big(\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\int_R\Big(\int_1^\infty f(\theta,\eta)\cdot\chi_{(f^*>a)_{y_1}}(\eta)\,\frac{1}{a}\,da\Big)\,d\eta\Big)\,d\theta\Big)\,dy_1$$

$$= C\int_R\Big(\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\int_R f(\theta,\eta)\,\mathrm{Log}^+ f^*(y_1,\eta)\,d\eta\Big)\,d\theta\Big)\,dy_1.$$

Now we define for $0\leqslant\eta\leqslant1$; $0\leqslant\theta\leqslant1$

$$g_N^1(\theta,\eta)=g_N(\theta,\eta)=\sum_{k=1}^N \chi_{I_k}(\eta)\,\frac{1}{|I_k|}\int_{I_k} f(\theta,y_2)\,dy_2=\sum_{k=1}^N\chi_{I_k}(\eta)\cdot h_k(\theta),$$

$|I_k|=1/N$ $(k=1,\dots N,)$, I_k being disjoint intervals in R such that $\bigcup_{k=1} I_k = [0,1]$. Clearly

$$f^*(y_1,\eta)\geqslant M_1 g_N(y_1,\eta)=\sum_{k=1}^N \chi_{I_k}(\eta)\, Mh_k(y_1)$$

since

$$M_1 g_N(y_1,\eta)=\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\sum_{k=1}^N\chi_{I_k}(\eta)\,\frac{1}{|I_k|}\int_{I_k}f(\theta,y_2)\,dy_2\Big)\,d\theta$$

$$=\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\frac{1}{|I_J|}\int_{I_J}f(\theta,y_2)\,dy_2\Big)\,d\theta = Mh_J(y_1)$$

for $\eta\in I_J$. Therefore

$$C\int_R\Big(\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\int_R f(\theta,\eta)\log^+ f^*(y_1,\eta)\,d\eta\Big)\,d\theta\Big)\,dy_1$$

$$\geqslant C\int_0^1\Big(\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\int_0^1 f(\theta,\eta)\log^+ M_1g_N(y_1,\eta)\,d\eta\Big)\,d\theta\Big)\,dy_1$$

$$= C\int_0^1\Big(\sup_{a<y_1<b}\frac{1}{b-a}\int_a^b\Big(\sum_{k=1}^N\log^+Mh_k(y_1)\int_{I_k}f(\theta,\eta)\,d\eta\Big)\,d\theta\Big)\,dy_1$$

$$= C\int_0^1\Big(\sup_{a<y_1<b}\frac{1}{b-a}\,\frac{1}{N}\int_a^b\Big(\sum_{k=1}^N\log^+Mh_k(y_1)\cdot\Big(\frac{1}{|I_k|}\int_{I_k}f(\theta,\eta)\,d\eta\Big)\Big)\,d\theta\Big)\,dy_1$$

$$= C\int_0^1\frac{1}{N}\,M\Big(\sum_{k=1}^N\big(\log^+Mh_k(y_1)\cdot h_k\big)\Big)(y_1)\,dy_1$$

$$\geqslant \frac{C}{2N^2}\int_0^1\sum_{k=1}^N h_k(y_1)\big(\log^+h_k(y_1)\big)^2\,dy_1.$$

Since

$$\frac{1}{N}\,M\big(\log^+Mh_1(y_1)\cdot h_1\big)+\dots+\frac{1}{N}\,M\big(\log^+Mh_N\cdot h_N\big)$$

$$\leqslant M\Big(\sum_{k=1}^N \mathrm{Log}^+Mh_k(y_1)\cdot h_k\Big)$$

and

$$\int_0^1\log^+Mh_k\cdot Mh_k\,dy_1=\int_1^\infty(1+\log^+a)\,|Mh_k>a|\,da$$

$$\geqslant\int_0^\infty\Big(\frac{1+\log^+a}{a}\int_{(h_k>a)}h_k(y_1)\,dy_1\Big)\,da \qquad (\text{see [1]})$$

$$\geqslant\int_0^1\Big(\int_1^{h_k}\frac{\log^+a}{a}\,da\Big)\,h_k\,dy_1=\tfrac{1}{2}\int_0^1 h_k(y_1)\big(\log^+h_k(y_1)\big)^2\,dy_1.$$

Consequently, if $K\colon R\to R$ is as in the hypothesis,

$$\frac{C}{2N^2}\int_0^1\sum_{k=1}^N h_k(y_1)\big(\log^+h_k(y_1)\big)^2\,dy_1$$

$$\geqslant\frac{C}{2N^2}\sum_{k=1}^N\int_{\{h_k\geqslant f-K\}\cap S}(f-K)\big(\log^+(f-K)\big)^2\,dy_1\,dy_2$$

$$\geqslant\frac{C}{2N^2}\int_{\bigcup_{k=1}^N\{h_k\geqslant f-K\}\cap S}(f-K)\big(\log^+(f-K)\big)^2\,dy_1\,dy_2$$

$$\geqslant\frac{C}{2N^2}\int_{\{g_N\geqslant(f-K)\}\cap S}(f-K)\big(\log^+(f-K)\big)^2\,dy_1\,dy_2$$

by definition of g_N.

Let $\varphi(x) = x(\log^+ x)^2$. Since φ is convex,

$$\frac{C}{N^2}\int_{\{g_N\geqslant f-K\}\cap S}\frac{1}{2}\varphi(f-K)\,dy_1dy_2 \geqslant \frac{C}{N^2}\int_{\{g_N\geqslant f-K\}\cap S}\left[\varphi(f/2)-\frac{\varphi(K)}{2}\right]dy_1dy_2$$

so that

$$\frac{C}{N^2}\int_{\{g_N\geqslant f-K\}\cap S}\varphi(f/2)\,dy_1dy_2 \leqslant \frac{C}{N^2}\int_{\{g_N\geqslant f-K\}\cap S}\frac{1}{2}\varphi(f-K)\,dy_1dy_2 + C\int\frac{\varphi(K)}{2N^2}dy_1$$

$$< \int_{(f^*>1)} f^*\,dy_1dy_2 + C\int\frac{\varphi(K)}{2N^2}dy_1 < +\infty.$$

Corollary 6. *Let f be a function satisfying $V_0^1[f(\theta,\cdot)] < K(\theta)$ for all $\theta\in[0,1]$, or $V_0^1[f(\cdot,\eta)] < K(\eta)$ for all $\eta\in[0,1]$, where $V_0^1[f(\theta,\cdot)]$ and $V_0^1[f(\cdot,\eta)]$ denote the variation in $[0,1]$ of $f(\theta,\cdot)$ and of $f(\cdot,\eta)$, respectively, and $K\colon R\to R$ is such that $\int_0^1 K(\log^+K)^2 < +\infty$, $K\geqslant 0$. Suppose further that $f\in L(\log^+L)$.*

Then $f\in L(\log^+L)^2$ if and only if f^* is integrable over $(f^*>1)$.

Proof. Let us suppose $V_0^1[f(\theta,\cdot)] < K(\theta)$ for all $\theta\in[0,1]$ and $\int_{\{f^*>1\}} f^* < +\infty$; then

$$|f(\theta,\eta)-g_N(\theta,\eta)| = \left|f(\theta,\eta)-\sum_{k=1}^{N}\frac{\chi_{I_k}(\eta)}{|I_k|}\int_{I_k} f(\theta,y_2)\,dy_2\right|$$

$$= \left|f(\theta,\eta)-\frac{1}{|I_J|}\int_{I_J} f(\theta,y_2)\,dy_2\right| \qquad (\text{if } \eta\in I_J)$$

$$\leqslant \frac{1}{|I_J|}\int_{I_J}|f(\theta,\eta)-f(\theta,y_2)|\,dy_2 \leqslant V_0^1[f(\theta,\cdot)] < K(\theta)$$

for all $(\theta,\eta)\in S$ hence $\{g_N\geqslant f-K\}\cap S = S$ for all $N\geqslant 1$. Consequently, $\int\varphi(f/2)\,dy_1dy_2 < +\infty$. Now if $f\in L(\log^+L)^2$, the inequality

$$|\{(y_1,y_2)|\ f^*(y_1,y_2)>\alpha\}| \leqslant \frac{C}{\alpha}\int f\log^+\frac{f}{\alpha}\,dy_1dy_2$$

proves the converse. See [3], p.64.

Section II. All functions considered will be non negative and supported in the unit cube. The support of a function f will be denoted by supp f.

Points of absolute eccentricity.

Remarks. Let $f\in L^1(R^2)$ and $f\neq 0$. If (y_1,y_2) is a point such that $S_{y_1} = S_{y_2} = \emptyset$, then (y_1,y_2) cannot be of *semi-absolute eccentricity*. Hence, neither can it be of *absolute eccentricity*.

Theorem 6. *Let*

$$L^1_{\geqslant 0}(R^2) = \{f\in L^1(R^2)|\ f\geqslant 0;\ \text{supp}\, f\subseteq S\},$$

$$H = \{f\in L^1_{\geqslant 0}(R^2)|\ \text{almost all points of } f \text{ in } S \text{ are of absolute eccentricity}\}.$$

Then H is of the second category in $L^1_{\geqslant 0}(R^2)$. More explicitly, H^c is of the first category in $L^1_{\geqslant 0}(R^2)$ ($H^c = \{f\in L^1(R^2)|\ f\notin H\}$).

Proof. Let

$$E_L^M = \{f\in L^1_{\geqslant 0}(R^2)|\ |A_f^M|\geqslant 1/L\},$$

where A_f^M denotes the set of all points (y_1,y_2) of *partial M-eccentricity* for f such that $(y_1,y_2)\in S$. Then the following statements are verified:

(i) *E_L^M is closed in $L^1(R^2)$.*

(ii) *E_L^M is nowhere dense in $L^1_{\geqslant 0}(R^2)$.*

(i): Let $(f_n)_{n\geqslant 1}\subseteq E_L^M$ and $f_n\xrightarrow{L^1} f$. Let

$$R_M = \{I \text{ intervals}|\ 1/M\leqslant |I_1|/|I_2|\leqslant M,\ I = I_1\times I_2\}$$

and let

$$f^M(y_1,y_2) = \sup_{\substack{(y_1,y_2)\in I\\ I\in R_M}}\frac{1}{|I|}\int_I f(\theta,\eta)\,d\theta\,d\eta.$$

Clearly, $(y_1,y_2)\in A_f^M$ if and only if $f^M(y_1,y_2) = f^*(y_1,y_2)$. Since

$$\left|\{(y_1,y_2)|\ |f_n^M-f^M|>\alpha\}\right| \leqslant |\{(y_1,y_2)|\ (f_n-f)^M>\alpha\}|$$

$$\leqslant\frac{C\cdot M}{\alpha}\int|f_n-f|\to 0,$$

C being a constant depending only on the dimension, $f_n^M\xrightarrow{m} f^M$. Hence there exists a subsequence $(f_n^M)_{n\geqslant 1}$ satisfying $f_n^M\to f^M$ a.e.

Let $\varphi_n = \inf_{k\geqslant n} f_k$, $0\leqslant\varphi_n\leqslant f_n$ and $\varphi_n\nearrow f$. Therefore $\varphi_n^*\nearrow f^*$. To see this let $(y_1,y_2)\in R^2$ and $\beta\in R$, $\beta>0$ such that

$$f^*(y_1,y_2)>\beta;$$

then there exists an interval I for which $\frac{1}{|I|}\int_I f>\beta$, $(y_1,y_2)\in I$, moreover, $\frac{1}{|I|}\int_I\varphi_n\to\frac{1}{|I|}\int_I f$, so that $\frac{1}{|I|}\int_I\varphi_n>\beta$ if $n\geqslant n_0$ meaning that

$\varphi_n^*(y_1, y_2) > \beta$ if $n \geqslant n_0$. For that reason $\lim_n \varphi_n^*(y_1, y_2) \geqslant f^*(y_1, y_2)$, hence $\lim_n \varphi_n^*(y_1, y_2) = f^*(y_1, y_2)$ for all $(y_1, y_2) \in R^2$. Let

$$A = \overline{\lim_n} A_{f_n}^M \quad \text{and} \quad B = \{(y_1, y_2)|\ f_n^M(y_1, y_2) \not\to f^M(y_1, y_2)\};$$

then $|B| = 0$. Since $|A| = |\overline{\lim} A_{f_n}^M| \geqslant \overline{\lim} |A_{f_n}^M| \geqslant 1/L$, then $|A-B| = |A| \geqslant 1/L$.

Let $(y_1, y_2) \in A-B$; therefore $(y_1, y_2) \in \bigcup_{J \geqslant 1} A_{f_{n_J}}^M$ and $(y_1, y_2) \notin B$, hence

$$\varphi_{n_J}^*(y_1, y_2) \leqslant f_{n_J}^*(y_1, y_2) = f_{n_J}^M(y_1, y_2);$$

and

$$f^*(y_1, y_2) = \lim_J \varphi_{n_J}^*(y_1, y_2) \leqslant \lim_J f_{n_J}^M(y_1, y_2) = f^M(y_1, y_2);$$

which implies that $f^*(y_1, y_2) = f^M(y_1, y_2)$ for all $(y_1, y_2) \in A-B$. Consequently $A-B \subseteq A_f^M$ and $|A_f^M| \geqslant |A-B| \geqslant 1/L$.

(ii): E_L^M is nowhere dense in $L^1_{\geqslant 0}(R^2)$: Given $\varepsilon > 0$, if $V_\varepsilon(f) = \{g \in L^1_{\geqslant 0}(R^2)|\ \|g-f\|_1 < \varepsilon\}$, let us see that $V_\varepsilon(f) \cap L\log^+ L \nsubseteq E_L^M \cap L\log^+ L$. We define for $N \geqslant 1$, $H \in N$ fixed:

$$g_N(y_1, y_2) = \begin{cases} \dfrac{1}{N}\dfrac{|S_k|}{|E_k|} & \text{if} \quad (y_1, y_2) \in E_k \quad (k \geqslant 1), \\ 0 & \text{if} \quad (y_1, y_2) \notin E_k \quad (k \geqslant 1), \end{cases}$$

where $|S_k|/|E_k| = H\alpha(H)$ $(k \geqslant 1)$ S_k, E_k intervals, $\alpha(H) = \sum_{k=1}^{H} 1/k \sim \log H$; $|\bigcup_{k \geqslant 1} S_k| = 1$ and $\bigcup_{k \geqslant 1} E_k \subseteq S$. (Saks' construction; [3], pp. 98, 99.) Given $\varepsilon > 0$, let $\tilde f$ be a simple function such that $0 \leqslant \tilde f \leqslant f$ and $\|\tilde f - f\|_1 < \varepsilon/2$; then $\tilde f + g_N \xrightarrow[N\to\infty]{L^1} \tilde f$ since $g_N \xrightarrow[N\to\infty]{L^1} 0$. Hence we have $\tilde f + g_N \in L\log^+ L$, $\tilde f + g_N \in L^1_{\geqslant 0}(R^2)$ and $\|(\tilde f + g_N) - f\|_1 < \varepsilon$ if $N \geqslant N_\varepsilon$. Now let us see that $\tilde f + g_N \notin E_L^M$ if $N \geqslant N_0$. We have

$$\text{(1)} \qquad |\{(y_1, y_2)|\ (g_N^M)(y_1, y_2) > \alpha\}| \xrightarrow[N\to\infty]{} 0, \qquad \text{for any } \alpha > 0,$$

and for almost all (y_1, y_2) there exists an interval I_N for which (for H big enough)

$$\text{(2)} \qquad \frac{1}{|I_N|}\int_{I_N} g_N > N \quad \text{and} \quad (y_1, y_2) \in I_N.$$

(See [3], p. 99.)

Now given $\alpha > 0$, $Q \in R_M$ with $(y_1, y_2) \in Q$, for a subsequence $(g_N)_{N \geqslant 1}$ $(N \geqslant 1)$ (1) yields

$$|\{(y_1, y_2)|\ g_N^M(y_1, y_2) > \alpha\}| < 1/2^{N+1} \cdot L.$$

Moreover, if $N > R + \alpha$, where $R = \|\tilde f\|_\infty$, we have using (2),

$$\frac{1}{|Q|}\int_Q g_N + \frac{1}{|Q|}\int_Q \tilde f \leqslant g_N^M(y_1, y_2) + R$$

$$\leqslant \alpha + R < N < \frac{1}{|I|}\int_I g_N < \frac{1}{|I|}\int_I g_N + \frac{1}{|I|}\int_I \tilde f$$

if $(y_1, y_2) \notin \{(y_1, y_2)|\ g_N^M(y_1, y_2) > \alpha$ for some $N \geqslant 1\} = B$. Hence, for almost all $(y_1, y_2) \in B^c = S - B$, given $Q \in R_M$ such that $(y_1, y_2) \in Q$, there exists an interval I_N with $(y_1, y_2) \in I_N$ satisfying

$$\frac{1}{|Q|}\int_Q (\tilde f + g_N) \leqslant \alpha + R \leqslant \frac{1}{|I_N|}\int_{I_N} (\tilde f + g_N) \quad \text{for} \quad N > R + \alpha.$$

$I_N \notin R_M$ if $N > R + \alpha$ since $\frac{1}{|I_N|}\int_{I_N} g_N > N$ and $I_N \in R_M$ for some $N > R + \alpha$ would give $g_N^M(y_1, y_2) > N > R + \alpha > \alpha$, where $(y_1, y_2) \notin B$, which is absurd. Then almost all $(y_1, y_2) \in B^c$ *are not* in $A_{(\tilde f + g_N)}^M$ if $N > R + \alpha$, and $|B^c| \geqslant 1 - 1/2L$ since $|B| \leqslant 1/2L$. Consequently, $|A_{(\tilde f + g_N)}^M| < 1/L$ if $N > R + \alpha$. Hence

$$\tilde f + g_N \notin E_L^M \quad \text{if} \quad N > \|\tilde f\|_\infty + 1$$

(we choose $\alpha < 1$). We conclude that $V_\varepsilon(f) \cap L\log^+ L \nsubseteq E_L^M \cap L\log^+ L$, as wanted; hence

$$V_\varepsilon(f) \nsubseteq \overline{E_L^M} = E_L^M.$$

Since $H^c = \bigcup_{L, M \geqslant 1} E_L^M$, the thesis is verified.

Open Problem. We conjecture that there exists $f \in L\log^+ L$, with $\operatorname{supp} f \subseteq S$ such that almost all points in S are of *absolute eccentricity*.

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On a theorem of Lebow and Mlak for several commuting operators

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Abstract. A result of Mlak concerning the spectral radius of an operator in a Hilbert space is extended to several commuting operators.

Let H be a complex Hilbert space. Denote by $L(H)$ the Banach algebra of all bounded linear operators in H. For an n-tuple of pairwise commuting operators $T_1, \ldots, T_n$ with the Taylor joint spectrum $\sigma(T_1, \ldots \ldots, T_n)$ contained in the open unit ball $B \subset C^n$ ($B = \{z \in C^n,\ |z| < 1\}$) we denote by

$$M(\xi, T) = I - \sum_{i=1}^{n} \xi_i T_i, \qquad \xi \in \partial B$$

the topological boundary of B.

Note that $M(\xi, T)$ is invertible for every $\xi \in \partial B$ (by the spectral mapping theorem for $\sigma(T_1, \ldots, T_n)$).

The operator-valued function $M(\xi, T)^{-n}$ plays the role of the Fredholm resolvent for the above system $T_1, \ldots, T_n$. In fact, it is easy to prove that for every function f holomorphic in B and continuous in $\bar{B}$ (the closure) we have the equality

$$f(T_1, \ldots, T_n) = \int_{\partial B} M(\xi, T)^{-n} f(\xi)\, \Omega(\xi),$$

where $\Omega(\xi)$ is the $(n-1, n)$ differential form given explicitly by Henkin; see [6] for the definition.

Let us recall some definitions and notations. Denote by $U = \{z \in C,\ |z| < 1\}$ the open unit disc. For $p \geqslant 1$ and $\alpha \geqslant 0$ let

$$A^{p,\alpha} = \Big\{f,\ f\colon U \to C \text{ is holomorphic and } \int_U |f|^p (1-|z|^2)^\alpha\, dx\, dy < +\infty\Big\}.$$

For $f \in A^{p,\alpha}$ let $\|f\|_{p,\alpha}^p = \int_U |f|^p (1-|z|^2)^\alpha\, dx\, dy$. The space $A^{p,\alpha}$ is called the *Bergman space* and has been investigated in detail by Horowitz [2], [3]