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Applications of a general comparison theorem for convolution integrals

by

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Abstract. Let X be a Banach space and B a Banach algebra of tempered distributions on R^d. For $f \in B$ put $f_{(\lambda)}(x) = \lambda^d f(\lambda x)$. The space of all φ such that $\|f_{(\lambda)} * \varphi; \Omega\|_X = O(\lambda^{-s})$ as $\lambda \to \infty$ is characterized when Ω is all of R^d or a bounded subset of R^d. The proofs are based on an extension of a comparison theorem by H. S. Shapiro [7].

0. Introduction. The aim of this paper is to extend the results of H. S. Shapiro [7] in two directions. First we replace the L_p-spaces by more general Banach spaces X of tempered distributions (for instance a weighted L_p-spaces). Secondly we give a localized version of Shapiro's results. In Shapiro's work the main role is played by the algebra of bounded measures. Here we work with a more general convolution algebra B related to the space X.

Given a function f we put $f_{(\lambda)}(x) = \lambda^d f(\lambda x)$ (d being the dimension). One of the subjects of the paper is to find the space A_f^s of all $\varphi \in X$ such that

$$\|f_{(\lambda)} * \varphi\|_X = O(\lambda^{-s}), \qquad \lambda \to \infty.$$

Another, closely connected subject is to find the space E^s of all $\varphi \in X$ whose best approximation with entire functions of exponential type λ is of the order λ^{-s} as $\lambda \to \infty$. Results of these types are found in Section 3, (Theorem 3 and Corollaries 1,2 in Sec. 3). They are deduced as more or less immediate corollaries of a general comparison theorem (Theorem 1, Sec. 1) which is very close to the corresponding theorem in Shapiro [6]. For other results of this kind we refer the reader to a series of works by J. Boman, see in particular [2] (and references given there), where this type of comparison theorem in $X = L_\infty$ has been penetrated very deeply.

In order to describe the localized version of the theory we let U be an open set. Let $\|\varphi; U\|_X$ be the infinum of $\|\chi\varphi\|_X$, where χ is infinitely differentiable with compact support and $\chi = 1$ on U, $0 \leqslant \chi \leqslant 1$ everywhere. For any open, bounded set Ω we let $A_f^s(U)$ be the space of all $\varphi \in X$ such that

$$\|f_{(\lambda)} * \varphi; U\|_X = O(\lambda^{-s}), \qquad \lambda \to \infty$$

for all U which are compactly included in Ω. A characterization of $A_f^s(\Omega)$ (as well as the localized version $E^s(\Omega)$ of E^s) is given in Sec. 3 (Theorem 4 and Corollaries 3,4) as a consequence of general comparison theorem (Theorem 2, Sec. 2). For other results of this kind we refer the reader to Löfström [4] (and references given there).

The Banach algebra B, to which the kernel f above should always belong, must satisfy several conditions, as must the space X. These conditions are named (B1)–(B5) and (X1)–(X2), respectively. Conditions (B1), (B2), (B3) and (X1) are listed in Section 1, conditions (B4) and (X2) in Section 2 and condition (B5) in Section 3. The most important condition is the local division property (B3) which we discuss in Section 5. In Sections 6 and 7 we discuss some concrete choices for the algebra B and the space X. In a forthcoming note we intend to extend the theory to (some cases of) eigenfunction expansions.

1. A general theorem. Let B be a Banach space of tempered distributions on R^d, such that every test function is a member of B. We shall assume that B is a Banach algebra with convolution as multiplication (with or without unit). Moreover, we shall make a few additional assumptions on B.

If φ is a test function we write

$$\varphi_{(\lambda)}(x) = \lambda^d \varphi(\lambda x) \quad \text{and} \quad \varphi^{(\lambda)}(x) = \varphi(x/\lambda).$$

Writing $\hat{\varphi}$ for the Fourier transform of φ we have

$$\widehat{\varphi_{(\lambda)}}(\xi) = \hat{\varphi}^{(\lambda)}(\xi).$$

If f is any tempered distribution we define $f_{(\lambda)}$ and $f^{(\lambda)}$ similarly by the formulas

$$f_{(\lambda)}(\varphi) = f(\varphi^{(\lambda)}), \quad f^{(\lambda)}(\varphi) = f(\varphi_{(\lambda)}).$$

Then

$$\widehat{f_{(\lambda)}} = \hat{f}^{(\lambda)}.$$

We now arrive at our first additional assumption on B, namely

(B1) If $f \in B$ and $\lambda > 0$ then $f_{(\lambda)} \in B$. Moreover, there are constants $A > 0$ and $a > 0$ such that

$$\|f_{(\lambda)}\|_B \leqslant A \max(1, \lambda^{-a}) \|f\|_B \quad (\lambda > 0).$$

We have already mentioned that the space $\mathcal{S}$ of test functions is assumed to be contained in B. Our next assumption describes an additional inclusion:

(B2) $\mathcal{S} \subset \hat{B}$ and $\hat{B} \subset L_{\infty,\mathrm{loc}}$, i.e. every element of $\hat{B}$ is locally an essentially bounded function.

Our third assumption is a kind of local division property:

(B3) Suppose that $f, g_1, \ldots, g_n \in B$ and that there is a compact set K such that $\hat{g}_1, \ldots, \hat{g}_n$ are continuous on K and

$$\sum_{j=1}^{n} |\hat{g}_j(\xi)| > 0 \quad \text{for all } \xi \in K,$$

$$\hat{f}(\xi) = 0 \quad \text{for all } \xi \notin K.$$

Then f belongs to the ideal generated by $g_1, \ldots, g_n$, i.e. there are $h_1, \ldots, h_n \in B$ such that

$$f = \sum_{j=1}^{n} h_j * g_j.$$

We shall also work with a second Banach space X of tempered distributions. This space is related to B by our next assumption:

(X1) For every $f \in B$ and every $\varphi \in X$ we have $f * \varphi \in X$. Moreover, there is a constant C such that

$$\|f*\varphi\|_X \leqslant C \|f\|_B \|\varphi\|_X.$$

Following Shapiro [6] we shall say that a function F on R^d satisfies the *Tauberian condition* if, on each half-ray through the origin, there is a point where F does not vanish. Thus for each $\xi \neq 0$, there is a $d > 0$ such that $F(\xi/d) \neq 0$. For continuous functions F the Tauberian condition is satisfied if and only if there are positive numbers $d_1, \ldots, d_r$ such that

$$(1) \qquad \sum_{i=1}^{r} |F(\xi/d_i)| > 0 \quad \text{for all } |\xi| = 1.$$

(See Shapiro [6], Lemma 11.)

The following theorem is given by Shapiro [6] when B is the algebra of bounded measures and $X = L_\infty$ (or L_p).

THEOREM 1. *Let B and X be spaces such that* (B1)–(B3) *and* (X1) *hold. Suppose that $f, g_1, \ldots, g_n \in B$, that $\hat{g}_1, \ldots, \hat{g}_n$ are continuous and that $\sum_j |\hat{g}_j|$ satisfies the Tauberian condition. Moreover, assume $\hat{f}$ belongs locally at the origin to the ideal generated by $\hat{g}_1, \ldots, \hat{g}_n$, that is, assume that there are $h_1, \ldots, h_n \in B$ such that*

$$(2) \qquad \hat{f}(\xi) = \sum_{j=1}^{n} \hat{h}_j(\xi) \hat{g}_j(\xi)$$

in a neighbourhood of the origin.

Then there are constants C, r and $c_0, c_1, \ldots, c_r$, depending on $f, g_1, \ldots, g_n$, and a constant $b > 1$, depending on $g_1, \ldots, g_n$ only, such that

$$\|f_\lambda * \varphi\|_X \leqslant C \max(1, \lambda_0^{-a}) \sum_{k=0}^{\infty} \sum_{j=1}^{n} \sum_{i=0}^{r} \|(g_j)_{(c_i b^k \lambda)} * \varphi\|_X$$

for all $\lambda \geqslant \lambda_0 > 0$.

Proof. By (1) there are constants $d_1, \ldots, d_r > 0$ such that

$$G(\xi) = \sum_{j=1}^{n} \sum_{i=1}^{r} |\hat{g}_j(\xi/d_i)| > 0 \quad \text{for} \quad |\xi| = 1.$$

Now G is continuous, so there is a constant $b > 1$, such that $G(\xi) \neq 0$ on $b^{-1} \leqslant |\xi| \leqslant b$. Next we choose a test function ϕ such that the support of ϕ is contained in the annulus $b^{-1} \leqslant |\xi| \leqslant b$ and

$$\sum_{-\infty}^{\infty} \hat{\phi}(b^{-k}\xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

(For the construction of ϕ see [1], Ch. 6.) Assuming that (2) holds for $|\xi| \leqslant \delta$ we choose a constant $c > 0$ such that $cb < \delta$. Then put

$$\hat{\phi}_0(\xi) = 1 - \sum_{k=0}^{\infty} \hat{\phi}(c^{-1} b^{-k} \xi).$$

Note that $\hat{\phi}_0(\xi) = 0$ for $|\xi| > \delta$ and $\phi_0 \in \mathscr{S}$. Thus $\phi_0 \in B$ and

(3) $$f * \phi_0 = \sum_{j=1}^{n} h_j * g_j * \phi_0.$$

Now

(4) $$f_{(\lambda)} * \varphi = (f * \phi_0)_{(\lambda)} * \varphi + \sum_{k=0}^{\infty} (f * \phi_{(cb^k)})_{(\lambda)} * \varphi.$$

Using the local division property (B3) we see that ϕ belongs to the ideal generated by $g_{i,j} = (g_j)_{(d_i)}$. Thus ϕ is a sum of elements of the form $h_{i,j} * g_{i,j}$, where $i = 1, \ldots, r$, $j = 1, \ldots, n$. Dropping the indices we conclude that each term in the infinite sum on the right-hand side of (4) can be written as a sum of elements of the form $(f_{(\lambda)} * h_{(cb^k\lambda)}) * (g_{(cb^k\lambda)} * \varphi)$. Using the properties (X1) and (B1) we can estimate the X-norm of every such term by a constant depending on b, times the X-norm of $g_{(cb^k\lambda)} * \varphi$. Here $g = (g_j)_{(d_i)}$ for some j and i.

Writing $c_i = cd_i$ for $i = 1, \ldots, r$ we conclude that

$$\|(f * \phi_{(cb^k)})_{(\lambda)} * \varphi\|_X \leqslant C \sum_{j=1}^{n} \sum_{i=1}^{r} \|(g_j)_{(c_i b^k \lambda)} * \varphi\|_X.$$

Similarly, using (3),

$$\|(f * \phi_0)_{(\lambda)} * \varphi\|_X \leqslant C \sum_{j=1}^{n} \|(g_j)_{(\lambda)} * \varphi\|_X.$$

If we put $c_0 = 1$ we therefore get the conclusion from (4).

2. Localization of the main result. We shall now give a localized version of Theorem 1, along the lines of Löfström [4]. We shall make two more assumptions on the spaces B and X, namely:

(X2) Suppose that $\chi \in \mathscr{S}$. Then $\chi\varphi \in X$ for every $\varphi \in X$ and

$$\|\chi\varphi\|_X \leqslant C \|\hat{\chi}\|_{L_1} \|\varphi\|_X.$$

(Remark: In many cases a stronger condition is satisfied, namely

$$\|\chi\varphi\|_X \leqslant C \|\chi\|_{L_\infty} \|\varphi\|_X.)$$

(B4) The space B satisfies condition (X2).

Let ζ be a fixed infinitely differentiable function such that $0 \leqslant \zeta \leqslant 1$, $\zeta(x) = 1$ for $|x| \geqslant 2$, $\zeta(x) = 0$ for $|x| < 1$. Then (B4) implies that

$$\|\zeta^{(\varepsilon)} f_{(\lambda)}\|_B \leqslant C \|f\|_B \quad (\lambda \geqslant 1).$$

If U is any subset of R^d we shall define the local semi-norm of φ at U by the formula

(1) $$\|\varphi; U\|_X = \inf\{\|\chi\varphi\|_X \colon \chi \in \mathscr{S}, 0 \leqslant \chi \leqslant 1, \chi = 1 \text{ on } U\}.$$

Clearly we have here implicitly used (X2).

Next we shall define a new Banach algebra $B(m)$ $(m > 0)$, to be the space of all $f \in B$ such that

$$\|\zeta^{(\varepsilon)} f_{(\lambda)}\|_B \leqslant C(\varepsilon\lambda)^{-m} \quad (\lambda \geqslant 1;\ \varepsilon \geqslant 1).$$

The norm on $B(m)$ is

$$\|f\|_{B(m)} = \|f\|_B + \sup_{\varepsilon,\lambda \geqslant 1} (\varepsilon\lambda)^m \|\zeta^{(\varepsilon)} f_{(\lambda)}\|_B.$$

In order to see that $B(m)$ is a convolution algebra we take $f, g \in B(m)$. Then, with $\delta = 8\varepsilon$,

$$\zeta^{(\delta)} \cdot (f * g)_{(\lambda)} = \zeta^{(\delta)} \cdot \big((\zeta^{(\varepsilon)} f_{(\lambda)}) * g_{(\lambda)}\big) + \zeta^{(\delta)} \big((f_{(\lambda)} - \zeta^{(\varepsilon)} f_{(\lambda)}) * \zeta^{(\varepsilon)} g_{(\lambda)}\big).$$

Thus using (B4) and (B1)

$$\begin{aligned} \|\zeta^{(\delta)} (f * g)_{(\lambda)}\|_B &\leqslant C \big(\|\zeta^{(\varepsilon)} f_{(\lambda)}\|_B \|g_{(\lambda)}\|_B + \|f_{(\lambda)}\|_B \|\zeta^{(\varepsilon)} g_{(\lambda)}\|_B\big) \\ &\leqslant C(\delta\lambda)^{-m} \big(\|f\|_{B(m)} \|g\|_B + \|f\|_B \|g\|_{B(m)}\big). \end{aligned}$$

This implies that $f * g \in B(m)$ and also that

$$\|f * g\|_{B(m)} \leqslant C \|f\|_{B(m)} \|g\|_{B(m)}.$$

The importance of the Banach algebra $B(m)$ is connected with the following lemma.

LEMMA 1. *Let U be a given set and U_ε the set of points of distance at most ε to U. Then if $f\in B(m)$ and $\varphi\in X$ we have for $\lambda\geqslant\lambda_0$, $\varepsilon\lambda\geqslant 4$*

$$\|f_{(\lambda)}*\varphi;\ U\|_X\leqslant C\max(1,\lambda_0^{-a})\big(\|\varphi;\ U_\varepsilon\|_X+(\varepsilon\lambda)^{-m}\|\varphi\|_X\big).$$

Here C is independent of $\varphi, f, \varepsilon, U$ and λ.

Proof. Choose infinitely differentiable functions χ and χ_ε such that $\chi=1$ on U, $\chi=0$ outside $U_{\varepsilon/2}$ and $\chi_\varepsilon=1$ on U_ε. Then

$$\chi\cdot(f_{(\lambda)}*\varphi)=\chi\cdot\big(f_{(\lambda)}*(\chi_\varepsilon\varphi)\big)+\chi\cdot\Big(\big(\zeta^{(\varepsilon/4)}\cdot f_{(\lambda)}\big)*\big((1-\chi_\varepsilon)\varphi\big)\Big).$$

Now write $\lambda=\lambda_0\mu$, where $\mu\geqslant 1$. Then

$$\|\zeta^{(\varepsilon/4)}f_{(\lambda)}\|_B=\|(\zeta^{(\varepsilon\lambda_0/4)}f_{(\mu)})_{(\lambda_0)}\|_B\leqslant A\max(1,\lambda_0^{-a})(\varepsilon\lambda/4)^{-m}\|f\|_{B(m)}.$$

The following theorem is a localized version of Theorem 1.

THEOREM 2. *Let B and X be spaces such that* (B1)–(B4) *and* (X1)–(X2) *hold. Moreover, assume that $m>0$ and that $B(m)$ has the local division property* (B3). *Let $f\in B(m)$, $g_1,\ldots,g_n\in B(m)$ be given, so that $\hat f=\sum_{j=1}^{n}\hat h_j\hat g_j$ in a neighbourhood of the origin, where $h_1,\ldots,h_n\in B(m)$. Moreover, assume that $\hat g_1,\ldots,\hat g_n$ are continuous and $\sum_j|\hat g_j|$ satisfies the Tauberian condition.*

Then there are constants $C, r, c_0,\ldots,c_r$, depending on $f, g_1,\ldots,g_n$, and a constant $b>1$, depending on $g_1,\ldots,g_n$ only, such that for $\lambda\geqslant\lambda_0$, $\varepsilon\lambda\geqslant 4$

$$\|f_{(\lambda)}*\varphi;\ U\|_X\leqslant C\max(1,\lambda_0^{-a})\Big(\sum_{k=0}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{r}\|(g_j)_{(c_ib^k\lambda)}*\varphi;\ U_\varepsilon\|_X+(\varepsilon\lambda)^{-m}\|\varphi\|_X\Big).$$

Here U is any set and U_ε the set of points with distance at most ε to U. Here $C\leqslant D\max_{j=1,\ldots,n}(\|f\|_{B(m)},\|h_j\|_{B(m)})$, where D is independent of f.

Proof. Just as in the proof of Theorem 1 we can write

$$f*\phi_0=\sum_{j=1}^{n}h_j*g_j*\phi_0,\qquad \phi_0,h_j,g_j\in B(m)$$

and

$$f_{(\lambda)}*\varphi=(f*\phi_0)_{(\lambda)}*\varphi+\sum_{k=0}^{\infty}(f*\phi_{(cb^k)})_{(\lambda)}*\varphi.$$

Now we observe that $\mathscr{S}\subset B(m)$. To show this write

$$\zeta_m(x)=|x|^{-m}\zeta(x),\qquad f_m(x)=|x|^m\zeta(2x)f(x),$$

where $f\in\mathscr{S}$. Then $f_m\in\mathscr{S}$ and hence $f_m\in B$. Since $\zeta^{(\varepsilon)}f_{(\lambda)}=(\zeta^{(\varepsilon\lambda)}f)_{(\lambda)}$, we get for $\lambda\geqslant 1$ that

$$\|\zeta^{(\varepsilon)}f_{(\lambda)}\|_B\leqslant A\|\zeta^{(\varepsilon\lambda)}f\|_B\leqslant A(\varepsilon\lambda)^{-m}\|\zeta_m^{(\varepsilon\lambda)}f_m\|_B.$$

Now the Fourier transform of $\zeta_m^{(\varepsilon\lambda)}$ is $(\hat\zeta_m)_{(\varepsilon\lambda)}$. Since $m>0$, we have $\hat\zeta_m\in L_1$. (This follows for instance from Löfström [3], Lemma 1.4.) Thus (B4) implies that

$$\|\zeta_m^{(\varepsilon\lambda)}f_m\|_B\leqslant C\|f_m\|_B.$$

Thus

$$\|\zeta^{(\varepsilon)}f_{(\lambda)}\|_B\leqslant AC(\varepsilon\lambda)^{-m}\|f_m\|_B;$$

i.e. $f\in B(m)$.

Now ϕ belongs to the ideal in $B(m)$ generated by $(g_j)_{(d_j)}$ (with the same notation as in the proof of Theorem 1). Using Lemma 1 we now get

$$\|(f*\phi_{(cb^k)})_{(\lambda)}*\varphi;\ U\|_X\leqslant C\Big(\sum_{j=1}^{n}\sum_{i=1}^{r}\|(g_j)_{(c_ib^k\lambda)}*\varphi;\ U_\varepsilon\|+(c_ib^k\lambda\varepsilon)^{-m}\|\varphi\|_X\Big),$$

and an analogous estimate for $\|(f*\phi_0)_{(\lambda)}*\varphi;\ U\|_X$. Since $\sum_{k=0}^{\infty}b^{-mk}<\infty$, we get the result.

In some cases the conditions on f in Theorem 2 can be relaxed. If $g_1,\ldots,g_n$ have compact supports one needs only to assume that $h_1,\ldots,h_n$ belong to B (not to $B(m)$).

THEOREM 2′. *Let B and X satisfy* (B1)–(B4) *and* (X1)–(X2) *and assume that $B(m)$ has the local division property. Suppose that $g_1,\ldots,g_n$ have compact supports and belong to B, that $\hat g_1,\ldots,\hat g_n$ are continuous and that $\sum_j|\hat g_j|$ satisfies the Tauberian condition. Moreover, assume that $f\in B(m)$ and that $\hat f=\sum_j\hat h_j g_j$ in a neighbourhood of the origin, where $h_j\in B$.*

Then there are constants $C, r, c_0,\ldots,c_r, b>1$ such that if λ is large enough

$$\|f_{(\lambda)}*\varphi;\ U\|_X\leqslant C\Big(\sum_{k=0}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{r}\|(g_j)_{(c_ib^k\lambda)}*\varphi;\ U_\varepsilon\|_X+(\varepsilon\lambda)^{-m}\|\varphi\|_X\Big).$$

Proof. Note first that if g_j has compact support then $g_j\in B(m)$ since $\zeta^{(\varepsilon)}(g_j)_{(\lambda)}=0$ if $\varepsilon\lambda$ is large enough. Now let χ be an infinitely differentiable function with compact support such that $\chi=1$ on $U_{\varepsilon/4}$ and $\chi=0$ outside $U_{\varepsilon/2}$. Put $\psi=\chi\varphi$. Then the proof of Lemma 1 implies that

$$\|f_{(\lambda)}*\varphi;\ U\|_X\leqslant C\big(\|f_{(\lambda)}*\psi\|_X+(\varepsilon\lambda)^{-m}\|\varphi\|_X\big).$$

Now Theorem 1 implies that

$$\|f_{(\lambda)}*\psi\|_X \leqslant C\sum_{k=0}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{r}\|(g_j)_{(c_ib^k\lambda)}*\psi\|_X.$$

Writing $g = g_j$, $\mu = c_ib^k\lambda$ we now note that $g_{(\mu)}*\psi = 0$ outside $U_{3\varepsilon/4}$ if λ (and hence μ) is large enough. Thus Lemma 1 implies

$$\|g_{(\mu)}*\psi\|_X = \|g_{(\mu)}*\psi;\ U_{3\varepsilon/4}\|_X \leqslant C\big(\|g_{(\mu)}*\varphi;\ U_\varepsilon\|_X + (\varepsilon\mu)^{-m}\|\varphi\|_X\big).$$

Now the result follows.

3. Applications to approximation theory. Let e_j be the jth unit vector in R^d and put

$$\delta_j(\varphi) = \varphi(e_j), \quad j = 1, \ldots, d,$$
$$\delta_0(\varphi) = \varphi(0).$$

Throughout this section we shall assume that

(B5) $$\delta_j \in B \quad \text{for} \quad j = 0, 1, \ldots, d.$$

Note that $(\delta_j)_{(\lambda)}*\varphi(x) = \varphi(x-e_j/\lambda)$. Thus if (B1) and (X1) hold then the translation operators along the unit vectors are bounded on X. Consequently, the difference operator

$$\Delta_{j,h}\varphi(x) = \varphi(x+he_j)-\varphi(x)$$

and its iterates $\Delta_{j,h}^M$, $M = 1, 2, 3, \ldots$ are bounded on X. In this section we shall work with the space X^s of all $\varphi\in X$ such that for some $M > s$ $(s > 0)$ we have

$$\|\Delta_{j,h}^M\varphi\|_X = O(h^s), \quad j = 1, \ldots, d \quad (h\to 0).$$

Similarly, given $f\in B$ we consider the approximation space A_f^s, $s > 0$, consisting of all $\varphi\in X$ such that

$$\|f_\lambda*\varphi\|_X = O(\lambda^{-s}) \quad (\lambda\to\infty).$$

Finally, we shall consider the space E^s of all $\varphi\in X$ whose best approximation $E(\lambda;\varphi)$ with entire functions of order λ is $O(\lambda^{-s})$. Thus

$$E_\lambda(\varphi) = \inf\{\|\psi-\varphi\|_X\colon \psi\in X,\ \hat\psi(\xi) = 0 \text{ for } |\xi| > \lambda\}$$

and

$$\varphi\in E^s \Leftrightarrow E_\lambda(\varphi) = O(\lambda^{-s}).$$

We shall prove that $E^s = X^s$ and that $A_f^s = X^s$ if f satisfies some additional conditions. Our presentation follows closely Shapiro [6] and [7].

THEOREM 3. *Suppose that* (B1)–(B3) *and* (X1) *are satisfied. Assume that $g_1, \ldots, g_n\in B$, that $\hat g_1, \ldots, \hat g_n$ are continuous and that $\sum_j|\hat g_j|$ satisfies the Tauberian condition. Moreover, assume that $f\in B$ and that $\hat f(\xi) = H(\xi)\hat v(\xi)$ in a neighbourhood of the origin. Here $v\in B$ and H is a positive homogeneous function of order t which locally at the origin belongs to $\hat B$. Let a be the exponent defined in* (B1) *and assume that $0 < s < t$, $t > a$. Then*

$$\bigcap_{j=1}^{n} A_{g_j}^s \subset A_f^s$$

Proof. Choose a test function ψ so that $\hat\psi$ vanishes outside an appropriate neighbourhood of the origin and so that $\hat\psi(\xi) = 1$ near $\xi = 0$. Put $\hat h = \hat f\hat\psi$ and $\hat g = (1-\hat\psi)\hat f$. Then $f = h+g$. Since g vanishes in a neighbourhood of the origin, $\hat g$ belongs locally at the origin to the ideal generated by $\hat g_1, \ldots, \hat g_n$. Thus Theorem 1 implies

$$\|g_{(\lambda)}*\varphi\|_X \leqslant C\sum_{k=0}^{\infty}\sum_{j,i}(c_ib^k\lambda)^{-s} \leqslant C\lambda^{-s}, \quad \lambda \geqslant 1.$$

Since

$$\|(g_j)_{(\lambda)}*\varphi\|_X = O(\lambda^{-s}), \quad \lambda\to\infty,$$

thus

$$\|f_{(\lambda)}*\varphi\|_X \leqslant C(\|h_{(\lambda)}*\varphi\|_X + \lambda^{-s}), \quad \lambda \geqslant 1.$$

It is no restriction to assume that $\hat h(\xi) = H(\xi)$ in a neighbourhood of the origin. For otherwise let $\hat h_0(\xi) = H(\xi)$ near the origin. Then $h = \psi*h_0$ so that

$$\|h_\lambda*\varphi\|_X \leqslant \|\psi\|_B\|(h_0)_{(\lambda)}*\varphi\|_X, \quad \lambda \geqslant 1.$$

Write $\hat k(\xi) = \hat h(\xi)-2^{-t}\hat h(2\xi)$. Then

$$\|k_{(2^{-l})}\|_B \leqslant A2^{al}\|k\|_B$$

and thus the series

$$\sum_{l=0}^{\infty}2^{-lt}k_{(2^{-l})}$$

converges in B (since $t > a$). But

$$\hat h(\xi) = \sum_{l=0}^{\infty}2^{-lt}\hat k(2^l\xi) \quad \text{in} \quad L_\infty$$

so (B2) implies

$$h = \sum_{l=0}^{\infty}2^{-lt}k_{(2^{-l})}.$$

Now $\hat{k}$ vanishes in a neighbourhood of the origin. Therefore

$$\|k_{(2^{-l}\lambda)}*\varphi\|_X \leqslant C\lambda^{-a}2^{al} \quad \text{if} \quad \lambda \leqslant 2^l,$$

$$\|k_{(2^{-l}\lambda)}*\varphi\|_X \leqslant C\lambda^{-s}2^{sl} \quad \text{if} \quad \lambda \geqslant 2^l,$$

the second estimate by a new application of Theorem 1. Consequently

$$\|h_\lambda*\varphi\|_X \leqslant C\Big(\lambda^{-a}\sum_{2^l\geqslant\lambda} 2^{-(t-a)l}+\lambda^{-s}\sum_{2^l<\lambda} 2^{-(t-a-s)l}\Big).$$

Since $t > s$, the conclusion follows.

COROLLARY 1. *Suppose that* (B1)–(B3), (B5) *and* (X1) *hold. Assume that $f \in B$ and that $\hat{f}$ is a continuous function satisfying the Tauberian condition. Moreover, let $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin, where $v \in B$ and H is positive homogeneous of order $t > a$ and belongs locally at the origin to $\hat{B}$. Then $A_f^s = X^s$ for $0 < s < t$.*

Proof. We use Theorem 3 with

$$\hat{g}_j(\xi) = (\exp(i\xi_j)-1)^M,$$

where $M > s$. Then (B5) implies that $g_j \in B$. Moreover, $\sum_{j=1}^{n}|\hat{g}_j|$ is a continuous function satisfying the Tauberian condition. Thus Theorem 3 implies $X^s \subset A_f^s$ if $t > s$, $t > a$. In order to get the converse inclusion we note that $(\exp(i\xi_j)-1)^M = (i\xi_j)^M\hat{U}_j(\xi)$ in a neighbourhood of the origin. Here $U_j \in B$. Now $(i\xi_j)^M$ is positive homogeneous of order M and is locally in $\hat{B}$ at the origin. Moreover, $\hat{f}$ is continuous and satisfies the Tauberian condition. Taking $M > s$, $M > a$, Theorem 3 implies $A_f^s \subset X^s$.

COROLLARY 2. *Suppose that* (B1)–(B3), (B5) *and* (X1) *hold. Then*

$$E^s = X^s \quad \textit{for all } s > 0.$$

Proof. A proper choice of f in Corollary 1 will give that $X^s = A^s \subset E^s$. Conversely, choose $f \in B$ so that $\hat{f}(\xi) = 0$ for $|\xi| \leqslant 1$. If $\varphi \in E^s$ there is for every $\lambda \geqslant 1$ a $\psi_\lambda \in X$ so that $\hat{\psi}_\lambda(\xi) = 0$ for $|\xi| \geqslant \lambda$ and

$$\|\varphi-\psi_\lambda\|_X \leqslant C\lambda^{-s}.$$

Then $f_\lambda*\varphi = f_\lambda*(\varphi-\psi_\lambda)$ so that

$$\|f_\lambda*\varphi\|_X \leqslant C\lambda^{-s}, \quad \lambda \geqslant 1.$$

If we choose $\hat{f}$ continuous and $\hat{f}(\xi) \neq 0$ for $|\xi| = 2$ we get from Corollary 1 that $\varphi \in X^s$. Thus $E^s \subset X^s$.

We shall now consider a localized version of the previous results. Let Ω be an open bounded set. We use the notation $U \Subset V$ to indicate that the closure of U has a positive distance to the complement of V. We let $X^s(\Omega)$ denote the space of all $\varphi \in X$ such that

$$\|\Delta_{j,h}^M\varphi;\, U\|_X = O(h^s) \quad \text{for all } U \Subset \Omega.$$

Similarly, $A_f^s(\Omega)$ is the space of all $\varphi \in X$ such that

$$\|f_{(\lambda)}*\varphi;\, U\|_X = O(\lambda^{-s}) \quad \text{for all } U \Subset \Omega.$$

Similarly, we shall let $E^s(\Omega)$ denote the space of all $\varphi \in X$ such that there is a family $(\psi_\lambda)_{\lambda>1}$, with $\sup_{\lambda\geqslant 1}\|\psi_\lambda\|_X < \infty$, $\hat{\psi}_\lambda(\xi) = 0$ for $|\xi| > \lambda$ and

$$\|\psi_\lambda-\varphi;\, U\|_X = O(\lambda^{-s}) \quad \text{for all } U \Subset \Omega.$$

THEOREM 4. *Suppose that* (B1)–(B4), (X1)–(X2) *hold and that $B(m)$ (for a fixed $m > 0$) has the local division property* (B3). *Let $g_1, \ldots, g_n \in B(m)$ and suppose that $\hat{g}_1, \ldots, \hat{g}_n$ are continuous and that $\sum_j|\hat{g}_j|$ satisfies the Tauberian condition. Moreover, assume that $f \in B(m)$ and that $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin. Here $v \in B(m)$ and H is a continuous positive homogeneous function of order $t > a$ which belongs locally at the origin to $\hat{B}(m)$. Let $s \leqslant m$ and $0 < s < t$.*

Then

$$\bigcap_{j=1}^{n} A_{g_j}^s(\Omega) \subset A_f^s(\Omega)$$

for every open bounded set Ω.

Proof. We write $f = h+g$, where $\hat{g}(\xi) = 0$ in a neighbourhood of the origin and where $\hat{h}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin. It is no restriction to assume that $\hat{h}$ is continuous and satisfies the Tauberian condition. Let $U \Subset \Omega$ and choose $\varepsilon > 0$ so that $U_\varepsilon \Subset \Omega$. Take $\varphi \in \bigcap_{j=1}^{n} A_{g_j}^s(\Omega)$. Then

$$\|(g_j)_{(\lambda)}*\varphi;\, U_\varepsilon\|_X \leqslant C\max(1, \lambda^{-a})\max(1, \lambda^{-s}).$$

By Theorem 2 we conclude that

$$\|g_{(\lambda)}*\varphi;\, U\|_X \leqslant C\Big(\sum_{k=0}^{\infty}\sum_{i,j}(c_ib^k\lambda)^{-s}+(\varepsilon\lambda)^{-m}\|\varphi\|_X\Big).$$

Since $m \geqslant s$, it follows that

$$\|g_{(\lambda)}*\varphi;\, U\|_X = O(\lambda^{-s}).$$

In order to estimate $h_{(\lambda)}*\varphi$ on U we write

$$\hat{h}(\xi) = \sum_{l=0}^{\infty}2^{-tl}\hat{k}(2^l\xi),$$

where

$$\hat{k}(\xi) = \hat{h}(\xi) - 2^{-t}\hat{h}(2\xi)$$

as in the proof of Theorem 3. Since $v \in B(m)$, it is no restriction to assume that $\hat{h}(\xi) = H(\xi)$ in a neighbourhood of the origin.

$$h = \sum_{l=0}^{\infty} 2^{-lt} k_{(2^{-l})}.$$

In order to show that this series converges in $B(m)$ we note that

$$\|\xi^{(s)} k_{(2^{-l}\lambda)}\|_B = \|(\xi^{(s2^l)} k_{(\lambda)})_{(2^{-l})}\|_B \leqslant A 2^{al} (\varepsilon\lambda 2^l)^{-m} \|k\|_{B(m)}$$

if $\lambda \geqslant 1$ and $\varepsilon\lambda \geqslant 1$. Thus if $l \geqslant 0$

$$\|k_{(2^{-l})}\|_{B(m)} \leqslant C \max(1, 2^{(a-m)l}).$$

Therefore the series converges if $t > a$. By Theorem 2 we now get (for $\lambda \geqslant 1$)

$$\|k_{(2^{-l}\lambda)} * \varphi;\, U\|_X \leqslant C\lambda^{-s} 2^{sl} \max(1, 2^{(a-m)l}), \qquad \lambda \geqslant 2^l.$$

Obviously,

$$\|k_{(2^{-l}\lambda)} * \varphi;\, U\|_X \leqslant C\lambda^{-a} 2^{al}, \qquad \lambda \leqslant 2^l.$$

Consequently,

$$\|h_{(\lambda)} * \varphi;\, U_{\varepsilon/2}\|_X \leqslant C_\varepsilon \Big(\lambda^{-a} \sum_{2^l \geqslant \lambda} 2^{-(t-a)l} + \lambda^{-s} \sum_{2^l < \lambda} 2^{-(t-s)l} \max(1, 2^{(a-m)l})\Big)$$

since $t > s$, $t > a$. If $g_1, \ldots, g_n$ have compact supports the conditions on v and H can be relaxed as a consequence of Theorem 2′.

THEOREM 4′. *Suppose that* (B1)–(B4) *and* (X1)–(X2) *hold and that* $B(m)$ *has the local division property. Let* $g_1, \ldots, g_n \in B$ *have compact supports and assume that* $\hat{g}_1, \ldots, \hat{g}_n$ *are continuous and that* $\sum |\hat{g}_j|$ *satisfies the Tauberian condition. Suppose that* $f \in B(m)$, $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ *in a neighbourhood of the origin, where* H *belongs locally at the origin to* $\hat{B}$ *and where* $v \in B$.

If H *is continuous and homogeneous of order* $t > a$ *and if* $0 < s \leqslant m$, $0 < s < t$ *then*

$$\bigcap_i A^s_{g_i}(\Omega) \subset A^s_f(\Omega)$$

for every open bounded set Ω.

COROLLARY 3. *Suppose that* (B1)–(B5), (X1)–(X2) *hold. Let* $f \in B(m)$ *and suppose that* $\hat{f}$ *is continuous and satisfies the Tauberian condition. Moreover, suppose that* $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ *in a neighbourhood of the origin. Here* $v \in B$ *and* H *is continuous positive homogeneous of order* t *and belongs locally at the origin to* $\hat{B}$.

Then $A^s_f(\Omega) = X^s(\Omega)$ *provided that* $0 < s \leqslant m$ *and* $t > s$, $t > a$.

COROLLARY 4. *Suppose that* (B1)–(B5), (X1)–(X2) *hold. Then*

$$E^s(\Omega) = X^s(\Omega) \quad \textit{for all } s > 0.$$

The proofs are almost word by word the same as the proofs of Corollaries 1, 2.

4. How to choose the space X. In the applications to approximation problems it is natural to choose the space X first and then try to find a suitable Banach algebra B. However it is also possible to go the other way round, that is, to start with the Banach algebra B and then try to find X so that (X1) and possibly (X2) hold. This is the situation we shall consider here.

Thus let B be a given Banach algebra. Then $X = B$ is a possible choice since then (X1) holds. If (B4) is satisfied then so is clearly (X2). The dual B^* of B is another possible choice for X, provided that B^* is a space of tempered distributions. In fact, if $\varphi \in X = B^*$ and $f \in B$ we have

$$\|f*\varphi\|_X = \sup_{\|g\|_B \leqslant 1} |g*f*\varphi(0)| \leqslant \|f\|_B \|\varphi\|_X.$$

If (B4) is satisfied then (X2) holds for $X = B^*$. To see this take a $\varphi \in X$ and let χ be an infinitely differentiable function with compact support such that $0 \leqslant \chi \leqslant 1$. Then (B4) implies

$$\|\chi\varphi\|_X = \sup_{\|g\|_B \leqslant 1} |g*(\chi\varphi)(0)| = \sup_{\|g\|_B \leqslant 1} |(\tilde{\chi} g)*\varphi(0)| \leqslant C\|\varphi\|_X.$$

(Here $\tilde{\chi}(x) = \chi(-x)$.)

Since B and B^* are possible choices for X, so are the interpolation spaces between B and B^*. We have proved the following result.

PROPOSITION 1. *Let B be a Banach algebra of tempered distributions such that* (B1) *holds and let B^*, the dual of B, be a Banach space of tempered distributions. Then any interpolation space X between B and B^* will satisfy* (X1). *If* (B4) *holds then so does* (X2).

Other choices for X are the closure of $\mathscr{S}$ in B or the closure of $\mathscr{S}$ in B^* (provided $\mathscr{S} \subset B^*$). In specific situations there can of course be still other possible choices for X. More about this in Section 6.

5. Discussion of the local division property. The most important condition on the Banach algebra B is the local division property (B3). This condition is well known in some cases, for instance when $B = L_1$. (See Rudin [8], Lemma 7.2.2.) In that case the characters, i.e. the non-trivial continuous linear and multiplicative functionals are known to be the point-evaluations of the Fourier transform. More generally we have the following lemma:

Lemma 2. *Suppose that B is a Banach algebra under convolution such that* (B2) *holds. Moreover, assume that the characters on B are just the point-evaluations of the Fourier transform. Then B has the local division property.*

Proof. Let $g_1, \ldots, g_n$ be given elements of B. Moreover, let K be a compact set and assume that $\hat{g}_1, \ldots, \hat{g}_n$ are continuous functions on K such that $|\hat{g}_1(\xi)| + \ldots + |\hat{g}_n(\xi)| > 0$ for all $\xi \in K$. For every $\xi_0 \in K$ there is a j such that $g_j(\xi_0) \neq 0$. Then there is a neighbourhood U of ξ_0 such that $\hat{g}_j(\xi_0) \neq 0$ on the closure V of U.

Now consider for a given $t \in B$ the equivalence class of all $T \in B$ such that $\hat{T} = \hat{t}$ on V. Let $\mathscr{B}$ be the Banach algebra of the equivalence classes, $\bar{t}$, with norm

$$\|\bar{t}\|_{\mathscr{B}} = \inf\{\|t\|_B \colon t \in \bar{t}\}$$

and multiplication defined by $\bar{t} * \bar{s} = \overline{(t*s)}$. Then $\mathscr{B}$ is a Banach algebra with unit. In fact there is a test function σ such that $\hat{\sigma}(\xi) = 1$ on V. Then $\bar{\sigma}$ is a unit on $\mathscr{B}$ since $(\sigma * t)\hat{} = \hat{t}$ on V and hence $\bar{\sigma} * \bar{t} = \overline{(\sigma * t)} = \bar{t}$.

Now let H be a character on $\mathscr{B}$, and put $G(t) = H(\bar{t})$. Then it is easily seen that G is a character on B and thus of the form $G(t) = \hat{t}(\eta)$ for some η. Now choose σ as above. Then

$$G(\sigma * t) = H(\bar{t}) = G(t),$$

i.e.

$$\hat{\sigma}(\eta)\hat{t}(\eta) = \hat{t}(\eta).$$

Since this is true for all test functions σ such that $\hat{\sigma}(\xi) = 1$ on V, we conclude that $\eta \in V$.

Now we have $\hat{g}_j(\xi) \neq 0$ on V. Hence $H(\bar{g}_j) \neq 0$ for all characters H on $\mathscr{B}$. Thus $\bar{g}_j$ has an invers $\bar{h}_j \in \mathscr{B}$ implying that $\bar{g}_j * \bar{h}_j * \bar{\phi} = \bar{\phi}$ for all test functions ϕ such that $\hat{\phi}$ has its support in V. Thus

$$g_j * h_j * \phi = \phi.$$

For any $\xi_0 \in K$ we have found a neighbourhood U, a number j and an $h_j \in B$ such that $g_j * h_j * \phi = \phi$ for all $\phi \in \mathscr{S}$ such that $\hat{\phi}$ has its support in U. Now cover K by finitely many such neighbourhoods $U_1, \ldots, U_k$ and assume that $\phi_1, \ldots, \phi_k$ are test functions such that $\hat{\phi}_i$ has its support in U_i and $\sum_i \hat{\phi}_i = 1$ on K. Then there are j_i and $h_i \in B$ such that

$$f = \sum \phi_i * f = \sum g_{j_i} * h_i * \phi_i * f$$

if $f \in B$ and $\hat{f}(\xi) = 0$ outside K. Thus f is in the ideal generated by $g_1, \ldots, g_n$. This completes the proof.

As an application of Lemma 2 consider a positive weight function w such that $w(x+y) \leqslant cw(x)w(y)$ and put $B = L_1(w)$ with norm

$$\|wf\|_1 = \int w(x)|f(x)|\,dx.$$

Then B is a Banach algebra. Assuming for instance that

$$w(x) \leqslant c(1+|x|)^N$$

for some N, we have a situation where Lemma 2 can be applied, since then the characters on B are just the point-evaluations of the Fourier transform.

Note that the local division property can be satisfied even if there are other characters than the point-evaluations. For instance put $B = \hat{L}_\infty$. If $g_1, \ldots, g_n \in B$ and $\hat{g}_1, \ldots, \hat{g}_n$ are continuous and $\sum |\hat{g}_j| > 0$ on K then we can write

$$\hat{f} = \sum_j \hat{h}_j \hat{g}_j, \quad \text{where} \quad \hat{h}_j = \frac{(\operatorname{sign} \hat{g}_j)\hat{f}}{\sum_\nu |\hat{g}_\nu|}$$

for any $f \in B$ such that $\hat{f} = 0$ outside K. Clearly $\hat{h}_j \in L_\infty$ and thus f is in the ideal generated by $g_1, \ldots, g_n$.

In many cases it is possible to extend the Banach algebra B slightly without loosing the local division property. Define the extended algebra $\tilde{B}$ as the space of all tempered distributions f such that $\phi * f \in B$ for all test functions with compact spectrum and

$$\|f\|_{\tilde{B}} = \sup\{\|\phi * f\|_B \colon \|\phi\|_B \leqslant 1,\ \hat{\phi} \in C_c^\infty\} < \infty.$$

Lemma 3. *If B has the local division property then so does $\tilde{B}$ and any Banach algebra B_1 such that $B \subset B_1 \subset \tilde{B}$.*

Proof. Assume that $g_1, \ldots, g_n \in B_1$, $\hat{g}_1, \ldots, \hat{g}_n$ are continuous and $\sum |\hat{g}_j| > 0$ on the compact K. Let $f \in \tilde{B}$ and assume $\hat{f} = 0$ outside K. Then choose a test function ϕ with compact spectrum so that $\hat{\phi} = 1$ on K. Then $\phi * g_j \in B$ and $\phi * f \in B$. Clearly the local division property on B implies the existence of $h_j \in B$ so that $\phi * f = \sum_j h_j * (\phi * g_j)$. Hence $f = \sum_j h_j * g_j$, where $h_j \in B \subset B_1$.

Lemma 3 can be used to extend the local division property from $L_1(w)$ (w as above) to $\mathscr{M}(w)$, the space of all f such that wf is a bounded measure. In fact,

$$\widetilde{L_1(w)} = \mathscr{M}(w).$$

To see this let ϕ be a test function with compact spectrum such that $\|w\phi\|_1 \leqslant 1$. Then

$$\|w(\phi * f)\|_1 \leqslant \|(w|\phi|) * (w|f|)\|_1 \leqslant \|wf\|_{\mathscr{M}}.$$

Thus $L_1(w) \subset \mathscr{M}(w) \subset \widetilde{L_1(w)}$ and thus $\mathscr{M}(w)$ has the local division property. (We have in fact $\mathscr{M}(w) = \widetilde{L_1(w)}$.)

As an additional illustration let B be the closure of $\mathscr{S}$ in $\hat{L}_\infty$. Then $B = \hat{C}_{(0)}$ ($C_{(0)}$ being the space of continuous function tending to zero at infinity). Clearly B has the local division property (by Lemma 2 or rather by direct inspection). Moreover, $\hat{B} = \hat{C}$ (C being the space of continuous bounded functions) which clearly has the local division property.

LEMMA 4. *Put $w(x) = (1+|x|)^a$ for some $a \geqslant 0$ and assume that*

$$L_1(w) \subset B \subset \hat{L}_\infty.$$

Then B and $B(m)$ has the local division property.

Proof. Since the characters on $L_1(w)$ are the point-evaluations of the Fourier transform, the same is true for B. Thus B has the local division property. Now put $w_m(x) = (1+|x|)^{a+m}$. Then it is easily seen that $L_1(w_m) \subset B(m)$. In fact, if $f \in L_1(w)$ then, for $\lambda \geqslant 1$,

$$\|\zeta^{(\varepsilon)} f_{(\lambda)}\|_B \leqslant A \|\zeta^{(\varepsilon\lambda)} f\|_B \leqslant A \int_{|x| \geqslant \varepsilon\lambda} (1+|x|)^a |f(x)|\,dx.$$

But if $f \in L_1(w_m)$ we can estimate the right-hand side by a constant times

$$(\varepsilon\lambda)^{-m} \int_{|x| \geqslant \varepsilon\lambda} |x|^m (1+|x|)^a |f(x)|\,dx \leqslant (\varepsilon\lambda)^{-m} \|f\|_{L_1(w_m)}.$$

Thus $L_1(w_m) \subset B(m) \subset B \subset \hat{L}_\infty$, so $B(m)$, too, has the local division property.

6. Some special choices for the algebra B. In Shapiro [7] the role of B is played by the algebra $\mathscr{M}$ of bounded measures. As we have seen $\mathscr{M}$ has the local division property. Moreover, $\|f_{(\lambda)}\|_{\mathscr{M}} = \|f\|_{\mathscr{M}}$ so that (B1) holds with $a = 0$. Clearly (B2) and (B4), (B5) are satisfied. Here we can take $X = L_p$, $1 \leqslant p \leqslant \infty$ as Shapiro does or more generally any interpolation space between L_1 and L_∞, for instance the Lorentz spaces $L_{p,q}$. We also get localized versions of Shapiro's results.

THEOREM 5. *Let X be any interpolation space between L_1 and L_∞. Then $E^s = X^s$ and $E^s(\Omega) = X^s(\Omega)$ for all $s > 0$ and all open bounded sets Ω. Moreover, let f be a bounded measure such that $\hat{f}$ satisfies the Tauberian condition. Assume that $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin, where v is a bounded measure and $H(\xi)$ is homogeneous of order t, infinitely differentiable and positive outside the origin.*

Then $A_f^s = X^s$ if $0 < s < t$. If, in addition, for some m such that $m \geqslant s$

$$(1) \qquad \int_{|x| \geqslant \mu} |df(x)| = O(\mu^{-m})$$

then we also have $A_f^s(\Omega) = X^s(\Omega)$, $0 < s < t$.

Proof. Note that (1) is equivalent to the condition $f \in \mathscr{M}(m)$. The theorem is clearly a consequence of Corollaries 1–4. The only point to check is that H belongs locally at the origin to $\hat{\mathscr{M}}$. In order to prove this put $\hat{h} = H\hat{\psi}$, where $\hat{\psi}$ is a test function with compact support. Assume that $\hat{\phi}$ is a test function with support on $2^{-1} \leqslant |\xi| \leqslant 2$ and with the property $\sum_k \hat{\phi}(2^{-k}\xi) = 1$ for $\xi \neq 0$. Put

$$\hat{h}_l(\xi) = \hat{\phi}(2^l\xi)\hat{h}(\xi).$$

Assume that $\hat{\psi}(\xi) = 0$ for $|\xi| > 1$. Then

$$\sum_{l \geqslant 0} \hat{h}_l(\xi) = \hat{h}(\xi) \quad \text{for all } \xi.$$

Since

$$|D^\alpha \hat{h}_l(\xi)| \leqslant C 2^{(|\alpha| - t)l},$$

we have

$$\|D^\alpha \hat{h}_l\|_{L_2} \leqslant C 2^{(|\alpha| - t - d/2)l}.$$

Now we have that

$$\|h_l\|_{L_1} \leqslant C \|\hat{h}_l\|_{L_2}^{1-\theta} \max_{|\alpha| = L} \|D^\alpha \hat{h}_l\|_{L_2}^{\theta},$$

where $L > d/2$ and $\theta = d/2L$. Hence

$$\|h_l\|_{L_1} \leqslant C 2^{-tl}$$

and thus

$$\|h\|_{L_1} \leqslant \sum_{l \geqslant 0} \|h_l\|_{L_1} \leqslant C \sum_{l \geqslant 0} 2^{-tl},$$

i.e. $h \in L_1$. Thus if $t > 0$ we have $h \in \mathscr{M}$. The proof is complete.

Next we shall consider weighted spaces. Let w be a positive continuous function such that there is a second positive continuous function $w^* \geqslant 1$ such that

$$(2) \qquad w(x+y) \leqslant C w^*(x) w(y),$$

and

$$(3) \qquad w^*(x/\lambda) \leqslant C \max(1, \lambda^{-a}) w^*(x) \qquad (a > 0).$$

An example of this situation is

$$w(x) = (1+|x|)^\sigma, \qquad \sigma \text{ real}$$

in which case $w^*(x) = (1+|x|)^a$, $a = |\sigma|$. Another example is

$$w(x) = \prod_{j=1}^{k} (1 + |\langle a_j, x\rangle|)^{\sigma_j},$$

where a_j are unit vectors and σ_j are real. Then

$$w^*(x) = \prod_{j=1}^{k} (1+|\langle a_j, x\rangle|)^{|\sigma_j|}$$

and

$$a = \sum_{j=1}^{k} |\sigma_j|.$$

Let $B = \mathcal{M}(w^*)$ denote the space of all f such that $w^*f \in \mathcal{M}$ with norm

$$\|f\|_B = \|w^*f\|.$$

Then (3) implies (B1). Since $1 \leqslant w^*$, we have $B \subset \mathcal{M}$ and thus $\hat{B} \subset L_\infty$. Moreover, (3) implies

$$w^*(x) \leqslant C(1+|x|)^a$$

since $w^*(e|x|) \leqslant C\max(a, |x|^a) w^*(e) \leqslant C(1+|x|)^a$ if $e = x/|x|$. Consequently $\mathcal{S} \subset B$ and thus (B2) holds. In order to prove the local division property we note that there is an integer N such that $w^*(x) \leqslant C(1+|x|^2)^N$. Thus the argument of the preceding section shows that $\mathcal{M}(w^*)$ has the local division property. Note also that (B4) and (B5) hold. As space X we choose for instance $L_p(w)$ defined by the norm $\|w\varphi\|_p$. More generally, we can take as X any interpolation space between $L_1(w)$ and $L_\infty(w)$. Clearly (X1) holds since $|w(f*\varphi)| \leqslant (w^*|f|)*(w|\varphi|)$ by (2). Condition (X2) is satisfied, too. We now get the following consequence of Corollaries 1–4.

THEOREM 6. *Let w be a positive continuous weight function such that* (2) *and* (3) *holds and let X be any interpolation space between $L_1(w)$ and $L_\infty(w)$. Then $E^s = X^s$ and $E^s(\Omega) = X^s(\Omega)$. Moreover, assume that w^*f is a bounded measure and that $\hat{f}$ satisfies the Tauberian condition. Suppose that $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin. Here w^*v is a bounded measure and H is homogeneous of order $t > a$, infinitely differentiable and positive outside the origin.*

Then $A_f^s = X^s$ if $0 < s < t$. If in addition

(4) $$\int_{|x|\geqslant\mu} |w^*(x)\,df(x)| = O(\mu^{-m}), \qquad \mu\to\infty$$

for some m such that $m \geqslant s$, then $A_f^s(\Omega) = X^s(\Omega)$.

Proof. Writing $\hat{h} = \hat{\psi}H$, where $\hat{\psi}$ is a test function with compact support, we need just to prove that

(5) $$\|w^*\zeta^{(\mu)}h\| \leqslant C\mu^{-t} \quad \text{if} \quad \mu\geqslant 1.$$

But $w^*(x) \leqslant C(1+|x|^2)^{a/2}$. Let J^a be the operator $(1+|D|^2)^{a/2}$. Then

$$\|J^{\hat{a}}\zeta_{(\mu)}\|_{L_1} \leqslant C\max(1, \mu^{-a})$$

and thus

$$\|w^*\zeta^{(\mu)}h\| \leqslant C\sum_{2^k\geqslant\mu} \|(J^{\hat{a}}\zeta_{(\mu)})*\hat{h}*\phi_{(2^k)}\|_{\hat{L}_1}$$
$$\leqslant C\max(1, \mu^{-a}) \sum_{2^k\geqslant\mu} \|\hat{h}*\phi_{(2^k)}\|_{\hat{L}_1}.$$

By the proof of Theorem 5 we get (5).

As an application of Theorem 6 we introduce the space $\mathcal{L}_p$ of all measurable functions φ such that there is a number $a \geqslant 0$ for which

$$\left(\int \left|\frac{\varphi(x)}{(1+|x|)^a}\right|^p dx\right)^{1/p} < \infty.$$

Let $\mathcal{E}$ be the space of all families $(\psi_\lambda)_{\lambda\geqslant 1}$ such that ψ_λ is an entire function of exponential type λ and such that there are positive constants C and c so that

$$|\psi_\lambda(x)| \leqslant C(1+|x|)^c, \qquad x\in R^d,\ \lambda\geqslant 1.$$

Then Theorem 6 implies

COROLLARY 5. *Assume that $\varphi \in \mathcal{L}_p$ and let Ω be open and bounded. Then*

(6) $$\Big(\int_U |\Delta_{j,h}^M \varphi(x)|^p dx\Big)^{1/p} = O(h^s) \quad \textit{for all } U \Subset \Omega$$

if and only if there is a family $(\psi_\lambda)_{\lambda>0}$ in $\mathcal{E}$ such that

(7) $$\Big(\int_U |\psi_\lambda(x) - \varphi(x)|^p dx\Big)^{1/p} = O(\lambda^{-s}) \textit{ for all } U \Subset \Omega.$$

Proof. If $\varphi\in\mathcal{L}_p$ then $\varphi\in L_p(w)$ if $w(x) = (1+|x|)^{-a}$ for some $a\geqslant 0$. Then Theorem 6 implies that (6) is equivalent to the existence of a family $(\psi_\lambda)_{\lambda>1}$ of entire functions of exponential type λ such that (7) holds and

$$\left(\int_{R^d} \left(\frac{\psi_\lambda(x)}{(1+|x|)^a}\right)^p dx\right)^{1/p} \leqslant C.$$

But then clearly $(\psi_\lambda)_{\lambda>1} \in \mathcal{E}$.

EXAMPLE 1. Consider a function φ such that

$$\varphi(x) = \begin{cases} |x| & \text{for} \quad |x|<1, \\ (1+|x|^2)^{a/2} & \text{for} \quad |x|>2 \ (a \text{ real}), \end{cases}$$

and assume that φ is infinitely differentiable for $|x| > 1/2$. Put $w(x) = (1+|x|)^{-a}$. Then $\varphi\in L_\infty(w)$. Moreover, $\varphi\in X^1$, i.e.

$$|\Delta_{j,h}^M \varphi(x)| \leqslant C|h|(1+|x|)^a \quad \text{for all } x.$$

Thus Theorem 6 implies that there exists a family ψ_λ of entire functions of exponential type λ such that

$$|\psi_\lambda(x)| \leqslant C(1+|x|)^a \quad \text{for all } x \text{ and } \lambda \geqslant 1,$$

and

$$|\psi_\lambda(x)-\varphi(x)| \leqslant C\lambda^{-1}(1+|x|)^a.$$

Theorem 6 also implies that λ^{-1} cannot be replaced by λ^{-s} for any $s > 1$.

EXAMPLE 2. Consider a function φ such that

$$\varphi(x) = |x| \quad \text{for} \quad |x| < 1,$$

$$|\varphi(x)| \leqslant C(1+|x|)^a \quad \text{for all } x \ (a \text{ real}).$$

Then Theorem 6 (or Corollary 5) implies that although φ may not belong to any space X^s (for $s > 0$) because of the lack of regularity outside the ball $|x| < 1$, we still have that there is a family ψ_λ of entire functions of exponential type λ such that

$$|\psi_\lambda(x)| \leqslant C(1+|x|)^a \quad \text{for all } x \text{ and } \lambda \geqslant 1$$

and

$$|\psi_\lambda(x)-\varphi(x)| \leqslant C_\varepsilon\lambda^{-1} \quad \text{for} \quad |x| < 1-\varepsilon,$$

for every ε such that $0 < \varepsilon < 1$.

EXAMPLE 3. Put $d = 1$ and

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad -\infty < x < \infty.$$

Clearly $f \in B = L_1(w^*)$ if $w^*(x) = (1+|x|)^a$ and $0 \leqslant a < 1$. Moreover, $f \in B(m)$ if $a = 1-m$, $0 < m < 1$. Since $\hat{f}(\xi) = \exp(-|\xi|)$, we can write

$$\hat{f}(\xi)-1 = |\xi|\hat{v}(\xi),$$

where

$$\hat{v}(\xi) = \int_0^1 \exp(-\tau|\xi|)\,d\tau.$$

Thus

$$v = \int_0^1 f_{(1/\tau)}\,d\tau.$$

Therefore $v \in B$ since

$$\|w^*v\|_{L_1} \leqslant \int_0^1 \|w^* f_{(1/\tau)}\|_{L_1} d\tau \leqslant C\int_0^1 \tau^{-a} d\tau \|f\|_{L_1} < \infty.$$

Thus the assumptions of Theorem 6 are satisfied with $d = 1$ and $0 \leqslant a < 1$. Thus suppose that for some real α with $|\alpha| < 1$ we have

$$(8) \qquad \Big(\int_{-\infty}^{\infty} |\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} < \infty.$$

Then

$$\Big(\int_{-\infty}^{\infty} |f_{(\lambda)}*\varphi(x)-\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} = O(\lambda^{-s}), \qquad \lambda\to\infty$$

if and only if

$$\Big(\int_{-\infty}^{\infty} |\Delta_h\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} = O(h^s), \qquad h\to 0,$$

provided that $0 < s < 1$, $(\Delta\varphi(x) = \varphi(x+h)-\varphi(x))$.

There is also a localized version of the last equivalence, stating that if (8) holds for some a such that $-1 < a < 1$ then for $0 < s \leqslant m = 1-a$

$$\Big(\int_{a+\varepsilon}^{b-\varepsilon} |f_{(\lambda)}*\varphi(x)-\varphi(x)|^p dx\Big)^{1/p} = O(\lambda^s), \qquad \lambda\to\infty$$

for all ε such that $0 < \varepsilon < b-a$ if and only if

$$\Big(\int_{a+\varepsilon}^{b-\varepsilon} |\Delta_h\varphi(x)|^p dx\Big)^{1/p} = O(h^s), \qquad h\to 0$$

for all such ε. Note that we can drop the weight function when integrating on finite intervals. Also note that the last result does not follow from Theorem 5 since an application of the theorem would require the global estimate

$$\Big(\int_{-\infty}^{\infty} |\varphi(x)|^p dx\Big)^{1/p} < \infty,$$

instead of (8), which is weaker if $-1 < a < 0$.

EXAMPLE 4. Consider the Gauss kernel

$$f(x) = c\exp(-|x|^2/2), \qquad x \in R^d.$$

Then $f \in B = L_1(w^*)$ if $w^*(x) = (1+|x|)^a$, $a \geqslant 0$, and $f \in B(m)$ for every $m > 0$. Moreover, $\hat{f}(\xi)-1 = |\xi|^2 v(\xi)$, where $v \in B$ if $0 \leqslant a < 2$. This is easily seen by writing

$$v = \tfrac{1}{2}\int_0^1 f_{(1/\sqrt{\tau})}\,d\tau.$$

Thus Theorem 6 implies that

$$\Big(\int_{R^d}|f_{(\lambda)}*\varphi(x)-\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} = \mathcal{O}(\lambda^{-s}), \quad \lambda\to\infty$$

if and only if

$$\Big(\int_{R^d}|\Delta^2_{j,h}\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} = \mathcal{O}(h^s), \quad h\to\infty,$$

provided that $0<s<2$ and $-2<a<2$ and that

(9) $$\Big(\int_{R^d}|\varphi(x)|^p(1+|x|)^{ap}dx\Big)^{1/p} < \infty.$$

As an illustration of the localized version of Theorem 6, let us now take

$$\Omega = \{x = (x_1, \ldots, x_{d-1}, x_d)\colon |x_d|<1\}.$$

Then suppose that $0<s<2$, $-2<a<2$ and that (9) holds. Put

$$w_0(x) = \big(1+(x_1^2+\ldots+x_{d-1}^2)^{1/2}\big)^a,$$

and

$$U_\varepsilon = \{x\colon |x_d|<1-\varepsilon\}, \quad 0<\varepsilon<1.$$

Then Theorem 6 implies that

$$\Big(\int_{U_\varepsilon}|f_{(\lambda)}*\varphi(x)-\varphi(x)|^p w_0(x)^p dx\Big)^{1/p} = \mathcal{O}(\lambda^{-s}), \quad \lambda\to\infty$$

for every ε if and only if

$$\Big(\int_{U_\varepsilon}|\Delta^2_{j,h}\varphi(x)|^p w_0(x)^p dx\Big)^{1/p} = \mathcal{O}(h^s), \quad h\to 0$$

(for every ε).

It is also possible to apply Theorem 6 to other weight-functions than $(1+|x|)^a$. Let us take

$$w(x) = (1+|x_1|)^{a_1}\ldots(1+|x_d|)^{a_d}.$$

Then (2) holds with $w^*(x) = (1+|x|)^a$, $a = |a_1|+\ldots+|a_d|$. Thus Theorem 6 implies that if $|a_1|+\ldots+|a_d|<2$ then

$$\Big(\int_{R^d}|f_{(\lambda)}*\varphi(x)-\varphi(x)|^p w(x)^p dx\Big)^{1/p} = \mathcal{O}(\lambda^{-s}), \quad \lambda\to\infty$$

if and only if

$$\Big(\int_{R^d}|\Delta^2_{j,h}\varphi(x)|^p w(x)^p dx\Big)^{1/p} = \mathcal{O}(h^s), \quad h\to 0,$$

and

$$\Big(\int_{R^d}|\varphi(x)|^p w(x)^p dx\Big)^{1/p} < \infty.$$

7. Some other choices for the space B. Let w be a positive continuous weight function such that

$$w(x+y)\leqslant Cw^*(x)w(y)$$

for some positive continuous function w^*. Then we consider the space $T_p(w)$ of all translation invariant operators on $L_p(w)$. In [5] we have proved that the operators $T_p(w)$ has the form $\varphi\to k*\varphi$, where k is a distribution. If w is symmetric, i.e. $w(-x) = w(x)$ then $T_p(w)\subset \hat{L}_\infty$ so in particular $T_p(w)$ is a space of tempered distributions. Clearly $T_p(w)$ is a convolution algebra, normed by the operator norm.

If we take $w = 1$ then $T_p(w) = T_p$ is the familiar space of bounded translation invariant operators on L_p. The space $\hat{T}_p = M_p$ is the space of Fourier multipliers on L_p. It is well known that

$$\|f_{(\lambda)}\|_{T_p} = \|\hat{f}^{(\lambda)}\|_{M_p} = \|\hat{f}\|_{M_p}$$

so that (B1) holds for $B = T_p$. Clearly (B2) and (B5) are satisfied. If χ is a test function such that $0\leqslant\chi\leqslant 1$ then

$$\|\chi f\|_{T_p} = \|\hat{\chi}*\hat{f}\|_{M_p}\leqslant\|\hat{\chi}\|_{L_1}\|\hat{f}\|_{M_p},$$

which implies that (B4) is true.

Next consider the closure B of $\mathcal{S}$ in T_p. Since $L_1\subset T_p\subset\hat{L}_\infty$, we have $L_1\subset B\subset \hat{C}_{(0)}$. Thus the characters on B are the point-evaluations of the Fourier transform. Consequently Lemmas 2, 3 imply that the space $\tilde{B}$ has the local division property (B3). We shall write $\mathcal{T}_p$ for the space $\tilde{B}$. Thus

$\mathcal{T}_p$ is the space of all tempered distributions f such that, for all test functions ϕ with compact spectrum, $\phi*f$ is a limit in T_p of test functions and $\|\phi*f\|_{T_p}\leqslant C$ if $\|\phi\|_{T_p}\leqslant 1$.

Note that $\hat{f}$ is continuous if $f\in\mathcal{T}_p$. Clearly $\mathcal{T}_p$ satisfies (B1)–(B5). Taking $X = L_p$ we can therefore use our general theorems. Since we have already proved that $E^s = X^s$ if $X = L_p$, the interesting applications concern the spaces A^s_f and $A^s_f(\Omega)$. Using Corollaries 1 and 3 we get following result.

THEOREM 7. *Suppose that $f\in\mathcal{T}_p$ and that $\hat{f}$ satisfies the Tauberian condition. Moreover, assume that $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$, where $v\in\mathcal{T}_p$ and H is homo-*

geneous of order $t > 0$, infinitely differentiable and positive outside the origin.

Then $A_f^s = (L_p)^s$ for $0 < s < t$. If in addition for some m, such that $m \geqslant s$,

$$\|(\zeta^{(\mu)}f)^\wedge\|_{M_p} = O(\mu^{-m}), \qquad \mu \to \infty \tag{1}$$

then we also have that $A_f^s(\Omega) = (L_p)^s(\Omega)$.

This result is very similar to Theorem 5 but the conditions on f are weaker if $1 < p < \infty$. For instance, if $p = 2$ then the conditions of the theorem reads:

$\hat{f}$ is bounded, continuous and satisfies the Tauberian condition,

$\hat{v}$ is bounded, continuous.

(Use the fact that $T_2 = \hat{L}_\infty$ and the remarks of sections.)

It is amusing to look at condition (1) in the light of our theory. Note that $(\zeta^{(\mu)}f)^\wedge = \hat{\zeta}_{(\mu)} * \hat{f}$. Now take $B = \mathscr{M}$ and $X = M_p$! Then we get the following result from Corollary 1.

PROPOSITION 2. *Condition* (1) *above is satisfied if and only if there is an integer $M > m$ such that*

$$\|\Delta_{j,h}^M \hat{f}\|_{M_p} = O(h^m), \qquad h \to 0$$

for $j = 1, \ldots, d$.

In the rest of this section we shall consider the weight function $w(x) = (1+|x|)^\sigma$, σ real. This is merely for convenience. The results we shall give are valid for more general weight functions, namely those weight functions which are symmetric ($w(-x) = w(x)$) and polynomially regular in the sense defined in [5]. A general example of a weight function of this type is

$$w(x) = \prod_{j=1}^{k} (1+|A_j x|)^{\sigma_j},$$

where A_j are bounded linear mappings on R^d and σ_j are real numbers. Put $w_\sigma(x) = (1+|x|)^\sigma$ and assume that a is an integer such that $|\sigma| \leqslant a$. Let B be the algebra of all f such that $f_a(x) = x^a f(x) \in \mathscr{T}_p$ for $|a| \leqslant a$, with norm

$$\|f\|_B = \sum_{|a|\leqslant a} \|f_a\|_{T_p}.$$

Then (B1)–(B5) hold ((B3) follows as usual from Lemma 2). In [5] we have proved that $T_p(w_\sigma) = T_{p'}(w_{-\sigma})$. Thus it is sufficient to consider the case $\sigma > 0$. First let σ be an integer. Then

$$w_\sigma(x) = \sum_{|\nu|\leqslant\sigma} a_\nu(x-y)y^\nu,$$

where a_ν are polynomials such that $|a_\nu(x)| \leqslant A_\nu w_\sigma(x)$. Thus

$$w_\sigma(f*\varphi) = \sum_{|\nu|\leqslant\sigma} f_\nu*(a_\nu\varphi), \qquad f_\nu(y) = y^\nu f(y),$$

and therefore

$$\|w_\sigma(f*\varphi)\|_{L_p} \leqslant \sum_{|\nu|\leqslant\sigma} \|D^\nu \hat{f}\|_{M_p} \|a_\nu\varphi\|_{L_p}$$

$$\leqslant C \sum_{|\nu|\leqslant\sigma} \|D^\nu \hat{f}\|_{M_p} \|w_\sigma\varphi\|_{L_p}.$$

If σ is not an integer we use the Stein–Weiss interpolation theorem. In fact, assume that $\sigma = \sigma_0 + \theta$, $0 < \theta < 1$, where σ_0 is a non-negative integer. Then

$$\|w_{\sigma_0}(f*\varphi)\|_{L_p} \leqslant C \sum_{|\nu|\leqslant\sigma_0} \|D^\nu \hat{f}\|_{M_p} \|w_{\sigma_0}\varphi\|_{L_p},$$

$$\|w_{\sigma_0+1}(f*\varphi)\|_{L_p} \leqslant C \sum_{|\nu|\leqslant\sigma_0+1} \|D^\nu \hat{f}\|_{M_p} \|w_{\sigma_0+1}\varphi\|_{L_p}.$$

Therefore

$$\|w_\sigma(f*\varphi)\|_{L_p} \leqslant C \sum_{|\nu|\leqslant\sigma_0+1} \|D^\nu \hat{f}\|_{M_p} \|w_\sigma\varphi\|_{L_p}.$$

This proves that

$$\|w_\sigma(f*\varphi)\|_{L_p} \leqslant C\|f\|_B \|w_\sigma\varphi\|_{L_p}.$$

Thus (X1) (and clearly (X2)) holds true if $X = L_p(w_\sigma)$.

THEOREM 8. *Suppose that $w_\sigma(x) = (1+|x|)^\sigma$ and let a be an integer such that $a \geqslant |\sigma|$. Put $f_a(x) = x^a f(x)$ and assume that $f_a \in \mathscr{T}_p$ for $|a| \leqslant a$ and that $\hat{f}$ satisfies the Tauberian condition. Moreover, let $\hat{f}(\xi) = H(\xi)\hat{v}(\xi)$ in a neighbourhood of the origin. Here $D^a \hat{v} \in M_p$ for $|a| \leqslant a$ and H is homogeneous of order $t > a$, infinitely differentiable and positive outside the origin.*

Then $A_f^s = (L_p(w_\sigma))^s$ for $0 < s < t$. If, in addition, there is an integer M so that $M > s$, and

$$\|\Delta_{j,h}^M D^a \hat{f}\|_{M_p} = O(h^s), \qquad h \to 0,\ j = 1, \ldots, d,$$

for all $|a| \leqslant a$, then we also have $A_f^s(\Omega) = (L_p(w_\sigma))^s(\Omega)$.

EXAMPLE 5. Let J_ν denote the Bessel function of order ν. Put

$$g(x) = cJ_{\beta+d/2}(x)$$

and choose c so that

$$\int_{R^d} g(x)\,dx = 1.$$

Then

$$\hat{g}(\xi) = (1-|\xi|^2)_+^\beta.$$

It is well known that $D^\alpha \hat{g}$ belongs to the closure of $\mathcal{S}$ in the space M_p of Fourier multipliers on L_p, provided that $\beta > |\alpha| + (d-1)|p^{-1}-2^{-1}|$. Moreover,

$$\|\Delta_{j,h}^M D^\alpha \hat{g}\|_{M_p} = O(h^m), \quad h \to 0$$

if $M > m > 0$, provided that $\beta > |\alpha| + m + (d-1)|p^{-1}-2^{-1}|$. (For a proof see Löfström [4].)

Let a be 0, 1 or 2, and assume that $\beta > a + m + (d-1)|p^{-1}-2^{-1}|$. Put $f = g - \delta_0$. Then $f \in J_p$ and if $\alpha \neq 0$, $f_\alpha = g_\alpha$ belongs to the closure of $\mathcal{S}$ in T_p, hence

$$f_\alpha \in J_p \quad \text{if} \quad |\alpha| \geq a.$$

Moreover,

$$\hat{f}(\xi) = |\xi|^2 \hat{v}(\xi),$$

where $D^\alpha \hat{v} \in M_p$ for $|\alpha| \leq a$ (see [4]). Thus we get the following conclusion from Theorem 8. Take

$$\beta > a + m + (d-1)|p^{-1}-2^{-1}|$$

and choose s so that $0 < s \leq m$, $0 < s < 2$. Assume that

$$\Big(\int_{R^d} |\varphi(x)|^p (1+|x|)^{\sigma p} dx\Big)^{1/p} < \infty \quad \text{for some } |\sigma| \leq a.$$

Then

$$\Big(\int_{R^d} |g_{(\lambda)} * \varphi(x) - \varphi(x)|^p (1+|x|)^{\sigma p} dx\Big)^{1/p} = O(\lambda^{-s}), \quad \lambda \to \infty$$

if and only if

$$\Big(\int_{R^d} |\Delta_{j,h}^M \varphi(x)|^p (1+|x|)^{\sigma p} dx\Big)^{1/p} = O(h^s), \quad h \to 0$$

for some $M > s$, $(0 < s \leq m,\ 0 < s < 2)$.

We leave to the reader to write down localized versions of this result. (Cf. Löfström [4].)

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