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An integral extrapolation theorem with applications

by

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Abstract. Let T be a sublinear operator mapping the simple, integrable functions on a σ-finite measure space (X_1, μ_1) to the measurable functions on another such space (X_2, μ_2). Then, for certain growth function b, we have the inequality

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\Big[\int_0^t [s^{-1}\int_0^s f^*(u)\,b(s/u)du + \int_s^\infty f^*(u)\,b(u/s)du/u]\,ds\Big],$$

involving the nonincreasing rearrangements of Tf and f if and only if

$$\int_{X_2} |(Tf)(y)|^p\,d\mu_2(y) \leqslant B_p^p \int_{X_1} |f(x)|^p\,d\mu_1(x)$$

with

$$B_p = O\left(\frac{p^2}{p-1}\, b\left(\exp\frac{p^2}{p-1}\right)\right).$$

This result is applied to the study of convolution operators with kernels $K(x) = b(1/|x|)\cot\pi x$, $-1/2 < x < 1/2$.

1. Introduction. The first extrapolation theorem, due to S. Yano [10], appeared in 1951. It asserts that, for a certain kind of transformation T, mapping $L^1(a, b)$ to the measurable functions on (a, b), and a $k > 0$, there exist positive constants A_k and B_k, independent of f, such that

$$\int_a^b |(Tf)(y)|\,dy \leqslant A_k \int_a^b |f(x)|\log^k\big(1+|f(x)|^2\big)\,dx + B_k,$$

provided

$$\int_a^b |(Tf)(y)|^p dy \leqslant B_p^p \int_a^b |f(x)|^p dx, \tag{1.1}$$

with $B_p = O\big(1/(p-1)^k\big)$ also independent of f. Using this theorem, Yano proved the boundedness from $L\log L$ to L^1 of the classical maximal function and conjugate function operators, results of Hardy–Littlewood and A. Zygmund, respectively.

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In 1963, O'Neil and Weiss [8] proved an inequality for the conjugate function closely related to the $L\,\mathrm{Log}\,L\to L^1$ result. Given f in $L^1(-1/2, 1/2)$, we define its conjugate function $\tilde f$ by

$$\tilde f(y) = \lim_{\varepsilon\to 0+} \int_{\varepsilon\leqslant|x|\leqslant 1/2} f(y-x)\cot\pi x\,dx, \qquad -1/2<y<1/2.$$

Transferred from $(-\pi,\pi)$ to $(-1/2, 1/2)$, and modified somewhat, the inequality of O'Neil and Weiss reads as follows:

$$\int_0^t \tilde f^*(s)\,ds \leqslant C\int_0^t (P+Q)f^*(s)\,ds, \qquad 0<t<1. \tag{1.2}$$

Here, f^* and $\tilde f^*$ are the nonincreasing rearrangements of f and $\tilde f$, and

$$(Pf^*)(s) = s^{-1}\int_0^s f^*(u)\,du, \qquad (Qf^*)(s) = \int_s^1 f^*(u)\,du/u.$$

Calderón [3] later showed that (1.2) holds for any linear operator T, which, together with its transpose, is of weak types (1,1) and (2,2). Such operators have the B_p in (1.1) of order $p^2/(p-1)$, $1<p<\infty$. A special case of our extrapolation theorem shows this latter property characterizes the T for which Tf can replace $\tilde f$ in (1.2).

Theorem 3.1 given, for now, on the interval (0, 1) shows that for certain growth functions $b(t)$, defined on $(1,\infty)$, and operators T, the inequality

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\left[\int_0^t \left[s^{-1}\int_0^s f^*(u)\,b(s/u)\,du + \int_s^1 f^*(u)\,b(u/s)\,du/u\right]ds\right] \tag{1.3}$$

is equivalent to

$$\int_0^1 |(Tf)(y)|^p dy \leqslant B_p^p \int_0^1 |f(x)|^p dx, \quad \text{with} \quad B_p = O\left(\frac{p^2}{p-1}\,b\left(\exp\frac{p^2}{p-1}\right)\right).$$

An important example of the growth functions we consider is $b(t) = (\log_e t)^{\alpha-1}$, $\alpha>0$, which corresponds to the order $p^{2\alpha}/(p-1)^\alpha$. The result (1.3) for such power growth rates has been obtained independently by B. Jawerth in an abstract setting.

In Section 4, Theorem 3.1 is applied to the study of the class of convolution operators with kernels $K(x) = b(1/|x|)\cot\pi x$, $-1/2<x<1/2$. There, we will need a pointwise version of (1.2):

$$\tilde f^*(t) \leqslant C(P+Q)f^*(t), \qquad 0<t<1.$$

This is seen to follow from Theorem 8 of [3], with $p_1 = q_1 = p$, $p_2 = q_2 = p/(p-1)$, on letting $p\to 1+$, in view of the order properties of the weak type norm of the mapping $f\to\tilde f$. (See [5].) A direct proof, using the methods of [3], is given in [1].

As usual, the constant C appearing throughout is not necessarily the same at each occurrence.

2. The operators P_b and Q_b. Given f in the class S of simple, integrable functions on $(0, l)$, $0<l\leqslant\infty$, we define

$$(P_b f)(t) = t^{-1}\int_0^t f(s)\,b(t/s)\,ds$$

and

$$(Q_b f)(t) = \int_t^1 f(s)\,b(s/t)\,ds/s.$$

The function b associated with these operators is positive and Lebesgue-measurable on $(1,\infty)$.

Such generalizations of the well-known Hardy operators P and Q, mentioned in Section 1, have been studied by many authors. See [2], #3, for example. Our concern is to study the growth with p of $\|P_b+Q_b\|_p$, the norm of P_b+Q_b as an element of the space $[L^p]$ of bounded linear transformations from $L^p(0, l)$ to itself. This norm is, of course, determined over $f\in S$.

A famous result of Hardy, Littlewood, and Polya, [4], p. 230, implies, in this instance, that

$$\|P_b\|_p \leqslant \int_0^1 b(1/s)\,s^{-1/p}\,ds = \int_1^\infty b(s)\,s^{1/p-2}\,ds,$$

and $\qquad 1\leqslant p\leqslant\infty.$

$$\|Q_b\|_p \leqslant \int_1^\infty b(s)\,s^{-1/p-1}\,ds,$$

Hence, $P_b\in[L^\infty]$ and $Q_b\in[L^1]$ whenever

$$\int_1^\infty b(s)\,ds/s^2 < \infty,$$

a condition which from here on we impose on b.

Since the operators P_b and Q_b are adjoints of one another and the order function claimed for $\|P_b+Q_b\|_p$ remains the same on replacing p by $p/(p-1)$, it will be enough to consider $\|P_b\|_p$ as $p\to 1+$ in the proofs below.

LEMMA 2.1. *The norm of P_b in $[L^p]$ has at least the order of*

$$\int_1^{\exp(p^2/(p-1))} b(s)\,ds/s \quad \textit{as} \quad p\to 1+.$$

Proof. The Marcinkiewicz interpolation theorem, as proved in [9], shows $\|P_b\|_p$ is equivalent to the least constant B_p so that

$$\sup_{0<t<1} t^{1/p-1}\int_0^t (P_b X_E^*)(s)\,ds \leqslant B_p|E|^{1/p}, \tag{2.1}$$

whenever E is a Lebesgue-measurable subset of $(0, l)$ and $|E|$ its Lebesgue measure. The supremum in (2.1) occurs when $t > |E|$ and the contribution of the integral of $P_b X_E^*$ is substantially that of

$$\int_{|E|}^{t} ds/s \int_0^{|E|} b(s/u)\,du. \tag{2.2}$$

The substitution $u = sv$ in (2.2), followed by a change in the order of integration, yields

$$\int_0^{|E|/t} b(1/v)[t-|E|]\,dv + |E|\int_{|E|/t}^{1} b(1/v)[1/v-1]\,dv,$$

and so, effectively, $\|P_b\|_p$ is the largest possible value of

$$t^{1/p-1}\int_1^t b(s)\,ds/s, \tag{2.3}$$

when $t > 1$. Now take $t = \exp\left(p^2/(p-1)\right)$ in (2.3).

The next two results deal with cases in which b is monotone.

Proposition 2.2. *We have*

$$\|P_b + Q_b\| \simeq \frac{p^2}{p-1}\, b\left(\exp\frac{p^2}{p-1}\right), \quad 1 < p < \infty,$$

if either b is nonincreasing and differentiable with $\lim_{t\to\infty} tb'(t)/b(t) = 0$ and

$$b(t^{1/2}) \leqslant Mb(t), \quad t > 1, \tag{2.4}$$

where $1 < M < 2$, or b is a nondecreasing, concave function satisfying

$$b(t^2) \leqslant Cb(t), \quad t > 1, \tag{2.5}$$

for some $C > 0$.

Proof. By Lemma 2.1, $\|P_b\|_p$ is at least as large as stated. The representation obtained for $\|P_b\|_p$ in that lemma now reduces the proof to the assertion

$$\sup_{t>1} t^{1/p-1} b(t)\log t \leqslant C\frac{p^2}{p-1}\, b\left(\exp\frac{p^2}{p-1}\right), \quad 1 < p \leqslant 2. \tag{2.6}$$

This is obvious when b is nondecreasing. To see it is enough when b is nonincreasing note (2.4) yields

$$\int_1^t b(s)\,ds/s \leqslant \left[b(t^{1/2})\frac{\log t}{2} + b(t^{1/4})\frac{\log t}{4} + \ldots\right] \leqslant Cb(t)\log t.$$

Giving the details for the case b nondecreasing only, we show that as $p \to 1$ the supremum in (2.6) occurs at a t_p with

$$\exp\frac{p}{p-1} \leqslant t_p \leqslant \exp\frac{2p}{p-1}, \tag{2.7}$$

The inequalities

$$b(t) = b(n) + \int_n^t b'(u)\,du \leqslant b(n) + tb'(n), \quad t > n,$$

n a fixed positive integer, and

$$b(t) \geqslant Cb(t^2) \geqslant Ct^2 b'(t^2), \quad t > 2,$$

together imply

$$\lim_{t\to\infty}\frac{b(t)}{t} = \lim_{t\to\infty} b'(t) = 0.$$

This ensures that $b(t)$ increases slower than any power of t, since

$$\frac{b(t)}{t^{2^{-n}}} \leqslant C^n\frac{b(t^{2^{-n}})}{t^{2^{-n}}}$$

for any positive integer n. Thus, $1 < t_p < \infty$.

Setting the derivative of $t^{1/p-1}b(t)\log t$ equal to 0 we find

$$t_p = \exp\frac{p}{p-1}\left(1 + t_p\frac{b'(t_p)}{b(t_p)}\right). \tag{2.8}$$

The bounds (2.6) on t_p now follow, as (2.7) entails $\lim_{p\to\infty} t_p = \infty$ and

$$\frac{tb'(t)}{b(t)} \leqslant 1 - \frac{b(2) - 2b'(t)}{b(t)} \leqslant 1$$

for sufficiently large t.

Examples. The functions $b(t) = (\log_e t)^{\alpha-1}$, $\alpha > 0$, satisfy the hypotheses of Proposition 2.2, as do those of the form $b(t) = s(\log t)$, where $s(u)$ is increasing and slowly varying.

In the proof of Theorem 4.1, certain auxilliary operators, involving b, arise. For $f \in S$ they are defined by

$$(P^b f)(t) = \frac{b(1/t)}{t}\int_0^t f(s)\,ds$$

and

$$(Q^b f)(t) = \int_t^1 f(s)\,b(1/s)\frac{ds}{s}.$$

LEMMA 2.3. *Suppose $b(t)$ is nonincreasing on $[1, \infty)$ and $b(2t) \leqslant Cb(t)$ there. Then, whenever $f \in S$ and $0 < t < 1$,*

(i) $$\int_0^t (P^b f^*)(s)\,ds \leqslant C\int_0^t (P_b f^*)(s)\,ds,$$

(ii) $$\int_0^t (Q^b f^*)(s)\,ds \leqslant C\int_0^t (Q_b f^*)(s)\,ds.$$

Proof. Inequality (i) is equivalent to

$$\int_0^t f(s)\,\Phi_1(s)\,ds \leqslant \int_0^t f(s)\,\Phi_2(s)\,ds,$$

where

$$\Phi_1(s) = \int_{s/t}^1 b(1/tu)\,du/u \quad \text{and} \quad \Phi_2(u) = \int_{s/t}^1 b(1/u)\,du/u.$$

But, b is nonincreasing, so $\Phi_1(s) \leqslant \Phi_2(s)$, $0 < s < t$.

By Theorem 7 of [3] it is enough to prove (ii) when $f = \chi_E$. Applying Fubini's theorem to the integrals in (ii), it is seen the proposed inequality will hold if

(2.9) $$\int_0^t b(1/u)\,du \leqslant Ct\int_0^1 b(1/u)\,du$$

and

(2.10) $$t\int_t^{|E|} b(1/u)\,du/u \leqslant C\int_t^{|E|} du/u\int_0^t b(u/s)\,ds,$$

for $0 < t \leqslant |E|$. Now, (2.9) is true since b is bounded. Again, observing that $b(1/u) \leqslant b(|E|/u)$ and $\int_0^t b(u/s)\,ds \geqslant Ctb(u/t)$, we see inequality (2.10) follows if

(2.11) $$\int_t^{|E|} b(|E|/u)\,du/u \leqslant \int_t^{|E|} b(u/t)\,du/u.$$

In fact, simple changes of variables show the two integrals in (2.11) are equal.

3. The extrapolation theorem.

THEOREM 3.1. *Let T be a sublinear operator from the class S of simple, integrable functions on a σ-finite, non-atomic measure space (X_1, μ_1) to the measurable ones on another such space (X_2, μ_2). Suppose b is a positive function on $(1, \infty)$ which is either decreasing with $b(t^{1/2}) \leqslant Mb(t)$, $t > 1$, $1 < M < 2$, or increasing and concave with $b(t^2) \leqslant Cb(t)$, $t > 1$. Then, the following are equivalent:*

(1) *There is a positive constant C, independent of $f \in S$, such that*

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\int_0^t (P_b + Q_b)f^*(s)\,ds, \qquad t > 0.$$

(2) $$\int_{X_2} |(Tf)(y)|^p\,d\mu_2(y) \leqslant B_p^p \int_{X_1} |f(x)|^p\,d\mu_1(x), \qquad f \in S,$$

where

$$B_p = O\left(\frac{p^2}{p-1}\,b\left(\exp\frac{p^2}{p-1}\right)\right), \qquad 1 < p < \infty.$$

Proof. That (1) implies (2) is a consequence of [6], Lemma 1, and Proposition 2.2.

To prove the converse we show there is a positive constant C so that for any set E of finite μ_2-measure

(3.1) $$\int_E |(Tf)(y)|\,d\mu_2(y) \leqslant C\int_0^{\mu_2(E)} (P_b + Q_b)f^*(s)\,ds,$$

which is sufficient, as

$$\int_0^t g^*(s)\,ds = \sup_{\mu_2(E)\leqslant t}\int_E |g(y)|\,d\mu_2(y).$$

Fix the set E. By Fubini's theorem

$$\int_0^{\mu_2(E)} (P_b+Q_b)f^*(s)\,ds = \int_0^\infty f^*(s)\Phi(s)\,ds,$$

where $\Phi(s)$ denotes the nonincreasing function

$$(P_b+Q_b)\chi_{(0,\mu_2(E))}(s).$$

Therefore, (3.1) amounts to showing T is bounded from the Lorentz space $\Lambda(\Phi)$ (with underlying measure space (X_1,μ_1)) to the Banach space of functions which are μ_2-integrable on E. We may restrict attention to $f=\chi_F$, F a set of finite μ_1-measure. See [3], Theorem 7.

Now,

$$\int_E |(T\chi_F)(y)|\,d\mu_2(y) \leqslant \Big[\int_{X_2} |(T\chi_F)(y)|^p d\mu_2(y)\Big]^{1/p}\mu_2(E)^{1-1/p},$$

and hence, from (2),

$$\int_E |(T\chi_F)(y)|\,d\mu_2(y) \leqslant B_p\mu_1(F)^{1/p}\mu_2(E)^{1-1/p}.$$

Thus, we require

$$B_p\mu_1(F)^{1/p}\mu_2(E)^{1-1/p} \leqslant C\int_0^{\mu_1(F)} \Phi(s)\,ds \tag{3.2}$$

for some $p>1$. We consider two cases, depending on the size of $a=\mu_2(E)/\mu_1(F)$.

(i) $a\geqslant 1$. The integral on the right side of (3.2) is no smaller than

$$\mu_1(F)\Big[\int_0^1 b(1/s)\,ds + \int_1^a b(s)\,ds/s\Big].$$

So, we need only show

$$\frac{p^2}{p-1}\,b\left(\exp\frac{p^2}{p-1}\right) \leqslant Ca^{1/p-1}\int_1^a b(s)\frac{ds}{s},$$

when $a>e$. The choice $p=1+1/\log a$ works, in view of (2.4) and the equivalence of the functions $\int_1^t b(s)\frac{ds}{s}$ and $b(t)\log t$.

(ii) $a<1$. Using symmetry, we obtain

$$\int_0^{\mu_1(F)} \Phi(s)\,ds = \int_0^{\mu_2(E)} (P_b+Q_b)\chi_{(0,\mu_1(F))}(s)\,ds,$$

the latter integral being bounded below by

$$\mu_2(E)\Big[\int_0^1 b(1/s)\,ds + \int_1^{a^{-1}} b(s)\,ds/s\Big].$$

This reduces the result to the inequality

$$\frac{p^2}{p-1}\,b\left(\exp\frac{p^2}{p-1}\right) \leqslant Ca^{1/p}\int_1^{a^{-1}} b(s)\frac{ds}{s},$$

which holds when $a<e^{-2}$, if we take $p=\log 1/a$.

4. Convolution operators. O'Neil, in [7], uses a generalization of (1.2) to prove, modulo a change of variable, that odd kernels of the form $K(x)=b(1/|x|)\cot\pi x$ define bounded convolution operators between certain pairs of Orlicz spaces on $(-1/2,1/2)$. Given Theorem 3.1, these results suggest the following

THEOREM 4.1. *Let $b(x)$ be a differentiable function on $[1,\infty)$ which decreases to 0 and which, together with $-xb'(x)$, is slowly varying and for which $b(x^{1/2})\leqslant Mb(x)$, $1<M<2$. Then the convolution operator*

$$(Tf)(y) = \lim_{\varepsilon\to 0+}\int_{\varepsilon\leqslant|x|\leqslant 1/2} f(y-x)K(x)\,dx$$

with odd kernel $K(x)=b(1/|x|)\cot\pi x$, satisfies the inequality

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\int_0^t (P_b+Q_b)f^*(s)\,ds, \qquad 0<t<1, \tag{4.1}$$

C a positive constant independent of $f\in S$.

The conditions put on the kernel K in [7] enable one to express it as a conjugate function. More precisely, $K=\tilde{k}$, where k is even, integrable, and decreasing on $(0,1/2)$. We now obtain something close to this basic representation of K under our somewhat different assumptions on b. These guarantee (see [11], p. 189) the well-known asymptotic estimates

$$\text{(i)}\qquad \sum_{n=1}^\infty b(n)\sin 2n\pi x \simeq \frac{1}{\pi}x^{-1}b(1/x),$$

$$\text{(ii)}\qquad \sum_{n=1}^\infty b(n)\cos 2n\pi x \simeq -\frac{1}{2}x^{-2}b'(1/x)$$

as $x\to 0+$.

We require more information on the error term in (i).

LEMMA 4.2. *If $b(x)$ satisfies the hypotheses of Theorem 4.1, then*

$$b(1/|x|)\cot\pi x = \sum_{n=1}^{\infty} b(n)\sin 2n\pi x + R(x),$$

where $R(x) = O\big(-x^{-2}b(1/|x|)\big)$, and hence is integrable on $(-1/2, 1/2)$.

Proof. It is enough to consider $x > 0$. We have

$$\sum_{n=1}^{\infty} b(n)\sin 2n\pi x = \tfrac{1}{2}\cot\pi x \sum_{n=1}^{\infty}\big(b(n)-b(n+1)\big)(1-\cos 2n\pi x) + O(1).$$

with

$$\sum_{n=1}^{[1/x]}\big(b(n)-b(n+1)\big)(1-\cos 2n\pi x) \leqslant C[1/x]^{-2}\sum_{n=1}^{[1/x]}\big(b(n)-b(n+1)\big)n^2.$$

The latter is no bigger than

$$C[1/x]^{-2}\sum_{n=1}^{[1/x]} -b'(n)n^2 \leqslant -C[1/x]b'([1/x]) = O\big(-x^{-1}b'(1/x)\big),$$

since $-xb'(x)$ is slowly varying. Again,

$$\sum_{n=[1/x]+1}^{\infty}\big(b(n)-b(n+1)\big)(1-\cos 2n\pi x)$$

equals

$$b(1/x) - \sum_{n=[1/x]+1}^{\infty}\big(b(n)-b(n+1)\big)\cos 2n\pi x + O(1).$$

The function $-x\big(b(x)-b(x+1)\big)$ is equivalent to $-xb'(x)$, and so, as in the proof of 2.15, p. 189 of [11], it is seen that

$$\sum_{n=[1/x]+1}^{\infty}\big(b(n)-b(n+1)\big)\cos 2n\pi x = O\big(x^{-1}[b(1/x)-b(1/x+1)]\big)$$
$$= O\big(-x^{-1}b'(1/x)\big).$$

Remark. The above lemma and (4.2), (ii), show K is the sum of the conjugate of an integrable function and a function which is itself in L^1. That the sum function of the series (i) in (4.2) is indeed conjugate to the one of series (ii) follows from the second part of the proof of Theorem 1.5, p. 183, and Theorem 3.25, p. 90, of [11].

Proof of Theorem 4.1. We show

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\int_0^t (P^b+QQ^b)f^*(s)\,ds, \quad 0<t<1. \tag{4.3}$$

By Lemma 2.3,

$$\int_0^t P^b f^*(s)\,ds \leqslant \int_0^t P_b f^*(s)\,ds \quad \text{and} \quad QQ^b f^* \leqslant CQ^2 f^*.$$

As Q is bounded on $L^p(0,1)$ for $1 \leqslant p < \infty$, this implies T has

$$B_p = O\left(\frac{p^2}{p-1}\,b\left(\exp\frac{p^2}{p-1}\right)\right) \quad \text{as} \quad p \to 1+.$$

But, T is essentially self-adjoint and so B_p has that order in general. This ensures (1.3), in view of Theorem 3.1.

The remark following Lemma 4.2, together with Young's convolution theorem, means it is enough to obtain (4.3) when T is defined to be the conjugate function of g, where

$$g(u) = (k^*f)(u) = \int_{-1/2}^{1/2} k(u-t)f(t)\,dt,$$

$$k(x) = \sum_{n=1}^{\infty} b(n)\cos 2n\pi x = O\big(x^{-2}b'(1/|x|)\big), \tag{4.4}$$

being even and integrable. From the pointwise inequality for the conjugate function operator,

$$(Tf)^*(s) \leqslant C(P+Q)g^*(s), \quad 0<s<1,$$

and so

$$P(Tf)^*(s) \leqslant CP(P+Q)g^*(s) = C(P+Q)Pg^*(s).$$

The basic lemma on convolution operators [7], p. 134, now yields

$$\int_0^t (Tf)^*(s)\,ds \leqslant C\Big[\int_0^t J(s)\,ds + t\int_t^1 J(s)\,ds/s\Big],$$

where

$$J(s) = \Big[s^{-1}\int_0^s f^*(u)\,du\Big]\Big[\int_0^s k^*(u)\,du\Big] + \int_s^1 f^*(u)k^*(u)\,du.$$

From (4.4) and

$$b(1/s) = \int_0^s -u^{-2}b'(1/u)\,du \geqslant -s^{-1}b'(1/s)$$

we get $J(s)$ dominated by a constant multiple of

$$(P^b+Q^b)f^*(s).$$

This gives the right bound for $\int_0^t J(s)\,ds$, because

$$\int_0^t Q^b f^*(s)\,ds \leqslant C\int_0^t QQ^b f^*(s)\,ds.$$

Again, $t\int_t^1 J(s)\,ds/s$ equals t times

$$\int_0^t f^*(u)\,du \int_t^1 b(1/s)\,ds/s^2 + \int_t^1 f^*(u)\,du \int_u^1 b(1/s)\,ds/s^2 + QQ^b f^*(t),$$

which is less than a constant multiple of

$$t(P^b+Q^b+QQ^b)f^*(t) \leqslant \int_0^t (p^b+Q^b+QQ^b)f^*(s)\,ds.$$

This completes the proof.

Remarks. 1. With the pointwise inequality for the conjugate function operator in mind, one might expect to have

$$(Tf)^*(t) \leqslant C(P_b+Q_b)f^*(t), \quad 0<t<1$$

for all $f \in S$. But, assuming this to be true and taking $f = n\chi_{(0,1/n)}$ for $n = 1, 2, 3, \ldots$ successively, we obtain

$$K^*(t) \leqslant C\lim_n b(nt) = 0, \quad 0<t<1.$$

2. Though the P_b and Q_b in (4.1) are the best possible choice from the class of maximal operators reflecting just the growth of the norm of T, they don't yield all its characteristic properties. Thus, for example, P_b+Q_b is not bounded between all the Orlicz spaces that are continuous pairs for T; the P^b+Q^b, and even P^b+QQ^b, operators, however, are. This, along with (4.3), suggests that, using other methods, $(P^b+Q^b)f^*(s)$ could be substituted in the integrand on the right side of (4.1). We hope to consider this in a future paper.

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