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On the existence of unitary representations of commutative nuclear Lie groups

by

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Abstract. A proof is given that if Γ is a discrete subgroup of a nuclear space X, then the quotient group X/Γ admits sufficiently many continuous characters.

In many situations nuclear spaces seem to be a more adequate generalization of finite-dimensional spaces than are normed spaces. Indeed, many important facts concerning finite-dimensional spaces remain valid in nuclear spaces but not in infinite-dimensional normed spaces. An example of this kind is given in the present paper.

Let us consider the following property of a topological vector space X:

(*) If Γ is a discrete subgroup of X, then the quotient group X/Γ admits sufficiently many continuous characters.

(The terminology is explained below.) Every finite-dimensional space X satisfies (*), which is trivial, and no infinite-dimensional normed space X satisfies (*), which has been proved in [1]. We shall prove here that every nuclear space X satisfies (*).

We begin with some notation and terminology. $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$ will denote the sets of positive integers, integers, reals and complexes, respectively. Vector spaces will often be regarded as additive topological groups. If A is a subset of a vector space X, then GA will denote the group generated by A, and $\operatorname{span} A$ — the linear span of A. The distance from a point u to a set A will be denoted by $d(u, A)$. For a topological vector space X the conjugate space will be denoted by X^*.

Let E be a real Hilbert space, and let $u_1, \ldots, u_n \in E$. Then Gram $(u_1, \ldots, u_n)$ will denote the Gram determinant of the vectors $u_1, \ldots, u_n$. If E is n-dimensional, and if K is a discrete subgroup of E which spans E, then K is an abelian free group with n linearly independent generators $u_1, \ldots, u_n$, and the number Gram$(u_1, \ldots, u_n)$ does not depend on the choice of generators; we denote this number by Gram K. A subgroup K of a Hilbert space will be called *r-discrete* if $\|u-w\| \geqslant r$ for any distinct $u, w \in K$.

Let G be a topological group. By a *character* of G we mean a homomorphism of G into the multiplicative group $\{z \in \boldsymbol{C}: z\bar{z} = 1\}$. We say that G

admits *sufficiently many* continuous characters if for any $1 \neq g \in G$ there is a continuous character χ of G such that $\chi(g) \neq 1$.

By a *unitary representation* of G we mean a homomorphism $V\colon G \to U(H)$ into the group of unitary operators on some Hilbert space H. V is called *faithful* when $\ker V = \{1\}$. A representation V is called *strongly* (*uniformly*) *continuous* when it is continuous in the strong (uniform) topology on $U(H)$.

LEMMA 1. *If K is an r-discrete subgroup of $\boldsymbol{R}^n$ which spans $\boldsymbol{R}^n$, then* $\operatorname{Gram} K \geqslant r^{2n} n^{-n}$.

Proof. Choose any generators $u_1, \ldots, u_n$ of K, and let

$$P = \Big\{ \sum_{k=1}^{n} t_k u_k\colon 0 \leqslant t_1, \ldots, t_n \leqslant 1 \Big\}.$$

Let m be the n-dimensional Lebesgue measure on $\boldsymbol{R}^n$, and B — the ball $\|u\| < r/2$. We have

$$\operatorname{Gram} K = \operatorname{Gram}(u_1, \ldots, u_n) = m^2(P),$$

and

$$\begin{aligned} m(P) &\geqslant m\Big(P \cap \bigcup (B+u)\Big) = m\Big(\bigcup [P \cap (B+u)]\Big) \\ &= \sum m\big(P \cap (B+u)\big) = \sum m\big((P+u) \cap B\big) \\ &\geqslant m\Big(\bigcup [(P+u) \cap B]\Big) = m\Big(B \cap \bigcup (P+u)\Big) = m(B \cap \boldsymbol{R}^n) = m(B), \end{aligned}$$

where all the sums are taken over all $u \in K$. As can easily be seen, $m(B) \geqslant r^n n^{-n/2}$, which completes the proof.

LEMMA 2. *Let E_1 and E_2 be real Hilbert spaces, and let $T\colon E_2 \to E_1$ be an infinite-dimensional linear compact operator. Let $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant \ldots$ be the full sequence of positive eigenvalues of the operator $(T^*T)^{1/2}$. Then for any $n \in \boldsymbol{N}$ and any $u_1, \ldots, u_n \in E_2$*

$$\operatorname{Gram}(Tu_1, \ldots, Tu_n) \leqslant \lambda_1^2 \ldots \lambda_n^2 \operatorname{Gram}(u_1, \ldots, u_n).$$

We omit the proof of this well-known fact.

LEMMA 3. *Under the assumptions of Lemma 2, suppose that $\lambda_k \leqslant e^{-2}k^{-2}$ for $k \in \boldsymbol{N}$, and let L be a discrete subgroup of E_2 such that* $\dim \operatorname{span} L = n < \infty$, $\operatorname{span} L \cap \ker T = \{0\}$, *and $T(L)$ is a 1-discrete subgroup of E_1. Then we can choose generators $w_1, \ldots, w_n$ of L such that*

$$d(w_k, \operatorname{span}\{w_i\}_{i<k}) \geqslant k \quad \textit{for} \quad k = 1, \ldots, n.$$

Proof. Choose any generators $u_1, \ldots, u_n$ of L. Since $\operatorname{span} L \cap \ker T = \{0\}$, the vectors $Tu_1, \ldots, Tu_n$ are linearly independent generators of the 1-discrete group $T(L)$. Thus, according to Lemma 1,

$$\operatorname{Gram}(Tu_1, \ldots, Tu_n) = \operatorname{Gram} T(L) \geqslant n^{-n}.$$

On the other hand, from Lemma 2 we obtain

$$\operatorname{Gram}(Tu_1, \ldots, Tu_n) \leqslant \lambda_1^2 \ldots \lambda_n^2 \operatorname{Gram}(u_1, \ldots, u_n).$$

Hence

$$\operatorname{Gram}(u_1, \ldots, u_n) \geqslant n^{-n} e^{4n} (n!)^4 \geqslant n^{3n},$$

because $n! \geqslant n^n e^{-n}$. Let $u_1^*, \ldots, u_n^* \in (\operatorname{span} L)^*$ be defined by $u_i^*(u_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. We have

$$\operatorname{Gram}(u_1^*, \ldots, u_n^*) = [\operatorname{Gram}(u_1, \ldots, u_n)]^{-1} \leqslant n^{-3n}.$$

From Lemma 1 easily follows the existence of a $w^* \in G\{u_k^*\}_{k=1}^n$ such that

$$0 < \|w^*\|^2 \leqslant n [\operatorname{Gram}(u_1^*, \ldots, u_n^*)]^{1/n};$$

hence $\|w^*\| \leqslant n^{-1}$. Let $w_n \in L$ be any element for which $w^*(w_n) = 1$ (if $w^*(L) = p\boldsymbol{Z}$ for some $p = 2, 3, \ldots$, then we would take w^*/p instead of w^*). Then $d(w_n, \ker w^*) = \|w^*\|^{-1} \geqslant n$, and $L = \boldsymbol{Z}w_n \oplus L_{n-1}$, where $L_{n-1} = L \cap \ker w^*$. Obviously $T(L_{n-1})$ is a 1-discrete subgroup of $T(\ker w^*)$. Applying the above argument to the group L_{n-1}, we shall obtain some $w_{n-1} \in L_{n-1}$ and a subgroup $L_{n-2} \subset L_{n-1}$, such that $d(w_{n-1}, \operatorname{span} L_{n-1}) \geqslant n-1$ and $L_{n-1} = \boldsymbol{Z}w_{n-1} \oplus L_{n-2}$. Proceeding by induction, we shall obtain elements $w_n, \ldots, w_1$, which satisfy the desired conditions. ∎

LEMMA 4. *Under the assumptions of Lemma 2, suppose that $\lambda_k \geqslant e^{-2}k^{-2}$ for $k \in \boldsymbol{N}$, and let K be a subgroup of E_2, such that* $\operatorname{span} K \cap \ker T = \{0\}$ *and $T(K)$ is a 1-discrete subgroup of E_1. Choose an arbitrary $a \in E_2 \setminus K$, put $d = d(a, K)$ and $t = \min(1/4, d/2)$. Then there is an $f_a \in E^*$ such that $\|f_a\| \leqslant 3$, $f_a(K) \subset \boldsymbol{Z}$, and $f_a(a) \in \langle t, 1-t \rangle + \boldsymbol{Z}$.*

Proof. Consider the sets

$$F_u = \{f \in E_2^*\colon f(u) \in \boldsymbol{Z}\}$$

for $u \in K$, and the set

$$F_a = \{f \in E_2^*\colon f(a) \in \langle t, 1-t \rangle + \boldsymbol{Z}\}.$$

Let B be the ball $\|f\| \leqslant 3$ in E_2^*; we shall prove that for each finite subset $J \subset K$ the intersection

$$B \cap F_a \cap \bigcap_{u \in J} F_u$$

is not empty.

To do this, choose any finite subset $J \subset K$. There are linearly independent $u_1, \ldots, u_n \in K$ with $L = G\{u_k\}_{k=1}^n \supset J$. By Lemma 3 we can choose generators $w_1, \ldots, w_n$ of L, such that

$$d(w_k, \operatorname{span}\{w_i\}_{i<k}) \geqslant k \quad \text{for} \quad k = 1, \ldots, n.$$

Let $e_1, \ldots, e_{n+1}$ be the system obtained by orthonormalization of the vectors $w_1, \ldots, w_n, a$. We may write

$$w_k = \sum_{i=1}^{k} w_{ki} e_i \quad \text{for} \quad k = 1, \ldots, n,$$

and

$$a = \sum_{i=1}^{n+1} a_i e_i.$$

We may assume that $w_{kk} > 0$ for $k = 1, \ldots, n$; then

$$w_{kk} = d(w_k, \operatorname{span}\{w_i\}_{i<k}) \geqslant k \quad \text{for} \quad k = 1, \ldots, n.$$

As can easily be seen, there is a $w_0 \in L$ such that if

$$a - w_0 = b = \sum_{i=1}^{n+1} b_i e_i,$$

then $|b_i| \leqslant w_{ii}/2$ for $i = 1, \ldots, n$. Obviously $\|b\| \geqslant d$. Now there are two possibilities. If $|b_{n+1}| \geqslant d/2$, then the mapping

$$E_2 \ni u \mapsto t|b_{n+1}|^{-1}(u, e_{n+1}),$$

where (u, e_{n+1}) is the scalar product, obviously belongs to

$$B \cap F_a \cap \bigcap_{u \in L} F_u \subset B \cap F_a \cap \bigcap_{u \in J} F_u.$$

If, on the other hand, $|b_{n+1}| < d/2$, then, taking $b' = \sum_{i=1}^{n} b_i e_i$, we must have $\|b'\| > d/2$. There is a linear functional h on $\operatorname{span} L$ such that $\|h\| \leqslant 1$ and $h(b') \in \langle t, 1-t \rangle$. Let $h_i = h(e_i)$ for $i = 1, \ldots, n$. As can easily be seen, there is an $f_1 \in \boldsymbol{R}$ such that $f_1 w_{11} \in \boldsymbol{Z}$, $|f_1 - h_1| \leqslant w_{11}^{-1}$, and

$$f_1 b_1 + \sum_{k=2}^{n} h_k b_k \in \langle t, 1-t \rangle.$$

Next, there is an $f_2 \in \boldsymbol{R}$ such that $f_1 w_{21} + f_2 w_{22} \in \boldsymbol{Z}$, $|f_2 - h_2| \leqslant w_{22}^{-1}$, and

$$f_1 b_1 + f_2 b_2 + \sum_{k=3}^{n} h_k b_k \in \langle t, 1-t \rangle.$$

Proceeding by induction we shall find $f_1, \ldots, f_n \in \boldsymbol{R}$ such that

$$f_1 b_1 + \ldots + f_n b_n \in \langle t, 1-t \rangle,$$

$$\sum_{i=1}^{k} f_i w_{ki} \in \boldsymbol{Z} \quad \text{for} \quad k = 1, \ldots, n,$$

and

$$|f_i - h_i| \leqslant w_{ii}^{-1} \quad \text{for} \quad i = 1, \ldots, n.$$

Let f' be the linear functional on $\operatorname{span}\{e_i\}_{i=1}^{n+1}$ defined by $f'(e_i) = f_i$ for $i = 1, \ldots, n$, and $f'(e_{n+1}) = 0$. Then $f'(b') \in \langle t, 1-t \rangle$ and $f'(L) \subset \boldsymbol{Z}$. Hence

$$f'(a) = f'(b + w_0) = f'(b') + f'(w_0) \in \langle t, 1-t \rangle + \boldsymbol{Z}.$$

Moreover,

$$\|f'\| \leqslant \|h\| + \|f' - h\| \leqslant 1 + \Big[\sum_{i=1}^{n} |f_i - h_i|^2\Big]^{1/2}$$

$$\leqslant 1 + \Big[\sum_{i=1}^{n} w_{ii}^{-2}\Big]^{1/2} \leqslant 1 + \Big[\sum_{i=1}^{n} i^{-2}\Big]^{1/2} \leqslant 3.$$

We may extend f' to an $f \in E_2^*$, with $\|f\| \leqslant 3$. Then

$$f \in B \cap F_a \cap \bigcap_{u \in J} F_u.$$

Since J was an arbitrary finite subset of K, and the sets F_a and F_u, $u \in K$, are weakly closed, from the weak compactness of B follows the existence of an

$$f_a \in B \cap F_a \cap \bigcap_{u \in K} F_u. \ \blacksquare$$

THEOREM. *Let Γ be a discrete subgroup of a real nuclear space X. Then Γ is an at most countably generated abelian free group, and the quotient group X/Γ admits sufficiently many continuous characters. Moreover, if the topology of X can be defined by a family of norms, then X/Γ admits a faithful uniformly continuous unitary representation.*

Proof. We may assume that $\dim \operatorname{span} \Gamma = \infty$. Take any $x_0 \in X \setminus \Gamma$. Since Γ is a discrete subgroup of X, and $x_0 \notin \Gamma$, there is a neighbourhood U of zero such that $U \cap \Gamma = \{0\}$ and $(x_0 + U) \cap \Gamma = \emptyset$. Then there is a continuous seminorm p_1 on X, such that $\{x: p_1(x) < 1\} \subset U$, and the space $X/p_1^{-1}(0)$ with the norm $\|[x]\|_1 = p_1(x)$ is a prehilbert space. Let E_1 be the completion of $X/p_1^{-1}(0)$. Since X is a nuclear space, there exists another continuous seminorm $p_2 \geqslant p_1$ on X, such that if E_2 is the corresponding Hilbert space, and $T: E_2 \to E_1$ — the natural nuclear operator, and if $\lambda_1 \geqslant \ldots \geqslant \lambda_k \geqslant \ldots$ is the full sequence of positive eigenvalues of the operator $(T^*T)^{1/2}$, then $\lambda_k \leqslant e^{-2} k^{-2}$ for $k \in \boldsymbol{N}$. We obtain the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} \\ E_2 & \xrightarrow{T} & E_1 \end{array}$$

where φ and ψ are the natural mappings.

Since $\{x\colon p_1(x)<1\}\subset U$ and $U\cap\Gamma=\{0\}$, $\varphi(\Gamma)$ is a 1-discrete subgroup of E_1, and $\Gamma\cap\ker\varphi=\{0\}$. We shall prove that

$$\text{(i)} \qquad \operatorname{span}\Gamma\cap\ker\varphi=\{0\}.$$

Suppose the contrary; then there are linearly independent $x_1,\ldots,x_n\in\Gamma$ and non-zero $a_1,\ldots,a_n\in\boldsymbol{R}$ such that $\varphi(a_1u_1+\ldots+a_nu_n)=0$. Let $M=\operatorname{span}\{x_k\}_{k=1}^n$; then $\dim M=n$, but $\dim\varphi(M)<n$. $\varphi(\Gamma\cap M)$ is a 1-discrete subgroup of $\varphi(M)$; therefore it is a group generated by m linearly independent elements, where $m\leqslant\dim\varphi(M)<n$. Since $\Gamma\cap\ker\varphi=\{0\}$, the mapping $\varphi|_{\Gamma\cap M}$ is a monomorphism of the group $\Gamma\cap M$ onto the group $\varphi(\Gamma\cap M)$, but it is impossible to map monomorphically $\boldsymbol{Z}^n$ into $\boldsymbol{Z}^m$ with $m<n$. The resulting contradiction proves (i).

As a discrete subgroup of a separable normed space E_1, the group $\varphi(\Gamma)$ is a free abelian group with a separable set of linearly independent generators, and, by (i), so is Γ.

Let $K=\psi(\Gamma)$. Since $T\psi=\varphi$, (i) implies that $\operatorname{span}K\cap\ker T=\{0\}$. Next, $T(K)=\varphi(\Gamma)$ is a 1-discrete subgroup of E_1. Let $a=\psi(x_0)$. By Lemma 4 there is an $f_a\in E_2^*$ such that $f_a(K)\subset\boldsymbol{Z}$ and $f_a(a)\notin\boldsymbol{Z}$. The mapping

$$X\ni x\mapsto\exp[2\pi i f_a\psi(x)]$$

is then a continuous character of the topological group X, trivial on Γ. It determines a continuous character χ of X/Γ, such that $\chi([x_0])\neq 1$. Thus, since $x_0\in X\setminus\Gamma$ was arbitrary, we have proved that X/Γ admits sufficiently many continuous characters.

Assume now that the topology of X can be defined by a family of norms. Then we may assume that p_2 is a norm. For every $a\in E_2\setminus K$ let χ_a be the character

$$E_2\ni u\mapsto\exp[2\pi i f_a(u)],$$

where f_a is the functional constructed above. The Hilbert sum V of the characters χ_a, $a\in E_2\setminus K$, is then a uniformly continuous unitary representation of E_2, and $\ker V=K$. The mapping $V\psi$ is a representation of X, and $\ker V\psi=\psi^{-1}(K)=\Gamma$, because p_2 is a norm. Thus we obtain a faithful uniformly continuous unitary representation of X/Γ. ∎

Remark. If a topological group admits sufficiently many continuous characters, then it also admits a faithful strongly continuous unitary representation (the Hilbert sum of those characters). On the other hand, there are groups of the form X/Γ, where Γ is a discrete subgroup of a nuclear space X, which do not admit any faithful uniformly continuous unitary representations. For example, let $X=\boldsymbol{R}^N$ and $\Gamma=\{0\}$. Any faithful uniformly continuous unitary representation of $\boldsymbol{R}^N$ would lead to an injective continuous linear operator acting from $\boldsymbol{R}^N$ to the Banach space of all real measurable essentially bounded functions on $(0,1)$ (see [2], Theorem 5), which is impossible.

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