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Continuous selections for a class of non-convex multivalued maps

by

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Abstract. Let S and T be compact spaces, Z a separable Banach space and $L_1(T, Z)$ the Banach space of μ_0-integrable functions $u: T \to Z$, where μ_0 is a nonnegative regular normed Borel measure on T.

We say that the multivalued map $K: S \to 2^{L_1(T,Z)}$ is decomposable if for each $s \in S$

(P) $\quad u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K(s)$ for each $u, v \in K(s)$ and A μ_0-measurable.

We prove the following generalization of a recent theorem of Antosiewicz and Cellina:

Assume that $K: S \to \mathrm{cl} L_1(T, Z)$ is decomposable and lower semicontinuos. Then there exists a countable family of continuous selections $k_n: S \to L_1(T, Z)$ such that

$$K(s) = \mathrm{cl}\{k_n(s),\ n = 1, 2, \ldots\}.$$

Introduction. Let $(T, \mathfrak{M})$ be a compact topological space with a σ-field of measurable sets $\mathfrak{M}$, given by a nonnegative and regular normed Borel measure μ_0. By Z we denote a separable Banach space with the norm $|\cdot|$ and by $L_1(T, Z)$ the Banach space of functions integrable in the Bochner sense, with the norm $\|u\| = \int_T |u(t)|\, d\mu_0$.

We call a set $K \subset L_1(T, Z)$ *decomposable* if for all $u, v \in K$ and $A \in \mathfrak{M}$

$$\text{(P)} \qquad \chi_A \cdot u + \chi_{T \setminus A} \cdot v \in K,$$

where χ_A stands for the characteristic function of set A.

The multivalued map $K(s)$ from the topological space S into space $N(X)$ of nonempty subsets of the topological space X is called *lower semicontinuous* (l.s.c.) if the set

$$\text{(0.1)} \qquad K^+F = \{s \in S: K(s) \subset F\}$$

is closed in S for every closed $F \subset X$.

The well-known theorem of Michael [6] gives us the existence of a continuous selection of the multivalued l.s.c. map $K: S \to \mathrm{cl} X$ ($\mathrm{cl} X$ denotes nonempty and closed subsets of X), where S is a paracompact topological space and the values of $K(s)$ are convex.

The purpose of this paper is to show that in the case where $X = L_1(T, Z)$, then an analogue of Michael's theorem holds with the convexity assumption replaced by condition (P). We prove this for compact S but it holds also for locally compact and separable S.

The first result of this type has been obtained by Antosiewicz and Cellina [1] for $K(s)$ given by

$$(0.2)\qquad K(s) = \{u \in L_1([0,1], R^m):\ u(t) \in P(t, s(t)) \text{ a.e. in } [0,1]\}$$

defined on a compact set S of continuous functions on $[0,1]$ into the Euclidean space R^m. Above $P(t, x)$ is a multivalued map from the Cartesian product $[0,1]\times R^m$ into compact subsets of R^m measurable in t, continuous in x and integrably bounded. Under these assumptions they proved that there exists a continuous map $\varphi\colon S\to L_1([0,1], R^m)$ such that $\varphi(s)(t) \in P(t, s(t))$ a.e. in $[0,1]$, that is, a continuous selection of $K(s)$ given by (0.2). This theorem was further extended by Bressan [2] and Łojasiewicz [5]. They weakened the condition of continuity in x of $P(t, x)$ replacing it by lower semicontinuity.

The above theorems were applied by those authors to prove the existence of a solution to the Cauchy Problem $\dot{x} \in P(t, x)$ and $x(0) = x_0$, where the values of $P(t, x)$ may be non-convex.

It is obvious that $K(s)$ given by (0.2) satisfies (P). This condition is a kind of substitute for convexity.

The existence of a continuous selection of $K(s)$ when the multivalued map K from a compact topological space S into $\mathrm{cl}L_1(T, Z)$ is l.s.c. and the sets $K(s)$ satisfy condition (P), which we prove in this paper, is an abstract version of the above-mentioned result of Antosiewicz and Cellina.

The main result and the construction of a continuous selection is given in Section 3. Section 1 contains a proposition which is a consequence of the Liapunov theorem on the range of a vector-valued measure and which is quite instrumental in solving the problem. In Section 2 we give some consequences of decomposability property (P).

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1. Some properties of a vector measure. Let us consider a nonatomic, complete vector measure $\vec{\mu} = (\mu_1, \ldots, \mu_m)$. We shall consider below the space M of such vector measures with the topology induced by the norm $\|\vec{\mu}\|$ equal to the variation of $\vec{\mu}$.

From the famous Liapunov theorem we know that the set $\mathcal{R} = \{\vec{\mu}(A)\colon A \in \mathfrak{M}\}$ is compact and convex. In particular, this theorem implies the following:

Remark 1.1. For an arbitrary $A \in \mathfrak{M}$ there is $B \in \mathfrak{M}$ such that $B \subset A$ and

$$\vec{\mu}(B) = \tfrac{1}{2}\cdot\vec{\mu}(A).$$

Using this we shall prove:

PROPOSITION 1.1. *For the above measure $\vec{\mu}$ there exists a family of measurable sets $\{A_\alpha\}_{\alpha\in[0,1]}$ such that*

(1.1) $A_\alpha \subset A_\beta$ *for* $\alpha < \beta$,

(1.2) $\vec{\mu}(A_\alpha) = \alpha\cdot\vec{\mu}(T)$.

Proof. From Remark 1.1 we may construct a family of sets A_α satisfying (1.1) and (1.2) for $\alpha = k/2^n$, where $n \in N$ and $k = 0, \ldots, 2^n$. Having this for arbitrary $\alpha \in [0,1]$ we put $A_\alpha = \bigcup_{k/2^n\leqslant\alpha} A_{k/2^n}$. Condition (1.1) holds by the definition of A_α, while condition (1.2) follows from the continuity of the measure. This completes the proof.

Remark 1.2. We may additionally require in Prop. 1.1 that $\mu_0(A_\alpha) = \alpha$. Indeed, it is enough to construct the family $\{A_\alpha\}$ for the measure $\vec{\nu} = (\mu_0, \mu_1, \ldots, \mu_m)$.

Let us consider a family of nonatomic complete measures $\vec{\mu}_s = (\mu_s^1, \ldots, \mu_m^s)$.

PROPOSITION 1.2. *Assume that the map $s\to\vec{\mu}_s$ from a compact topological space S into space M is continuous. Then for every $\varepsilon > 0$ there exists a family of measurable sets $\{A_\alpha\}_{\alpha\in[0,1]}$ with the properties*

(1.3) $A_\alpha \subset A_\beta$ *for* $\alpha < \beta$,

(1.4) $|\vec{\mu}_s(A_\alpha) - \alpha\cdot\vec{\mu}_s(T)| < \varepsilon$ *for all* $\alpha \in [0,1]$ *and* $s \in S$,

(1.5) $\mu_0(A_\alpha) = \alpha$.

Proof. Let us take an $\varepsilon > 0$. The family of open sets $\{V_{s_0}\}_{s_0\in S}$ given by the formula

$$(1.6)\qquad V_{s_0} = \{s\colon \|\vec{\mu}_s - \vec{\mu}_{s_0}\| < \varepsilon/2\}$$

is an open covering of the compact space S. Let $s_1, \ldots, s_k$ be such elements of S that $S = V_{s_1}\cup\ldots\cup V_{s_k}$. From Prop. 1.1 for the measure $\vec{\nu} = (\vec{\mu}_{s_1}, \ldots, \vec{\mu}_{s_k}, \mu_0)$ there exists a family of measurable sets $\{A_\alpha\}_{\alpha\in[0,1]}$ such that (1.3) holds and

$$(1.7)\qquad \vec{\nu}(A_\alpha) = \alpha\cdot\vec{\nu}(T) \quad \text{for all } \alpha \in [0,1].$$

To end the proof we show that the family $\{A_\alpha\}$ satisfies (1.4). For an arbitrary $\alpha \in [0,1]$ and $s \in S$ we have

$$|\vec{\mu}_s(A_\alpha) - \alpha\cdot\vec{\mu}_s(T)| \leqslant |\vec{\mu}_s(A_\alpha) - \vec{\mu}_{s_i}(A_\alpha)| + |\vec{\mu}_{s_i}(A_\alpha) - \alpha\cdot\vec{\mu}_{s_i}(T)| + \\ + |\alpha(\vec{\mu}_{s_i}(T) - \vec{\mu}_s(T))|,$$

where s_i is such that $s \in V_{s_i}$. The first and the last terms of the right-hand side of the above inequality are estimated by $\varepsilon/2$ because of (1.6), while the middle term is equal to 0 because of (1.7). Hence (1.4) holds.

We shall now prove

PROPOSITION 1.3. *Let $\{A_\alpha\}_{\alpha\in[0,1]}$ be a family of measurable sets with the following properties*

(1.8) $A_\alpha \subset A_\beta$ *for* $\alpha < \beta$,

(1.9) $\mu_0(A_\alpha) = \alpha$,

and let $p\colon S\to[0,1]$ and $k\colon S\to L_1(T,Z)$, where S is a topological space, be continuous maps. Then the map $l(s) = k(s)\cdot\chi_{A_{p(s)}}$ is continuous.

Proof. The continuity of map $l(s)$ follows from the inequalities

$$\|k(s)\cdot\chi_{A_{p(s)}} - k(s_0)\cdot\chi_{A_{p(s_0)}}\| \leqslant \|k(s)\cdot\chi_{A_{p(s)}} - k(s_0)\cdot\chi_{A_{p(s)}}\| + \|k(s_0)\cdot\chi_{A_{p(s)}} - k(s_0)\cdot\chi_{A_{p(s_0)}}\| \leqslant \|k(s)-k(s_0)\| + \int_{A_{p(s)}\dot{-}A_{p(s_0)}} |k(s_0)(t)|\,d\mu_0$$

and the equality $\mu_0(A_{p(s)}\dot{-}A_{p(s_0)}) = |p(s)-p(s_0)|$, which is true for arbitrary s_0 and any s from S.

2. The decomposability property. For an arbitrary set $\mathfrak{A}$ of measurable real-valued functions defined on $(T,\mathfrak{M})$, we denote by $\operatorname{ess\,inf}_{a\in\mathfrak{A}} a(t)$ the essential infimum. It is known (see [3]) that there exists a sequence $\{a_n\}\subset\mathfrak{A}$ such that

$$\operatorname*{ess\,inf}_{a\in\mathfrak{A}} a(t) = \inf_n a_n(t) \text{ a.e. in } T. \tag{2.1}$$

Consider now a nonempty and closed set $K\subset L_1(T,Z)$ which fulfils the decomposability property (P). We denote

$$\psi(t) = \operatorname*{ess\,inf}_{u\in K} |u(t)|. \tag{2.2}$$

There exist functions $u_n\in K$, for $n\in N$, such that a.e. in T

$$|u_1(t)| \geqslant |u_2(t)| \geqslant \ldots \tag{2.3}$$

and

$$\psi(t) = \lim_{n\to\infty} |u_n(t)|. \tag{2.4}$$

Let $v_n\in K$ be such that (2.1) holds; $\psi(t) = \inf_n |v_n(t)|$ a.e. in T. Let us put $u_1 = v_1$ and inductively $u_{n+1} = u_n\cdot\chi_{T_n} + v_{n+1}\cdot\chi_{T\setminus T_n}$ where $T_n = \{t\colon |u_n(t)| < |v_{n+1}(t)|\}$. Then (2.3) and (2.4) are implied by the inequality

$$|u_{n+1}(t)| \leqslant \inf\{|v_1(t)|,\ldots,|v_n(t)|\}. \tag{2.5}$$

PROPOSITION 2.1. *Let $K\subset L_1(T,Z)$ be a closed and nonempty set which satisfies condition* (P). *Then there exists an element $u_0\in K$ such that*

$$|u_0(t)| = \psi(t) = \operatorname*{ess\,inf}_{u\in K}|u(t)| \text{ a.e. in } T. \tag{2.6}$$

Proof. Let $u_n\in K$ be a sequence satisfying (2.3) and (2.4). Then the multivalued map $P(t) = \operatorname{cl}\{u_n(t),\ n\in N\}\cap\bar{B}(0,\psi(t))$ ($\bar{B}(0,r)$ denotes a closed ball with the centre 0 and radius r) is measurable and has nonempty values a.e. in T.

Let u_0 be a measurable selection of $P(t)$. We shall prove that $u_0\in K$. Fix $i\in N$ and for $n\in N$ put $T_n = \{t\colon |u_n(t)-u_0(t)| \leqslant 1/i\}$. Then $\bigcup_{n=1}^{\infty} T_n$ is a set of full measure. From property (P) and (2.3) we see that v_i given by the formula

$$v_i(t) = \begin{cases} u_1(t), & t\in T_1,\\ u_2(t), & t\in T_2\setminus T_1,\\ \ldots & \ldots\\ u_n(t), & t\in T_n\setminus\bigcup_{k<n} T_k\\ \ldots & \ldots \end{cases}$$

belongs to K and the inequality $|v_i(t)-u_0(t)| \leqslant 1/i$ holds a.e. in T. And so $u_0 = \lim_{i\to\infty} v_i$ belongs to K. Clearly u_0 satisfies (2.6).

We will now pass to the investigation of a multivalued map $K\colon S\to \operatorname{cl} L_1(T,Z)$, where S is a topological space.

DEFINITION 2.1. We will say that the multivalued map $K\colon S\to \operatorname{cl} L_1(T,Z)$ is *decomposable* if for all $s\in S$ the sets $K(s)$ satisfy property (P).

PROPOSITION 2.2. *Assume that the map $K\colon S\to \operatorname{cl} L_1(T,Z)$ is* l.s.c. *and decomposable and put*

$$\psi_s(t) = \operatorname*{ess\,inf}_{u\in K(s)} |u(t)|. \tag{2.7}$$

Then the multivalued map

$$P(s) = \{v\in L_1(T,R^1)\colon v(t)\geqslant\psi_s(t) \text{ a.e. in } T\} \tag{2.8}$$

is l.s.c. *and decomposable.*

Proof. Let F be an arbitrary closed set in $L_1(T,R^1)$. It is enough to show that if for a sequence $s_n \underset{n\to\infty}{\to} s_0$ we have $P(s_n)\subset F$, then $P(s_0)\subset F$, too. For this purpose take an arbitrary $v_0\in P(s_0)$. From Prop. 2.1 there exists a function $u_0\in K(s_0)$ such that

$$v_0(t) \geqslant |u_0(t)| = \psi_{s_0}(t) \text{ a.e. in } T.$$

Let $u_n\in K(s_n)$ be a sequence such that $\lim_{n\to\infty} u_n = u_0$ (such a sequence exists because $K(s)$ is l.s.c.). Then the sequence $v_n = |u_n| + v_0 - |u_0|$ belongs to $P(s_n) \subset F$ and converges to v_0. Since F is closed and $v_n \in F$, $v_0 \in F$ also. But v_0 is an arbitrary point of $P(s_0)$; hence $P(s_0) \subset F$, which was to be proved.

Let $K\colon S \to \mathrm{cl}\,L_1(T, Z)$ be a decomposable and l.s.c. multivalued map. We shall prove that these properties are preserved where we take an intersection with certain special multivalued maps. We have the following

PROPOSITION 2.3. *Let $K\colon S \to \mathrm{cl}\,L_1(T, Z)$ be an* l.s.c. *and decomposable multivalued map and $\varphi\colon S \to L_1(T, R^1)$ and let $k\colon S \to L_1(T, Z)$ be such continuous maps that the set*

$$L(s) = \{u \in K(s)\colon\ |u(t) - k(s)(t)| < \varphi(s)(t) \ \text{a.e. in } T\}$$

is nonempty for any $s \in S$. Then the map $L\colon S \to N(L_1(T, Z))$ is decomposable and l.s.c.

Proof. Let F be an arbitrary closed subset in $L_1(T, Z)$. It is enough to show that if the inclusion $L(s_n) \subset F$ holds for the sequence $s_n \xrightarrow[n\to\infty]{} s_0$, then $L(s_0) \subset F$. For this purpose take an arbitrary $u_0 \in L(s_0)$. Because of the lower semicontinuity of $K(s)$ there exists a sequence $u_n \in K(s_n)$ such that $\lim_{n\to\infty} u_n = u_0$. Without any loss of generality we may assume that $u_n(t)$, $k(s_n)(t)$ and $\varphi(s_n)(t)$ converges to $u_0(t)$, $k(s_0)(t)$, and $\varphi(s_0)(t)$ a.e. in T. For each $i \in N$, let T_i be such a compact set that the functions u_n, $k(s_n)$ and $\varphi(s_n)$ restricted to T_i are continuous and converge uniformly and that the following inequality holds:

$$\int_{T \setminus T_i} \varphi(s_0)(t)\, d\mu_0 < 1/i. \tag{2.9}$$

Since for $t \in T_i$, $|u_0(t) - k(s_0)(t)| < \varphi(s_0)(t)$, there exists n_i such that for $n \geqslant n_i$ and all $t \in T_i$ we have the inequality

$$|u_n(t) - k(s_n)(t)| < \varphi(s_n)(t). \tag{2.10}$$

We may additionally assume that $n_1 < n_2 < \ldots$ Put $v_n = u_n \cdot \chi_{T_i} + w_n \cdot \chi_{T \setminus T_i}$ for $n_i \leqslant n < n_{i+1}$, where w_n are arbitrary but fixed elements from $L(s_n)$ for $n \in N$. Then the sequence v_n is converging to u_0, because for $n_i \leqslant n < n_{i+1}$ we have the inequalities

$$\begin{aligned}\|v_n - u_0\| &\leqslant \int_{T\setminus T_i} |w_n(t) - k(s_n)(t)|\, d\mu_0 + \int_{T\setminus T_i} |k(s_n)(t) - k(s_0)(t)|\, d\mu_0 + \\ &\quad + \int_{T\setminus T_i} |k(s_0)(t) - u_0(t)|\, d\mu_0 + \int_{T_i} |u_n(t) - u_0(t)|\, d\mu_0 \\ &\leqslant 2\cdot \int_{T\setminus T_i} \varphi(s_0)(t)\, d\mu_0 + \|\varphi(s_n) - \varphi(s_0)\| + \|k(s_n) - k(s_0)\| + \|u_n - u_0\| \\ &< 2/i + \|\varphi(s_n) - \varphi(s_0)\| + \|k(s_n) - k(s_0)\| + \|u_n - u_0\|.\end{aligned}$$

It is easy to check that v_n belongs to $L(s_n) \subset F$. Since $v_n \in F$ and F is closed, $u_0 \in F$ also. But u_0 is an arbitrary point of $L(s_0)$; hence $L(s_0) \subset F$, which was to be proved.

3. Construction of a continuous selection. The scheme of the construction is analogous to the proof of Michael's theorem [6]. Namely, we shall construct a sequence of approximate selections which, in the limit, will give a continuous selection. We begin with the following

LEMMA 3.1. *Take a decomposable and* l.s.c. *multivalued map $K\colon S \to \mathrm{cl}\,L_1(T, Z)$. Then for every $\varepsilon > 0$ there exist continuous maps $k\colon S \to L_1(T, Z)$ and $\varphi\colon S \to L_1(T, R^1)$ such that*

$$\int_T \varphi(s)(t)\, d\mu_0 < \varepsilon \quad \textit{for each } s \tag{3.1}$$

and the set

$$L(s) = \{u \in K(s)\colon\ |u(t) - k(s)(t)| < \varphi(s)(t) \ \text{a.e. in } T\} \tag{3.2}$$

is nonempty for each $s \in S$.

Proof. Fix $\varepsilon > 0$. From Proposition 2.2 and Michael's theorem we see that for every fixed $s_0 \in S$ and $u_0 \in K(s_0)$ there exists a continuous function $\varphi_{s_0,u_0}\colon S \to L_1(T, R^1)$ such that

$$\varphi_{s_0,u_0}(s)(t) \geqslant \operatorname*{ess\,inf}_{u\in K(s)} |u(t) - u_0(t)| \ \text{ a.e. in } T \tag{3.3}$$

and

$$\varphi_{s_0,u_0}(s_0) = 0. \tag{3.4}$$

Consider the family of sets $\{V_{s_0,u_0}\}_{s_0\in S,\, u_0\in K(s_0)}$ given by the formula

$$V_{s_0,u_0} = \Big\{s\colon \int_T \varphi_{s_0,u_0}(s)(t)\, d\mu_0 < \varepsilon/4\Big\}. \tag{3.5}$$

It is an open covering of the compact space S. We can establish a finite partition of unity $p_1(s), \ldots, p_r(s)$ subordinate to this covering. Let V_{s_i,u_i} denote such sets that

$$p_i^{-1}(0, 1] \subset V_{s_i,u_i} \quad \text{for} \quad i = 1, \ldots, r. \tag{3.6}$$

Then for every $s \in S$ and $i = 1, \ldots, r$ the following inequalities are satisfied:

$$p_i(s) \cdot \int_T \varphi_i(s)(t)\, d\mu_0 \leqslant (\varepsilon/4)\cdot p_i(s), \quad \text{where} \quad \varphi_i = \varphi_{s_i,u_i}. \tag{3.7}$$

Consider measures $\vec{\mu}_s$ with the Radon–Nikodým derivatives

$$\big(\varphi_1(s)(t), \ldots, \varphi_r(s)(t)\big). \tag{3.8}$$

Since $\varphi_i(s)$ are continuous in the norm topology of $L_1(T, R^1)$, $\vec{\mu}_s$ is continuous in M. Therefore from Prop. 1.3 we have the existence of a family $\{A_a\}_{a\in[0,1]}$ of measurable sets such that

(3.9) $A_a \subset A_\beta$ for $a < \beta$,

(3.10) $|\vec{\mu}_s(A_a) - a\cdot\vec{\mu}_s(T)| < \varepsilon/4r$ for all $s \in S$ and $a \in [0, 1]$ and

(3.11) $\mu_0(A_a) = a$.

Define functions $\varphi(s)$ and $k(s)$ by the formulas

$$\varphi(s) = \sum_{i=1}^{r} \big(\varphi_i(s) + \varepsilon/4\big)\cdot\chi_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}}, \tag{3.12}$$

$$k(s) = \sum_{i=1}^{r} u_i\cdot\chi_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}}, \tag{3.13}$$

where $z_0(s) = 0$ and $z_i(s) = p_1(s) + \ldots + p_i(s)$ for $i = 1, \ldots, r$. From Prop. 1.3 it follows that $k(s)$ and $\varphi(s)$ are continuous. We shall prove that $\int_T \varphi(s)(t)d\mu_0 < \varepsilon$. From (3.10) we have

$$\Big|\int_{A_a}\varphi_i(s)(t)\,d\mu_0 - a\cdot\int_T \varphi_i(s)(t)\,d\mu_0\Big| < \varepsilon/4r \quad \text{for} \quad a \in [0, 1].$$

Therefore

$$\int_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}} \varphi_i(s)(t)\,d\mu_0 = \int_{A_{z_i(s)}} \varphi_i(s)(t)\,d\mu_0 - \int_{A_{z_{i-1}(s)}} \varphi_i(s)(t)\,d\mu_0$$
$$< \big(z_i(s) - z_{i-1}(s)\big)\cdot\int_T \varphi_i(s)(t)\,d\mu_0 + \varepsilon/2r.$$

Since $p_i(s) = z_i(s) - z_{i-1}(s)$, by (3.12) we have

$$\int_T \varphi(s)(t)\,d\mu_0 < \sum_{i=1}^{r} p_i(s)\cdot\int_T \varphi_i(s)(t)\,d\mu_0 + 3\varepsilon/4$$

and by (3.7) we have the required estimate.

It remains to establish that $L(s) \neq \emptyset$ for $s \in S$. From Prop. 2.1 there is $u_s^i \in K(s)$ such that for each $s \in S$ and $i = 1, \ldots, r$

$$|u_s^i(t) - u_i(t)| = \operatorname*{ess\,inf}_{u\in K(s)} |u(t) - u_i(t)| \text{ a.e. in } T. \tag{3.14}$$

Then the function $u_s = \sum_{i=1}^{r} u_s^i\cdot\chi_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}}$ belongs to $K(s)$ because of property (P). On the other hand, we have by (3.12)–(3.14)

$$|u_s(t) - k(s)(t)| = \sum_{i=1}^{r} |u_i(t) - u_s^i(t)|\cdot\chi_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}} \tag{3.15}$$
$$\leqslant \sum_{i=1}^{r} \varphi_i(s)(t)\cdot\chi_{A_{z_i(s)}\setminus A_{z_{i-1}(s)}} < \varphi(s)(t).$$

Therefore $u_s \in L(s)$, which completes the proof of Lemma 3.1.

Now we are able to prove the main result of the paper.

THEOREM 3.1. *Let a multivalued map K: $S \to \mathrm{cl}L_1(T, Z)$ be* l.s.c. *and decomposable. Then K has a continuous selection.*

Proof. We shall define by induction a decreasing sequence of multivalued maps $K_n(s)$, for $n = 0, 1, \ldots$, which are decomposable and l.s.c., and sequences of continuous maps k_n: $S \to L_1(T, Z)$ and φ_n: $S \to L_1(T, R^1)$ for $n = 1, 2, \ldots$ with the properties:

$$\int_T \varphi_n(s)(t)\,d\mu_0 < 1/2^n \tag{3.16}$$

and

$$L_{n+1}(s) = \{u \in K_n(s): |u(t) - k_n(s)(t)| < \varphi_n(s)(t) \text{ a.e. in } T\} \tag{3.17}$$

is nonempty for all $s \in S$.

For $n = 0$, put $K_0(s) = K(s)$.

If, for a fixed $n \geqslant 0$, the multivalued maps $K_n(s)$ are defined, then the continuous maps $k_{n+1}(s)$ and $\varphi_{n+1}(s)$ are defined by Lemma 3.1 with $\varepsilon = 1/2^{n+1}$, so that for $s \in S$ the sets

$$L_{n+1}(s) = \{u \in K_n(s): |u(t) - k_{n+1}(s)(t)| < \varphi_{n+1}(s)(t) \text{ a.e. in } T\}$$

are nonempty and $\int_T \varphi_{n+1}(s)(t)\,d\mu_0 < 1/2^{n+1}$. Then from Prop. 2.3 we can put for every $s \in S$

$$K_{n+1}(s) = \mathrm{cl}L_{n+1}(s).$$

It is clear that $K_{n+1}(s) \subset K_n(s)$. For each $s \in S$ and $n \in N$, let u_n^s be an arbitrary point of $K_n(s)$. Since $K_{n+p}(s) \subset K_n(s)$, we have $u_{n+p}^s \in K_n(s)$ for each $p \geqslant 0$. Therefore by (3.17) we have, for each n and $p \geqslant 0$, the inequality

$$|k_n(s)(t) - u_{n+p}^s(t)| \leqslant \varphi_n(s)(t) \text{ a.e. in } T. \tag{3.18}$$

Inequality (3.18) implies that

$$|k_n(s)(t) - k_{n+p}(s)(t)| \leqslant \varphi_n(s)(t) + \varphi_{n+p}(s)(t).$$

Because of (3.16) the above inequality implies that $k_n(s)$ converges uniformly in the $L_1(T, Z)$-norm to a continuous map $k_0(s)$. Again from (3.16) and (3.18) it follows that $\|k_n(s) - u_n^s\|$ tends to zero; hence $k_0(s) \in K(s)$. Thus $k_0(s)$ is a continuous selection of $K(s)$, which completes the proof.

COROLLARY 3.1. *Theorem 3.1 is also true when S is a locally compact separable metric space.*

Proof. Let S_n be such a family of compact sets that $S_n \subset \mathrm{Int}S_{n+1}$ and $\bigcup_{n=1}^{\infty} S_n = S$. Theorem 3.1 applied to the mapping $K(s)$ restricted to S_1 gives us the existence of a continuous function k_1: $S_1 \to L_1(T, Z)$ which is a selection of $K(s)$ for $s \in S_1$.

Let us define map $K_1(s)$ for $s \in S$ by the formula

$$K_1(s) = \begin{cases} \{k_1(s)\}, & s \in S_1 \\ K(s), & s \in S \setminus S_1. \end{cases}$$

It may easily be proved that $K_1(s)$ is l.s.c. and decomposable. Restricting $K_1(s)$ to S_2 and applying Theorem 3.1, we get a continuous $k_2\colon S_2 \to L_1(T, Z)$, which is a selection of $K_1(s)$ for $s \in S_2$. Obviously, for $s \in S_1$, $k_1(s) = k_2(s)$, and so k_2 is a continuation of k_1 to the set S_2. In this way, by induction, we get a selection defined on the whole S.

Corollary 3.2. *Let K satisfy the assumption of Theorem 3.1. Fix $s_0 \in S$ and $u_0 \in K(s_0)$. Then there exists a continuous selection $k_0(s)$ of the $K(s)$ such that*

$$k_0(s_0) = u_0. \tag{3.19}$$

Proof. It can easily be verified that the multivalued map

$$\tilde{K}(s) = \begin{cases} K(s) & \text{if} \quad s \neq s_0, \\ \{u_0\} & \text{if} \quad s = s_0 \end{cases}$$

satisfies the assumptions of Theorem 3.1. Then each selection of $\tilde{K}(s)$ fulfils (3.19).

In the case where, additionally, the values of K are assumed to be convex, it is known that there exists a denumerable sequence $k_n(s)$ of continuous selections such that $\{k_n(s)\}$ is a dense subset of $K(s)$ for each s. A similar statement is true also in the case considered here. Namely, we have the following

Theorem 3.2. *For a decomposable and* l.s.c. *multivalued map $K\colon S \to \mathrm{cl}\, L_1(T, Z)$ there exists a countable family of continuous functions $k_n\colon S \to L_1(T, Z)$ such that*

$$K(s) = \mathrm{cl}\{k_n(s)\colon n \in N\} \quad \textit{for all} \quad s \in S. \tag{3.20}$$

Proof. The space C of continuous maps $k\colon S \to L_1(T, Z)$ with the norm $|||k||| = \sup_{s \in S} \|k(s)\|$ is a separable Banach space. The set $\mathscr{K} = \{k \in C\colon k(s)$ is a selection of $K(s)\}$ is closed in the norm topology. There exist selections k_n from $\mathscr{K}$ for each n such that

$$\mathscr{K} = \mathrm{cl}\{k_n\colon n \in N\}. \tag{3.21}$$

We claim that, for an arbitrary s, (3.20) holds for this sequence. To show this let $k_0 \in K$ be a continuous map such that $k_0(s_0) = u_0$ for arbitrary but fixed $s_0 \in S$ and $u_0 \in K(s_0)$. For every $i \in N$ there exists n_i such that $|||k_{n_i} - k_0||| < 1/i$. In particular, it follows that $\|k_{n_i}(s_0) - u_0\| < 1/i$, and this means that $u_0 \in \mathrm{cl}\{k_n(s_0)\colon n \in N\}$. This completes the proof.

Corollary 3.3 (Bressan [2], Łojasiewicz [5]). *Suppose that $P\colon [0,1] \times R^k \to \mathrm{cl}\, R^m$ satisfies the conditions*

(a) *P is $\mathscr{L} \otimes B$-measurable,*

(b) *$P(t, \cdot)$ is* l.s.c.,

(c) *there exists a $p \in L_1([0,1], R^1)$ such that for every $x \in R^m$*

$$\sup\{|z|\colon z \in P(t, x)\} \leqslant p(t) \quad \textit{a.e. in} \quad [0,1].$$

Let S be a compact subset of Banach space $C([0,1], R^k)$ of continuous functions from $[0,1]$ into R^k, and for $s \in S$ put

$$K(s) = \{u \in L_1([0,1], R^m)\colon u(t) \in P(t, s(t)) \ \textit{a.e. in} \ [0,1]\}.$$

Then there exists a continuous selection $k\colon S \to L_1([0,1], R^m)$ of $K(s)$.

Proof: It is enough to prove that conditions (a), (b), (c) imply the lower semicontinuity of $K(s)$. Let F be an arbitrary closed set in $L_1([0,1], R^m)$. We need to prove that if $s_n \xrightarrow[n\to\infty]{} s_0$ uniformly, $K(s_n) \subset F$, then $K(s_0) \subset F$ also. For this purpose take $u_0 \in K(s_0)$ and define $u_n(t)$ so that $u_n \in K(s_n)$ and

$$|u_n(t) - u_0(t)| = d\big(u_0(t), P(t, s_n(t))\big) \ \text{a.e. in} \ [0,1]. \tag{3.22}$$

Because of (a) such an u_n exists, is measurable and, because of (c), integrable. There is a set $T' \subset T$ of full measure such (3.22) holds for each n on T'. For each fixed $t \in T'$, (3.22) and (b) imply that $u_n(t) \xrightarrow[n\to\infty]{} u_0(t)$. Hence because of (c) $u_n \xrightarrow[n\to\infty]{} u_0$ in L_1-norm. Since $u_n \in F$ and F is closed, $u_0 \in F$ also. But u_0 an arbitrary point of $K(s_0)$. Hence $K(s_0) \subset F$, which was to be proved.

Remark. If we assume additionally that the values of P are convex, then the corollary easily follows from the fact that there exists a selection $p(t, x)$ of $P(t, x)$ which is measurable in t and continuous in x (see [4]).

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On the existence of unitary representations of commutative nuclear Lie groups

by

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Abstract. A proof is given that if Γ is a discrete subgroup of a nuclear space X, then the quotient group X/Γ admits sufficiently many continuous characters.

In many situations nuclear spaces seem to be a more adequate generalization of finite-dimensional spaces than are normed spaces. Indeed, many important facts concerning finite-dimensional spaces remain valid in nuclear spaces but not in infinite-dimensional normed spaces. An example of this kind is given in the present paper.

Let us consider the following property of a topological vector space X:

(*) If Γ is a discrete subgroup of X, then the quotient group X/Γ admits sufficiently many continuous characters.

(The terminology is explained below.) Every finite-dimensional space X satisfies (*), which is trivial, and no infinite-dimensional normed space X satisfies (*), which has been proved in [1]. We shall prove here that every nuclear space X satisfies (*).

We begin with some notation and terminology. $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{R}$, $\boldsymbol{C}$ will denote the sets of positive integers, integers, reals and complexes, respectively. Vector spaces will often be regarded as additive topological groups. If A is a subset of a vector space X, then GA will denote the group generated by A, and span A — the linear span of A. The distance from a point u to a set A will be denoted by $d(u, A)$. For a topological vector space X the conjugate space will be denoted by X^*.

Let E be a real Hilbert space, and let $u_1, \ldots, u_n \in E$. Then Gram $(u_1, \ldots, u_n)$ will denote the Gram determinant of the vectors $u_1, \ldots, u_n$. If E is n-dimensional, and if K is a discrete subgroup of E which spans E, then K is an abelian free group with n linearly independent generators $u_1, \ldots, u_n$, and the number Gram$(u_1, \ldots, u_n)$ does not depend on the choice of generators; we denote this number by Gram K. A subgroup K of a Hilbert space will be called *r-discrete* if $\|u-w\| \geqslant r$ for any distinct $u, w \in K$.

Let G be a topological group. By a *character* of G we mean a homomorphism of G into the multiplicative group $\{z \in \boldsymbol{C}: z\bar{z} = 1\}$. We say that G