STUDIA MATHEMATICA, T. LXXV. (1983)

On the relation of the bounded approximation property and a finite dimensional decomposition in nuclear Fréchet spaces

by

ANDREAS BENNDORF (Darmstadt)

Abstract. A Fréchet Schwartz space has the bounded approximation property (resp. the bounded projection approximation property) iff the identity operator on the space is the pointwise limit of a sequence of finite rank continuous linear operators (resp. of finite rank continuous linear projections). The space has an absolute unconditional partition of the identity (resp. an absolute finite dimensional decomposition) iff there exists a series of finite rank continuous linear operators (resp. a series $\sum_n B_n$ of finite rank continuous linear projections with $B_nB_m = 0$ for every n, m with $n \neq m$) converging pointwise absolutely to the identity operator.

In the present paper we show that every Fréchet Schwartz space with the bounded approximation property has an absolute unconditional partition of the identity, that every Fréchet Schwartz space with a continuous norm and the bounded projection approximation property has an absolute finite dimensional decomposition and that every nuclear Fréchet space (resp. Fréchet Schwartz space) with the bounded approximation property is isomorphic to a complemented subspace of a nuclear Fréchet space (resp. Fréchet Schwartz space) with an absolute finite dimensional decomposition.

Introduction. In the present paper we are interested in nuclear Fréchet spaces with the bounded approximation property.

It was proved by Dubinsky [2], [3] that not every nuclear Fréchet spaces has the bounded approximation property and by Mitjagin and Zobin [9], [13] that not every nuclear Fréchet with a finite dimensional decomposition, and hence with the bounded approximation property, has a basis.

We deal in this paper with the question if the existence of the bounded approximation property implies that of a finite dimensional decomposition in nuclear Fréchet spaces.

We show the following main results:

(1) every Fréchet Schwartz space with the bounded approximation property has an absolute unconditional partition of the identity,

(2) every nuclear Fréchet space (resp. Fréchet Schwartz space) with the bounded approximation property is isomorphic to a complemented

subspace of a nuclear Fréchet space (resp. Fréchet Schwartz space) with a finite dimensional decomposition,

(3) every Fréchet Schwartz space with a continuous norm and the bounded approximation property has an absolute finite dimensional decomposition.

Fréchet Schwartz spaces seem to be a natural framework for most of the methods used in this paper, hence we state our results in this context, though we are mainly interested in nuclear Fréchet spaces.

Our first result solves a problem posed in [2]. The proof uses a result of Pełczyński and Wojtaszczyk [11], who show that every separable Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a Fréchet space with a finite dimensional decomposition. A slightly stronger version of our fist result and the embedding method of Pełczyński and Wojtaszczyk in [11] yield our second result.

This result is related to a result of Pełczyński [10], who observed that every separable Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a Fréchet space with a basis (see [8] for a proof), and to a result of Djakov and Mitjagin [1] who proved that not every nuclear Fréchet space with the bounded approximation property is isomorphic to a complemented subspace of a nuclear Fréchet space with a basis.

A result of Johnson, Rosenthal and Zippin [7] is that every reflexive Banach space with the bounded projection approximation property has a finite dimensional decomposition. Their methods and a modified version of our second result allow us to show our third result.

In [5] Johnson shows a result weaker than our third result for the more general case of separable Fréchet spaces with a continuous norm. An application of a method of Johnson [6] allows then to prove the following result:

For every nuclear Fréchet space (resp. Fréchet Schwartz space) E with a continuous norm and the bounded approximation property there exists a nuclear Fréchet space (resp. Fréchet Schwartz space) F with a continuous norm and a finite dimensional decomposition such that the product space $E\times F$ has a finite dimensional decomposition.

It remains an open problem if every nuclear Fréchet space with the bounded approximation property has a finite dimensional decomposition.

This paper is part of the author's Ph.D. thesis prepared under supervision of Ed Dubinsky during the author's stay at Clarkson College of Technology in Potsdam, New York, during the academic year 1979/80. The author likes to express his gratitude to Ed Dubinsky for his invitation, his interest, his suggestions and his constant encouragement and to the whole functional analysis group in Potsdam for the very stimulating atmosphere.

Definitions and terminology. A *Fréchet space* is a complete metrizable locally convex topological vecor space. Hence, its topology is defined by an increasing sequence $(p_k)_k$ of seminorms. We call $(p_k)_k$ a *fundamental sequence of seminorms*. Two sequences of seminorms which determine the same topology are called *equivalent*.

Let E be a Fréchet space. We will write $(E, (p_k)_k)$ instead of E only if we want to stress that we consider the linear space E with the topology generated by $(p_k)_k$. If a Fréchet space has a fundamental sequence of norms then we say that the space *admits a continuous norm*. If E is a Fréchet space and $(A_n)_n$ is a sequence of continuous linear operators on E and $B_1 := A_1$ and $B_{n+1} := A_{n+1}-A_n$ for every n then the sequence $(B_n)_n$ is called the *associated sequence to* $(A_n)_n$.

A separable Fréchet space E has the *bounded approximation property*, shortly BAP (resp. the *bounded projection approximation property*, shortly BPAP) iff there exists a sequence $(A_n)_n$ of finite rank continuous linear operators (resp. of finite rank continuous linear projections) from E into E such that $\lim_n A_n(x) = x$ for every $x \in E$. We say $(A_n)_n$ is a BAP (resp. a BPAP) *determining sequence of operators*.

The space E has an *unconditional partition of the identity*, shortly UPI (resp. a *finite dimensional decomposition*, shortly FDD) iff there exists a sequence $(B_n)_n$ of finite rank continuous linear operators (resp. finite rank continuous linear projections with $B_nB_m = 0$ for every n, m with $n \neq m$) such that $\sum_n B_n(x) = x$ and the convergence of $\sum_n B_n(x)$ is unconditional (resp. conditional) in E for every $x \in E$. If in addition $\sum_n B_n(x)$ converges absolutely for every $x \in E$ then we say that E has an *absolute* UPI (resp. an *absolute* FDD). We say that $(A_n)_n$ with $A_n := \sum_{i=1}^{n} B_i$ is a UPI (resp. FDD, resp. *absolute* UPI, resp. *absolute* FDD) *determining sequence of operators*.

An infinite matrix $(a_n^k)_{k,n}$ of real numbers with $0 \leqslant a_n^k \leqslant a_n^{k+1}$ and $\sup_n a_n^k > 0$ for every $k, n \in N$ is called *Köthe matrix* and the sequence space

$$K(a) = \Big\{x\colon\ x = (x_n)_n,\ x_n \text{ is a scalar for every } n,\quad p_k(x) := \sum_n a_n^k |x_n| < \infty \text{ for every } k\Big\}$$

with the topology defined by the sequence $(p_k)_k$ of seminorms is called

Köthe sequence space. If E is a Fréchet space, $(p_k)_k$ is a fundamental sequence of seminorms on E and m is a natural number then (E, p_m) means that we consider the linear space E with the topology generated by the seminorm p_m. We denote by A' the dual operator of a continuous linear operator A from $(E(p_k)_k)$ into $(E, (p_k)_k)$, by ${}^mB'$ the dual operator of a continuous linear operator B from (E, p_m) into (E, p_m), by E' the dual space of E and by p'_m with

$$p'_m(y) = \sup\{|y(x)|:\ x \in E,\ p_m(x) \leqslant 1\}$$

for every $y \in (E, p_m)'$ the dual norm of p_m.

If A is a subset of a locally convex linear topological vector space E then we mean by $\mathrm{cl}(A)$ the closed hull of A, by $\mathrm{span}\{A\}$ the linear hull of A and by $E^\wedge$ the completion of E.

From now on the term "operator" stands always for "continuous linear operator" and "subspace" means always a "closed subspace" if not otherwise specified.

1. Every Fréchet Schwartz space with the BAP has an absolute UPI. The following theorem is well known.

THEOREM 1 [11]. *Every Fréchet space (with a continuous norm and) with the* BAP *is isomorphic to a complemented subspace of a Fréchet space (with a continuous norm and) with an* FDD.

LEMMA 1. *Let F be a Fréchet space with an* FDD, *let $(p_k)_k$ be a fundamental sequence of seminorms on F, let $(A_n)_n$ be an* FDD *determining sequence of operators on F and let E be a complemented subspace of F. If E is a Fréchet Schwartz space then for every k there exists a $j(k)$ such that*

$$\limsup_n \{p_k(A_n(x)-x):\ x \in E,\ p_{j(k)}(x) \leqslant 1\} = 0.$$

Proof. Define $q_k(x) := \sup_n p_k(A_n(x))$ for every k and every $x \in F$. Then $(q_k)_k$ and $(p_k)_k$ are equivalent fundamental sequences of seminorms. Since E is a Fréchet Schwartz space, for every k there exists a $j(k)$ such that for every $\varepsilon > 0$ there exists a finite set $\{x_1, \ldots, x_m\}$ in E such that for every $x \in \{x:\ x \in E,\ q_{j(k)}(x) \leqslant 1\}$ there exists an $x_i \in \{x_1, \ldots, x_m\}$ with $q_k(x-x_i) \leqslant \varepsilon$.

The rest of the proof follows then from a standard argument using the facts that $(A_n)_n$ converges pointwise to the identity operator on E, that $\{A_n:\ n \in N\}$ is an equicontinous set of operators on (F, q_k) for every k and that $(p_k)_k$ and $(q_k)_k$ are equivalent fundamental sequences of seminorms on F.

We are now ready to state and prove the main result of this section.

THEOREM 2. *Let E be a Fréchet Schwartz space with the* BAP *(resp. with an* FDD*) and let $(p_k)_k$ be a fundamental sequence of seminorms on E. If $(a_n^k)_{k,n}$ is a Köthe matrix then there exists a* BAP *(resp. an* FDD*) determining sequence of operators on E which has an associated sequence $(B_n)_n$ such that*

$$\sum_n a_n^k p_k(B_n(x)) < \infty \quad \textit{for every } x \in E \textit{ and every } k.$$

In particular, every Fréchet Schwartz space with the BAP *(resp. an* FDD*) has an absolute* UPI *(resp. an absolute* FDD*).*

Proof. Suppose E has the BAP. We can assume by Theorem 1 that E is a complemented subspace of a Fréchet space F with an FDD. Let $(A_n)_n$ determine the FDD of F. Lemma 1 implies now that for every k there exists a $j(k)$ such that

$$\limsup_n \{p_k(A_n(x)-x):\ x \in E,\ p_{j(k)}(x) \leqslant 1\} = 0.$$

Hence we can find a strictly increasing sequence $(n(i))_i$ of indices such that

$$\sup\{\max\{a_i^1, a_{i+1}^1\}p_1(A_n(x)-x):\ x \in E,\ p_{j(1)}(x) \leqslant 1\} < 2^{-(i+2)}, \ldots,$$
$$\sup\{\max\{a_i^{i+1}, a_{i+1}^{i+1}\}p_{i+1}(A_n(x)-x):\ x \in E,\ p_{j(i+1)}(x) \leqslant 1\} < 2^{-(i+2)}$$

for every i and every $n \geqslant n(i)$.

Define $D_{n(1)} := A_{n(1)}$ and $D_{n(i+1)} := A_{n(i+1)} - A_{n(i)}$ for every i. We obtain

$$\sup\{a_i^k p_k(D_{n(i)}(x)):\ x \in E,\ p_{j(k)}(x) \leqslant 1\} < 2^{-i}$$

for every i and $k = 1, 2, \ldots, i-1$ and we conclude

$$\sum_i a_i^k p_k(D_{n(i)}(x)) \leqslant \sum_{i=1}^{k} a_i^k p_k(D_{n(i)}(x)) + \sum_{i=k+1}^{\infty} 2^{-i} < \infty$$

for every k and every $x \in E$ with $p_{j(k)}(x) \leqslant 1$. But this implies

$$\sum_i a_i^k p_k(D_{n(i)}(x)) < \infty \quad \text{for every } k \text{ and every } x \in E.$$

The sequence $(PA_{n(i)})_i$ is a BAP determining sequence of operators on E, where P is a continuous linear projection from F onto E. Obviously, for every k there exists an $m(k) > k$ and an $M_k > 0$ such that

$$\sum_i a_i^k p_k(PD_{n(i)}(x)) \leqslant M_k \sum_i a_i^{m(k)} p_{m(k)}(D_{n(i)}(x)) < \infty \quad \text{for every } x \in E.$$

This implies that E has an absolute UPI, if one considers the case that $a_n^k = 1$ for every n and every k.

If E has an FDD then a repetition of the proof with $E = F$ and $P = I$ implies that E has an absolute FDD.

2. Every nuclear Fréchet space (resp. Fréchet Schwartz space) with the BAP is a complemented subspace of a nuclear Fréchet space (resp. Fréchet Schwartz space) with an FDD. Let E be a Fréchet space, let $(p_k)_k$ be a fundamental sequence of seminorms on E, let $(E_n)_n$ be a sequence of subspaces of E and let A be a subset of $\boldsymbol{N}$ with the natural order.

If $K(a)$ is a Köthe sequence space then the space

$$K(a)\big((E_n)_{n\in A}\big) := \Big\{x\colon x = (x_n)_{n\in A},\ x_n \in E_n \text{ for every } n \in A,\quad q_k(x) := \sum_{n\in A} a_n^k p_k(x_n) < \infty \text{ for every } k\Big\}$$

with the fundamental sequence $(q_k)_k$ of seminorms is called the *sum of* $(E_n)_{n\in A}$ *in the sense of* $K(a)$. We write $K(a)\big((E_n)_n\big)$ instead of $K(a)\big((E_n)_{n\in\boldsymbol{N}}\big)$.

We have the following fact:

Lemma 2 [12]. *If $K(a)$ is a Köthe sequence space and $(E_n)_n$ is a sequence of subspaces of a Fréchet space E then $K(a)\big((E_n)_n\big)$ is isomorphic to a subspace of the completion of the projective tensor product of $K(a)$ and E.*

Theorem 2 and arguments of Pełczyński and Wojtaszczyk [11] allow to prove the following proposition.

Proposition 1. *If E is a Fréchet Schwartz space with the* BAP *and $K(a)$ is a Köthe sequence space with $a_n^k \geqslant 1$ for every k, n then there exists a sequence of* BAP *determining operators on E with an associated sequence $(B_n)_n$ such that E is isomorphic to a complemented subspace of $K(a)\big((B_n(E))_n\big)$.*

Proof. Let $(p_k)_k$ be a fundamental sequence of seminorms on E. According to Theorem 2 there exists a sequence of BAP determining operators with an associated sequence $(B_n)_n$ such that

$$p_k^a(x) := \sum_n a_n^k p_k\big(B_n(x)\big) < \infty \quad \text{for every } k \text{ and every } x \in E.$$

$(p_k^a)_k$ and $(p_k)_k$ are equivalent fundamental sequences of seminorms on E.

It is then easy to see that the operator U form E into $K(a)\big((B_n(E))_n\big)$ with $U(x) = \big(B_n(x)\big)_n$ for every $x \in E$ is an isomorphic embedding and that the operator P from $K(a)\big((B_n(E))_n\big)$ onto $U(E)$ with

$$P\big((x_i)_i\big) = \Big(B_n\Big(\sum_i x_i\Big)\Big)_n \quad \text{for every } (x_i)_i \in K(a)\big((B_i(E))_i\big)$$

is a continuous linear projection. This completes the proof of the proposition.

Theorem 3. *If E is a Fréchet Schwartz space (resp. a nuclear Fréchet space) with the* BAP *then E is isomorphic to a complemented subspace of a Fréchet Schwartz space (resp. nuclear Fréchet space) F with an* FDD. *Moreover, if in addition E has a continuous norm then F can be chosen to admit a continuous norm.*

Proof. Choose a nuclear Köthe sequence space $K(a)$ with $a_n^k \geqslant 1$ for every k, n. According to Proposition 1 there exists a BAP determining sequence of operators with an associated sequence $(B_n)_n$ such that E is isomorphic to a complemented subspace of $K(a)\big((B_n(E))_n\big)$. Obviously $K(a)\big((B_n(E))_n\big)$ has an FDD. Using Lemma 2 it follows easily that $K(a)\big((B_n(E))_n\big)$ is a Fréchet Schwartz space (resp. a nuclear Fréchet space) iff E is a Fréchet Schwartz space (resp. a nuclear Fréchet space) and that $K(a)\big((B_n(E))_n\big)$ has in addition a continuous norm iff E has in addition a continuous norm.

Remark. Theorem 3 has besides the version for Fréchet Schwartz spaces and nuclear Fréchet spaces also a version for every subclass of Fréchet Schwartz spaces which contains with two spaces the projective tensor product, which contains with each space also each of its subspaces and each isomorphic space and which contains a Köthe sequence space $K(a)$ with $a_n^k \geqslant 1$ for every n, k. For example, modifications of the concept of nuclearity can lead to such classes of spaces [4].

3. Every Fréchet Schwartz space with a continuous norm and the BPAP has an absolute FDD. We start with a modification of Theorem 3.

Proposition 2. *Every Fréchet Schwartz space with a continuous norm and the* BAP *is isomorphic to a complemented subspace of a Fréchet space F with an* FDD *such that on F there exists a fundamental sequence $(p_k)_k$ of norms and an* FDD *determining sequence $(A_n)_n$ of operators such that for every k the space $(F, p_k)^\wedge$ is reflexive, that the sequence of the extentions of the A_n onto $(F, p_k)^\wedge$ determines an* FDD *of $(F, p_k)^\wedge$ and that $({}^kA_n')_n$ determines an* FDD *for $(F, p_k)'$.*

Proof. Let G be a Fréchet Schwartz space with the BAP and let $(q_k)_k$ be a fundamental sequence of norms on G. For the choice $a_n^k = n^k$ for every k, n there exists according to Proposition 1 a BAP determining sequence of operators on G with an associated $(B_n)_n$, such that G is isomorphic to a complemented subspace of

$$F := K(a)\big((B_n(G))_n\big),$$

where $(q_k^1)_k$ with $q_k^1\big((x_n)_n\big) := \sum_n n^k q_k(x_n)$ for every k and every $(x_n)_n \in F$ is a fundamental sequence of norms on F. Define

$$q_k^2\big((x_n)_n\big) := \Big(\sum_n \big(n^k q_k(x_n)\big)^2\Big)^{1/2} \quad \text{for every } k \text{ and every } (x_n)_n \in F.$$

Then it is easy to see that $(q_k^2)_k$ and $(q_k^1)_k$ are equivalent fundamental sequences of norms on F. Set $p_k = q_k^2$ for every k. Then $(F, p_k)\hat{}$ is reflexive. Let A_n be the canonical projection from F onto $K(a)\big((B_i(G))_{i\leq n}\big)$ for every n. It is obvious that F, $(p_k)_k$ and $(A_n)_n$ have the required properties.

In the following part of this section we will study more closely the situation described in the next remark. It is the purpose of the remark to fix for later references the notation of the terms which occur in this situation.

Remark 1. We show in Proposition 2 that every Fréchet Schwartz space with a continuous norm and the BAP is isomorphic to a complemented subspace E of a Fréchet space F with an FDD, where on F there exists a fundamental sequence $(p_k)_k$ of norms and an FDD determining sequence $(A_n)_n$ of operators such that $(F, p_k)\hat{}$ is reflexive for every k, that the sequence of the extentions of the A_n onto $(F, p_k)\hat{}$ determines an FDD of $(F, p_k)\hat{}$ and that $({}^kA_n')_n$ determines an FDD for $(F, p_k)'$. Since E is isomorphic to a space with the BAP, E itself has the BAP. $(P_n)_n$ may determine the BAP of E. Since E is a complemented subspace of F, there exists a continuous linear projection P from F onto E. Since P is a continuous linear projection and $\{P_n\colon n \in N\}$ and $\{A_n\colon n \in N\}$ are equicontinuous sets of operators, there exist, for every k, $l(k)$, $L(k) > 0$, $m(k)$, $M(k) > 0$, $j(k)$, $J(k) > 0$ such that

$$p_k\big(P(x)\big) \leqslant L(k)p_{l(k)}(x) \qquad \text{for every } x \in F,$$

$$p_k\big(A_n(x)\big) \leqslant M(k)p_{m(k)}(x) \qquad \text{for every } n \text{ and every } x \in F$$

and

$$p_k\big(P_n(x)\big) \leqslant J(k)p_{j(k)}(x) \qquad \text{for every } n \text{ and every } x \in E.$$

We choose in addition the sequences $\big(l(k)\big)_k$, $\big(L(k)\big)_k$, $\big(m(k)\big)_k$, $\big(M(k)\big)_k$, $\big(j(k)\big)_k$ and $\big(J(k)\big)_k$ such that they are strictly increasing and termwise larger than 1 and that $k < l(k)$, $k < m(k)$ and $k < j(k)$ for every k.

Lemma 3. *Let E be a Fréchet space and let $(p_k)_k$ be a fundamental sequence of norms on E. If A is a finite rank operator from E into E and if there exists an m such that A is also a continuous map from (E, p_m) into (E, p_m) then*

$$A'(y) = {}^mA'(y) \quad \text{for every } y \in (E, p_m)' \quad \text{and} \quad A'(E') = {}^mA'\big((E, p_m)'\big).$$

Proof. We have that $\dim\big(A(E)\big) = r$ for an $r \in N$. Hence A has a representation

$$A(x) = \sum_{n=1}^{r} f_n(x)x_n \qquad \text{for every } x \in E$$

where $x_n \in E$ and $f_n \in (E, p_m)'$ for $n = 1, 2, \ldots, r$.

It follows that

$$A'(y) = \sum_{n=1}^{r} y(x_n)f_n \qquad \text{for every } y \in E'$$

and

$${}^mA'(y) = \sum_{n=1}^{r} y(x_n)f_n \qquad \text{for every } y \in (E, p_m)'$$

which implies the assertion.

The following lemma gives an information how the spaces $P_n'(E')$ are embedded in $P'^{-1}(F')$ in the situation of Remark 1.

Lemma 4. *Let F, $(p_k)_k$, E, P and $(P_n)_n$ be chosen as in Remark 1. Then there exists a k_0 such that $P_n'(E') \subset P'^{-1}\big((F, p_{\hat{k}_0})'\big)$ for every n and every $\hat{k}_0 \geqslant k_0$.*

Proof. $\{P_nP\colon n \in N\}$ is an equicontinuous set of operators from F into E. Hence there exists a k_0 and an $M > 0$ such that

$$p_1\big(P_nP(x)\big) \leqslant Mp_{k_0}(x) \qquad \text{for every } x \in E.$$

Since $\dim\big(P_nP(E)\big) < \infty$ for every n, there exists for every n a $K_n > 0$ such that

$$K_np_{k_0}\big(P_nP(x)\big) \leqslant p_1\big(P_nP(x)\big) \qquad \text{for every } x \in E.$$

Hence P_nP is a continuous map from (F, p_{k_0}) into (E, p_{k_0}). Therefore we have ${}^{k_0}(P_nP)'\big((E, p_{k_0})'\big) \subset (F, p_{k_0})'$ and by Lemma 3

$${}^{k_0}(P_nP)'\big((E, p_{k_0})'\big) = (P_nP)'(E') = P'P_n'(E')$$

and thus

$$P_n'(E') \subset P'^{-1}\big((F, p_{k_0})'\big) \qquad \text{for every } n.$$

The only restriction on k_0 is that it has to be sufficiently large. This completes the proof.

Proposition 2 and Lemma 4 allow now an application of the methods of Johnson, Rosenthal and Zippin [7]. The purpose of the following four lemmata is to ensure essentially that in a Fréchet Schwartz space E with a continuous norm and the BAP for every finite dimensional subspace G of E and certain finite dimensional subspaces H of E' there exists a finite rank continuous linear projection Q on E, which is the identity on G, whose dual operator is the identity on H and which is bounded in a certain way. With the help of this fact it is then possible to prove the main result of this section.

The Lemmata 5–8 and Theorem 4 are modifications of corresponding results for Banach spaces presented in [7]. In spite of the fact that the proofs we give are in many parts identical to those given in [7] we feel legitimated to give these proofs.

LEMMA 5. *Let $\{X_j: j = 1, 2, \ldots, l\}$ be a set of linear vector spaces such that $X_j \subset X_{j+1}$ for $j = 1, 2, \ldots, l-1$ and let p_j be a norm on X_j for $j = 1, 2, \ldots, l$ such that $p_j(x) \geqslant p_{j+1}(x)$ for every $x \in X_j$ and $j = 1, 2, \ldots, l-1$. Let G be a k-dimensional subspace of X_1 and let*

$$\{(x_i, f_i): i = 1, 2, \ldots, k\} \subset G \times (X_l, p_l)'$$

be a biorthogonal system with $p_l'(f_i) = 1$ for $i = 1, 2, \ldots, k$. Let $\varepsilon > 0$ with $\varepsilon < 1$ and $\varepsilon(1+\varepsilon)(1-\varepsilon)^{-2} M < 1$, where

$$M = \max\Big\{\sum_{i=1}^{k} p_j(x_j): j = 1, 2, \ldots, l\Big\}.$$

If T is an operator from (X_l, p_l) into (X_l, p_l) with $\dim(T(X_l)) = n$, where $n \geqslant k$, and $T(X_l) \subset X_1$ such that

$$\sup\{p_j(T(x)-x): x \in G, p_j(x) \leqslant 1\} < \varepsilon \quad \text{for} \quad j = 1, 2, \ldots, l$$

then there exists an operator S form (X_l, p_l) into (X_l, p_l) such that

(a) $\dim(S(X_l)) = n$, $S(X_l) \subset X_1$ *and* $S(x) = x$ *for every* $x \in G$,

(b) ${}^lS'((X_l, p_l)') = {}^lT'((X_l, p_l)')$,

(c) *S is a projection if T is a projection,*

(d) $p_j(S(x)) \leqslant 2p_j(T(x))$ *for every* $x \in X_l$ *and* $j = 1, 2, \ldots, l$.

Proof. $U := T|_G$ is an invertible operator form G onto $T(G)$, because of the assumption

$$\sup\{p_j(T(x)-x): x \in G, p_j(x) \leqslant 1\} < \varepsilon \quad \text{for} \quad 1, 2, \ldots, l.$$

One has furthermore for $j = 1, 2, \ldots, l$ the estimates $\|U\|_j < 1+\varepsilon$ and $\|U^{-1}\|_j < (1-\varepsilon)^{-1}$ and hence $\|U^{-1} - I|_{T(G)}\|_j < \varepsilon(1-\varepsilon)^{-1}$, where $\|\ \|_j$ is the usual operator norm, if G and $T(G)$ are considered with the norm p_j. For $i = 1, 2, \ldots, k$ we have $|f_i(U^{-1}(x))| < (1-\varepsilon)^{-1} p_l(x)$ for every $x \in T(G)$ and therefore we can extend every functional $(f_i U^{-1})$ from $T(G)$ onto $T(X_l)$ to a functional $(f_i U^{-1})^{\sim}$ such that

$$p_l'((f_i U^{-1})^{\sim}) < (1-\varepsilon)^{-1}$$

and thus

$$p_j'((f_i U^{-1})^{\sim}) \leqslant p_l'((f_i U^{-1})^{\sim}) < (1-\varepsilon)^{-1} \quad \text{for} \quad j = 1, 2, \ldots l.$$

Consider the projection $P^{\sim}$ from $T(X_l)$ onto $T(G)$ with

$$P^{\sim}(x) = \sum_{i=1}^{k} (f_i U^{-1})^{\sim}(x) U(x_i).$$

We have

$$p_j(P^{\sim}(x)) \leqslant (1+\varepsilon)(1-\varepsilon)^{-1} M p_j(x) \quad \text{for} \quad j = 1, 2, \ldots, l.$$

Set $V = U^{-1}P^{\sim} + I|_{T(X_l)} - P^{\sim}$ and $S = VT$. The map S is obviously an operator from $(T(X_l), p_l)$ into $(\operatorname{span}\{F, T(X_l)\}, p_l)$. It is easy to check that $S(x) = x$ for every $x \in G$. We have furthermore for $j = 1, 2, \ldots, l$

$$\sup\{p_j((V - I|_{T(X_l)})(x)): x \in T(X_l), p_j(x) \leqslant 1\} \leqslant \varepsilon(1-\varepsilon)^{-2}(1+\varepsilon)M < 1.$$

This shows that V is injective and thus $\dim(T(X_l)) = n$ implies $\dim(S(X_l)) = n$. From the fact that $S^{-1}(0)$ and $T^{-1}(0)$ are finite codimensional and that $S^{-1}(0) = T^{-1}(0)$ it follows that

$${}^lS'((X_l, p_l)') = {}^lT'((X_l, p_l)').$$

If T is a projection, it is easy to calculate that $S(S(x)) = x$ for every $x \in X_l$ and hence S is a projection.

The assertion (d) follows by another simple calculation. This completes the proof.

LEMMA 6. *Let F, $(p_k)_k$, E, P, $(A_n)_n$, $(l(k))_k$, $(L(k))_k$, $(m(k))_k$ and $(M(k))$ be chosen as in Remark 1. Then there exists a k_1 such that for every $\hat{k_1} \geqslant k_1$, for every k and every finite dimensional subspace H of $P'^{-1}((F, p_{\hat{k_1}})')$ there exists a finite rank operator T from E into E such that*

$$T'(y) = y \quad \text{for every } y \in H, \quad T'(E') \subset P'^{-1}((F, p_{\hat{k_1}})')$$

and

$$\sup\{p_{\hat{k_1}+i}(T(x)): x \in E, p_{m(l(\hat{k_1}+i))}(x) \leqslant 1\} \leqslant 2L(\hat{k_1}+i)M(l(\hat{k_1}+i))$$
$$\text{for} \quad i = 1, 2, \ldots, k.$$

Proof. Let J be the canonical embedding of E as a complemented subspace of F. Choose $k_1 \geqslant l(1)$ and $M > 0$ such that $p_{l(1)}(A_n JP(x)) \leqslant Mp_{k_1}(x)$ for every $x \in F$ and every n. Since $\dim(A_n JP(F)) < \infty$ for every n, there exists for every n a $K_n > 0$ such that

$$K_n p_{k_1}(A_n JP(x)) \leqslant p_{l(1)}(A_n JP(x)) \quad \text{for every } x \in F.$$

Let $k \in \mathbf{N}$, $G := P'(H)$, $d := \dim(G)$ and

$$\{(x_r, f_r): r = 1, 2, \ldots, d\} \subset G \times (F, p_{k_1+k})''$$

be a biorthogonal system with $p_{k_1+k}''(f_r) = 1$ for $r = 1, 2, \ldots, d$.

Choose $\varepsilon > 0$ with $\varepsilon < 1$ and $\varepsilon(1+\varepsilon)(1-\varepsilon)^{-2}\, M < 1$, where

$$M = \max\Big\{\sum_{i=1}^{d} p'_{k_1+j}(x_i):\ j = 1, 2, \ldots, k\Big\}.$$

Since $({}^{k}A'_n)_n$ determines a BAP in $(F, p_k)'$ we can choose an $n(G)$ such that

$$\sup\big\{p'_{m(k_1+i)}\big({}^{m(k_1+k)}A'_n(y) - y\big):\ y \in G,\ p'_{m(k_1+i)}(y) \leqslant 1\big\} < \varepsilon$$

for every $n = n(G), n(G)+1, \ldots$ and $i = 1, 2, \ldots, k$.

By Lemma 5 there exists a finite rank operator R from $(F, p_{m(k_1+k)})'$ into $(F, p_{m(k_1+k)})'$ such that $R(y) = y$ for every $y \in G$,

$$R'\big((F, p_{m(k_1+k)})''\big) = ({}^{m(k_1+k)}A'_{n(G)})'\big((F, p_{m(k_1+k)})''\big)$$

and

$$\sup\big\{p'_{m(k_1+i)}\big(R(x)\big):\ x \in (F, p_{k_1+i})',\ p'_{k_1+i}(x) \leqslant 1\big\}$$
$$\leqslant 2\sup\big\{p'_{m(k_1+i)}\big({}^{m(k_1+k)}A'_{n(G)}(x)\big):\ x \in (F, p_{k_1+i})',\ p'_{k_1+i}(x) \leqslant 1\big\}$$
$$\text{for every } i = 1, 2, \ldots, k.$$

Since we have $({}^{m(k_1+k)}A'_{n(G)})'\big((F, p_{m(k_1+k)})''\big) \subset F$, we can define S to be the restriction $R'|_F$ of R' onto F, where $R'|_F$ is considered as an operator from F into F.

It is easy to see that ${}^{m(k_1+k)}S' = R$ and hence $S'(y) = y$ for every $y \in G$. Furthermore we have the following estimates:

$$\sup\big\{p_{k_1+i}\big(S(x)\big):\ x \in F,\ p_{m(k_1+i)}(x) \leqslant 1\big\}$$
$$\leqslant 2\sup\big\{p_{k_1+i}\big(A_{n(G)}(x)\big):\ x \in F,\ p_{m(k_1+i)}(x) \leqslant 1\big\} \leqslant 2\ M(k_1+i)$$
$$\text{for}\quad i = 1, 2, \ldots, k.$$

Define $T := PSJ$. Then T is a finite rank operator from E into E and we have $T'(y) = J'S'P'(y) = J'P'(y) = y$ for every $y \in H$. Our choice of k_1 implies that $PSJP$ is a continuous map from (F, p_{k_1}) into (E, p_{k_1}). We argue as follows:

Since $\dim\big(PSJP(F)\big) < \infty$, there exists a $K > 0$ with

$$Kp_{k_1}\big(PSJP(x)\big) \leqslant p_1\big(PSJP(x)\big) \leqslant L(1)p_{l(1)}\big(SJP(x)\big) \leqslant L(1)p_{k_1}\big(SJP(x)\big)$$
$$\leqslant 2\ L(1)p_{k_1}\big(A_{n(G)}JP(x)\big) \leqslant 2\ L(1)M(K_{n(G)})^{-1}p_{k_1}(x)$$
$$\text{for every } x \in F,$$

where $M > 0$ and $K_{n(G)} > 0$ were chosen in the beginning of the proof. Hence we have ${}^{k_1}(PSJP)'\big((E, p_{k_1})'\big) \subset (F, p_{k_1})'$ and because of ${}^{k_1}(PSJP)'\big((E, p_{k_1})'\big) = P'T'(E')$ we can conclude

$$T'(E') \subset P'^{-1}\big((F, p_{k_1})'\big).$$

Furhermore an easy computation gives

$$p_{k_1+i}\big(T(x)\big) \leqslant 2L(k_1+i)M\big(l(k_1+i)\big)\, p_{m(l(k_1+i))}(x)$$
$$\text{for every } x \in E \quad\text{and}\quad i = 1, 2, \ldots, k.$$

The only restriction on k_1 is that it has to be sufficiently large. This completes the proof.

LEMMA 7. *Let F, $(p_k)_k$, E, $(P_n)_n$, $\big(j(k)\big)_k$ and $\big(J(k)\big)_k$ be chosen as in Remark 1. Then for every k and every finite dimensional subspace G of E there exists a finite rank operator S from E into E such that*

$$S(x) = x \quad \textit{for every } x \in G,\quad S'(E') \subset \bigcup_n P'_n(E'),$$

$$\sup\big\{p_i\big(S(x)\big): x \in E,\ p_{j(i)}(x) \leqslant 1\big\} \leqslant 2\ J(i) \quad \textit{for every } i = 1, 2, \ldots, k$$

and that S is a projection if P_n is a projection for every n.

Proof. Let $k \in \mathbf{N}$, G be a finite dimensional subspace of E, $d := \dim(G)$ and $\{(x_r, f_r):\ r = 1, 2, \ldots, d\} \subset G \times (E, p_1)'$ be a biorthogonal system with $p'_1(f_r) = 1$ for $r = 1, 2, \ldots, d$.

Choose $\varepsilon > 0$ with $\varepsilon < 1$ and $\varepsilon(1+\varepsilon)(1-\varepsilon)^{-2}M < 1$, where

$$M = \max\Big\{\sum_{r=1}^{d} p_j(x_r):\ j = 1, 2, \ldots, k\Big\}.$$

Since $(P_n)_n$ determines a BAP (resp. a BPAP) of E, we can find an $n(G)$ such that

$$\sup\big\{p_j\big(P_n(x) - x\big):\ x \in G,\ p_j(x) \leqslant 1\big\} < \varepsilon$$
$$\text{for}\quad n = n(G), n(G)+1, \ldots \text{ and }\quad j = 1, 2, \ldots, k.$$

An application of Lemma 5 implies then the assertion.

LEMMA 8. *Let F, $(p_k)_k$, E, P, $(P_n)_n$, $\big(L(k)\big)_k$, $\big(l(k)\big)_k$, $\big(M(k)\big)_k$, $\big(m(k)\big)_k$, $\big(J(k)\big)_k$ and $\big(j(k)\big)_k$ be chosen as in Remark 1 and let P_n in addition be a projection for every n. Then there exists a k_2 such that for every k, every finite dimensional subspace H of $P'^{-1}\big((F, p_{k_2})'\big)$ and every finite dimensional subspace G of E there exists a continuous linear projection Q from E into E such that*

$$Q(x) = x \quad \textit{for every } x \in G,$$

$$Q'(y) = y \quad \textit{for every } y \in H,\quad Q'(E') \subset P'^{-1}\big((F, p_{k_2})'\big)$$

and

$$\sup\{p_{k_2+i}(Q(x)):\ x\in E,\ p_{j(m(l(k_2+i)))}(x)\leqslant 1\}$$
$$\leqslant 8\ L(k_2+i)M\big(l(k_2+i)\big)\,J\big(m\big(l(k_2+i)\big)\big)\quad \textit{for}\quad i=1,2,\ldots,k.$$

Proof. Choose k_2 larger than k_0 of Lemma 4 and larger than k_1 of Lemma 6. Let $k\in N$. According to Lemma 6 there exists a finite rank operator T from E into E such that $T'(y)=y$ for every $y\in H$,

$$T'(E')\subset P'^{-1}\big((F,p_{k_2})'\big)$$

and

$$\sup\{p_{k_2+i}(T(x)):\ x\in E,\ p_{m(l(k_2+1))}(x)\leqslant 1\}\leqslant 2\ L(k_2+i)M\big(l(k_2+i)\big)$$
$$\text{for}\quad i=1,2,\ldots,k.$$

Choose $D:=\operatorname{span}\{G,T(E)\}$. According to Lemma 7 there exists a finite rank continuous linear projection S from E into E such that $S(x)=x$ for every $x\in D$,

$$S'(E')\subset\bigcup_n P_n'(E')$$

and

$$\sup\{p_{k_2+i}(S(x)):\ x\in E,\ p_{j(k_2+i)}(x)\leqslant 1\}\leqslant 2\ J(k_2+i)$$
$$\text{for}\quad i=1,2,\ldots,k.$$

Set $Q:=T+S-TS$. We will show now that Q has all required properties.

Since S is a projection and $T(E)\subset S(E)$, it follows from an easy calculation that Q is a projection on E. We have $(I-T')(y)=0$ for every $y\in H$ and therefore $(I-Q')(y)=0$ and hence $Q'(y)=y$ for every $y\in H$ and we have $(I-Q)(x)=0$ for every $x\in D$ and thus $Q(x)=x$ for every $x\in G\subset D$. The Lemmata 4 and 7 imply

$$S'(E')\subset\bigcup_n P_n'(E')\subset P'^{-1}\big((F,p_{k_2})'\big)$$

and Lemma 6 implies $T'(E')\subset P'^{-1}\big((F,p_{k_2})'\big)$ and hence

$$Q'(E')\subset P'^{-1}\big((F,p_{k_2})'\big).$$

Moreover, it follows easily from Lemmata 6 and 7 that

$$\sup\{p_{k_2+i}(Q(x)):\ x\in E,\ p_{j(m(l(k_2+i)))}(x)\leqslant 1\}$$
$$\leqslant 8\ L(k_2+i)M\big(l(k_2+i)\big)\,J\big(m\big(l(k_2+i)\big)\big)\quad\text{for}\quad i=1,2,\ldots,k.$$

This completes the proof.

THEOREM 4. *Every Fréchet Schwartz space with a continuous norm and the* BPAP *has an absolute* FDD.

Proof. A Fréchet Schwartz space with a continuous norm and the BPAP is according to Theorem 3 isomorphic to a complemented subspace E of a Fréchet space F, where E and F can be chosen as in Remark 1. The space E has then also the BPAP.

Therefore, let F, $(p_k)_k$, E, $(P_n)_n$, $\big(L(k)\big)_k$, $\big(l(k)\big)_k$, $\big(M(k)\big)_k$, $\big(m(k)\big)_k$, $\big(J(k)\big)_k$ and $\big(j(k)\big)_k$ be chosen as in Remark 1. Since E has the BPAP, we can assume that P_n is a projection for every n. Let k_2 be chosen as in Lemma 8.

Since E is a Fréchet Schwartz space, it is separable, so we can choose in E a dense subset $\{x_n:\ n\in N\}$ with $x_1=0$. Now an FDD determining sequences $(Q_n)_n$ of operators on E will be constructed inductively. Let $Q_1:=P_1$. We have

$$\sup\{p_{k_2+1}(Q_1(x)):\ x\in E,\ p_{j(m(l(k_2+1)))}(x)\leqslant 1\}$$
$$\leqslant 8\ L(k_2+1)M\big(l(k_2+1)\big)\,J\big(m\big(l(k_2+1)\big)\big).$$

Let $k\in N$ and assume that the projections $Q_1,\ldots,Q_k$ on E are already chosen such that the following conditions are fulfilled for $i=1,2,\ldots,k$ and $j=1,2,\ldots,k$:

(a) $$Q_iQ_j=Q_jQ_i=Q_{\min\{i,j\}},$$

(b) $$Q_i(E)\supset\{x_1,\ldots,x_i\},$$

(c) $$Q_i'(E')\subset P'^{-1}\big((F,p_{k_2})'\big)$$

and

(d) $$\sup\{p_{k_2+r}(Q_i(x)):\ x\in E,\ p_{j(m(l(k_2+r)))}(x)\leqslant 1\}$$
$$\leqslant 8\ L(k_2+r)M\big(l(k_2+r)\big)\,J\big(m\big(l(k_2+r)\big)\big)\quad\text{for}\quad r=1,2,\ldots,i.$$

Set

$$G_{k+1}:=\operatorname{span}\{x_{k+1},Q_k(E)\}\quad\text{and}\quad H_{k+1}:=Q_k'(E').$$

According to Lemma 8 there exists a finite rank continuous linear projection Q_{k+1} on E such that

$$Q_{k+1}(x)=x\quad\text{for every}\quad x\in G_{k+1},$$
$$Q_{k+1}'(y)=y\quad\text{for every } y\in H_{k+1},\qquad Q_{k+1}'(E')\subset P'^{-1}\big((F,p_{k_2})'\big)$$

and

$$\sup\{p_{k_2+i}(Q_{k+1}(x)):\ x\in E,\ p_{j(m(l(k_2+i)))}(x)\leqslant 1\}$$
$$\leqslant 8\ L(k_2+i)M\big(l(k_2+i)\big)\,J\big(m\big(l(k_2+i)\big)\big)\quad\text{for}\quad i=1,2,\ldots,k+1.$$

This shows (d). It follows immediately from the construction of Q_{k+1} that (b) and (c) are fulfilled. One has $Q_{k+1}Q_i=Q_i$ and $Q_{k+1}'Q_i'=Q_i'$ for

$i = 1, 2, \ldots, k+1$ and thus (a) is fulfilled. It follows from (d) that $\{Q_n : n \in N\}$ is equicontinuous.

Now it is easy to see that $(Q_n)_n$ determines an FDD of E, hence E has an absolute FDD by Theorem 1 and so does every Fréchet space isomorphic to E. This completes the proof.

The following theorem is a modification of a result of Johnson [6].

THEOREM 5. *Let E be a Fréchet Schwartz space with a continuous norm and the* BAP *and let λ be a Köthe sequence space. Then there exists a sequence $(G_n)_n$ of finite dimensional subspaces of E such that $E \times \lambda((G_n)_n)$ has an* FDD.

Proof. Since E is separable, there exists a sequence $(H_n)_n$ of finite dimensional subspaces of E with $H_1 \subset H_2 \subset \ldots$ and $E = \mathrm{cl}(\bigcup_n H_n)$. It follows from Lemma 7 that there exists an equicontinuous set $\{T_n : n \in N\}$ of finite rank operators from E into E such that we have for every n that $T_n(x) = x$ for every $x \in H_n$. Set

$$G_n := (I - T_n)(T_n(E)) \quad \text{for every } n \quad \text{and} \quad F := E \times \lambda((G_n)_n).$$

We have

$$F = \mathrm{cl}\Big(\bigcup_{i=2}^{\infty} \big(H_i \times \lambda((G_n)_{n \leqslant i-1})\big)\Big).$$

Let Q_i be the canonical projection from $\lambda((G_n)_n)$ onto $\lambda((G_n)_{n \leqslant i-1})$ and let P_i be the canonical projection from $\lambda((G_n)_n)$ onto G_i for every i. Define a map S_{i-1} from F into F by

$$S_{i-1}(x, y) = \big(T_i(x) + P_i(y), (I - T_i)(T_i(x)) + (I - T_i)(P_i(y)) + Q_i(y)\big)$$
$$\text{for every } (x, y) \in F \text{ and } i = 2, 3, \ldots$$

It is easy to check that $\{S_i : i \in N\}$ is equicontinuous, that $S_i(x, y) = (x, y)$ for every $(x, y) \in H_{i+1} \times \lambda((G_n)_{n \leqslant i})$ and that S_i is a projection for every i. Hence $(S_i)_i$ determines a BPAP of F and thus F has an FDD by Theorem 4.

COROLLARY 1. *If E is a Fréchet Schwartz space (resp. a nuclear Fréchet space) with a continuous norm and the BAP then there exists a Fréchet Schwartz space (resp. a nuclear Fréchet space) F with an* FDD *and a continuous norm such that the product space of E and F has an* FDD.

The proof follows from Theorem 5 and Lemma 2 and is similar to that of Theorem 3.

References

[1] P. B. Djakov, B. S. Mitjagin, *Modified construction of a nuclear Fréchet space without basis*, J. Functional Analysis 23 (1976), 415–423.
[2] Ed Dubinsky, *The structure of nuclear Fréchet spaces*, Lecture Notes in Math. 720, Springer-Verlag, Berlin–Heidelberg–New York 1979.

[3] — *Nuclear Fréchet spaces without the bounded approximation property*, Studia Math. 71 (1981), 85–105.
[4] Ed Dubinsky, M. S. Ramanujan, *On λ-nuclearity*, Mem. Amer. Math. Soc. 128 (1972).
[5] W. B. Johnson, *Finite dimensional decompositions in certain Fréchet spaces*, Collect. Math. 23 (1971), 269–272.
[6] — *Factoring compact operators*, Israel J. Math. 9 (1971), 337–345.
[7] W. B. Johnson, H. P. Rosenthal, M. Zippin, *On bases, finite dimensional decompositions and weaker structures in Banach spaces*, ibid. 9 (1971), 488–506.
[8] C. Matyszczyk, *Approximation of analytic and continuous mappings by polynomials in Fréchet spaces*, Studia Math. 60 (1977), 223–238.
[9] B. S. Mitjagin, N. M. Zobin, *Contre-example à l'existence d'une base dans un espace de Fréchet nucléaire*, C. R. Acad. Sci. Paris Sér. A 279 (1974), 255–256; 325–327.
[10] A. Pełczyński, *Any separable Banach space with the bounded aproximation property is a complemented subspace of a Banach space with a basis*, Studia Math. 40 (1971), 239–243.
[11] A. Pełczyński, P. Wojtaszczyk, *Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces*, Studia Math. 40 (1971), 91–108.
[12] A. Pietsch, *Zur Theorie der topologischen Tensorprodukte*, Math. Nachr. 25 (1963), 19–33.
[13] N. M. Zobin, B. S. Mitjagin, *Examples of nuclear linear metric spaces without a basis*, Functional Anal. Appl. 8, 4 (1975), 304–313.

FACHBEREICH MATHEMATIK
TECHNISCHE HOCHSCHULE
6100 DARMSTADT, FEDERAL REPUBLIC OF GERMANY

Received November 5, 1980 (1650)