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# Condensation principles with rates

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**Abstract.** The purpose of this paper is to supply the classical condensation principle of Banach–Steinhaus (1927) with rates. The method of proof consists in the familiar gliding hump method, but now equipped with rates. First applications concerning the sharpness of (pointwise) error bounds are given in connection with Fourier partial sums, Bernstein polynomials, Lagrange interpolation, and numerical quadrature.

**1. Introduction.** Let $X$ be a Banach space, $Y$ a normed linear space (with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively), and let $[X, Y]$ be the space of bounded linear operators of $X$ into $Y$. Then one version of the classical condensation principle (CP) reads ($\mathbf{N}$: = set of natural numbers):

CP. *Let* $\{T_{n,p}\}_{n,p\in\mathbf{N}} \subset [X, Y]$. *Suppose that for each* $n, p \in \mathbf{N}$ *there exists* $h_{n,p} \in X$ *such that*

$$\|h_{n,p}\|_X \leqslant C, \tag{1.1}$$

$$\limsup_{n\to\infty} \|T_{n,p} h_{n,p}\|_Y = \infty. \tag{1.2}$$

*Then there exists* $f_0 \in X$, *independent of* $n, p \in \mathbf{N}$, *such that*

$$\limsup_{n\to\infty} \|T_{n,p} f_0\|_Y = \infty \tag{1.3}$$

*simultaneously for all* $p \in \mathbf{N}$.

Note that conditions (1.1, 2) are equivalent to the fact that for each $p \in \mathbf{N}$

$$\limsup_{n\to\infty} \|T_{n,p}\|_{[X,Y]} = \infty, \tag{1.4}$$

which, in view of the classical uniform boundedness principle (UBP), is also equivalent to the existence of a sequence $\{f_p\}_{p\in\mathbf{N}} \subset X$ such that for each $p \in \mathbf{N}$

$$\limsup_{n\to\infty} \|T_{n,p} f_p\|_Y = \infty. \tag{1.5}$$

Of course, these remarks just reflect alternative, perhaps more familiar versions of the classical CP.

The CP was shown in 1927 by Banach–Steinhaus [2] via Baire category arguments; see also the account given in [1], p. 81 ff, [11], p. 23 ff in connection with a number of applications. For a proof via the gliding hump method see [20]. Of course, in view of the implication (1.5)⇒(1.3), the CP is indeed an extension of the UBP, which just corresponds to the implication (1.1,2)⇒(1.5), thus delivering the existence of a sequence $\{f_p\}_{p\in N}\subset X$ rather than a single element $f_0\in X$, independent of $p$, such that (1.5), (1.3), respectively, hold true.

Continuing our previous work on UBP's with rates (cf. [5], [6], [7]), it is the purpose of this paper to develop CP's with rates in the sense that (1.1–1.3) are equipped with rates. To this end, Sec. 2 first considers the case of countable index families. The method of proof will be the gliding hump method but now equipped with rates. Following classical work of Orlicz [15], the results of Sec. 2 are then extended in Sec. 3 to uncountable (topologized) index families, using category arguments. Finally, Sec. 4 outlines some first illustrating examples concerned with Fourier partial sums, Bernstein polynomials, Lagrange interpolation, and numerical quadrature of indefinite integrals.

## 2. Condensation principles with rates for countable index families.

Let $U\subset X$ be a seminormed linear subspace of $X$ with seminorm $|\cdot|_U$. Then the $K$-functional $(t\geqslant 0)$

$$K(t,f):=K(t,f;X,U):=\inf\{\|f-g\|_X+t|g|_U;\ g\in U\} \tag{2.1}$$

serves as an abstract measure of smoothness for the element $f\in X$ (cf. (4.1,4)). Let $\omega$ be a continuous function on $[0,\infty)$ with

$$\begin{aligned} &0<\omega(t)\leqslant\omega(s) && (0<t\leqslant s),\\ &\omega(t+s)\leqslant\omega(t)+\omega(s) && (s,t\geqslant 0). \end{aligned} \tag{2.2}$$

Note that necessarily $\liminf_{t\to 0+}\omega(t)/t>0$, in fact (cf. [18], p. 96 ff)

$$0<\omega(s)/s\leqslant 2\omega(t)/t\qquad (0<t\leqslant s). \tag{2.3}$$

Thus, if $\omega(0)=0$, then $\omega$ is an abstract modulus of continuity in the usual sense. Consider the intermediate spaces $(t\to 0+)$

$$X_\omega^0=\{f\in X;\ K(t,f)=o_f(\omega(t))\}, \tag{2.4}$$

$$X_\omega=\{f\in X;\ K(t,f)=O_f(\omega(t))\}, \tag{2.5}$$

and let $\{\varphi_{n,p}\}_{n,p\in N}$ denote a double sequence of (strictly) positive numbers with

$$\lim_{n\to\infty}\varphi_{n,p}=0 \tag{2.6}$$

for each $p\in N$. Then one has the following version of a CP with large $O$-rates.



THEOREM 1. *Let $\{T_p\}_{p\in N}$, $\{T_{n,p}\}_{n,p\in N}\subset[X,Y]$ and let $\omega$, $\{\varphi_{n,p}\}_{n,p\in N}$ satisfy (2.2), (2.6), respectively. Suppose that for each $n,p\in N$ there exists $h_{n,p}\in U$ such that*

$$\|h_{n,p}\|_X\leqslant C_1, \tag{2.7}$$

$$|h_{n,p}|_U\leqslant C_2/\varphi_{n,p}, \tag{2.8}$$

$$\limsup_{n\to\infty}\|T_{n,p}h_{n,p}\|_Y=\infty\qquad (p\in N). \tag{2.9}$$

(a) *If $\omega$ satisfies additionally*

$$\lim_{t\to 0+}\omega(t)/t=\infty, \tag{2.10}$$

*then there exists $f_\omega\in X_\omega^0$, independent of $p\in N$, such that for each $p\in N$*

$$\limsup_{n\to\infty}\|T_{n,p}f_\omega-T_pf_\omega\|_Y/\omega(\varphi_{n,p})=\infty. \tag{2.11}$$

(b) *If $\omega(t)=\omega_1(t):=t$ (i.e., essentially, one does not have (2.10) (cf. (2.3))), then there exists $f_\omega\in X_\omega$, independent of $p\in N$, such that (2.11) holds true (for $\omega=\omega_1$).*

*Thus, in any case, there exists an element $f_\omega$ as specified above such that for each $p\in N$*

$$\|T_{n,p}f_\omega-T_pf_\omega\|_Y\neq O(\omega(\varphi_{n,p}))\qquad (n\to\infty). \tag{2.11*}$$

Proof. Let the sequence $\{p_k\}_{k\in N}\subset N$ be given via

$$p_1:=1,\qquad p_{k+1}:=\begin{cases}p_k-1, & \text{if}\quad p_k>1,\\ k+1, & \text{if}\quad p_k=1,\end{cases} \tag{2.12}$$

in other words, if $k=2^l+m$, $0\leqslant m<2^l$, $m,l\in N\cup\{0\}$, then $p_k=2^l-m$. It follows that for each $p\in N$ there are infinitely many $k\in N$ with $p_k=p$. Now one may successively construct sequences $\{a_k\}_{k\in N}\subset(0,1/2]$, $\{\delta_k\}_{k\in N}\subset\{0,1\}$, $\{n_k\}_{k\in N}\subset N$ with $a_1=1/2$, $\delta_1=n_1=1$ such that for $k\geqslant 2$

$$0<a_k\leqslant a_{k-1}\leqslant 1/2, \tag{2.13}$$

$$\text{(i)}\qquad n_k>n_{k-1},\quad \varphi_{n_k,p_k}<\varphi_{n_{k-1},p_{k-1}}, \tag{2.14}$$

and in case (a) additionally

$$\text{(ii)}\qquad \sum_{j=1}^{k-1}a_j^j\,\omega(\varphi_{n_j,p_j})/\varphi_{n_j,p_j}\leqslant a_k^k\,\omega(\varphi_{n_k,p_k})/\varphi_{n_k,p_k},$$

$$\|T_{n_{k-1},p_{k-1}}-T_{p_{k-1}}\|_{[X,Y]}\,a_k^k\,2C_1\leqslant(k-1)/2, \tag{2.15}$$

$$\|T_{n_k,p_k}g_k-T_{p_k}g_k\|_Y\geqslant k\omega(\varphi_{n_k,p_k}), \tag{2.16}$$

$$g_k:=\sum_{j=1}^{k}\delta_ja_j^j\,\omega(\varphi_{n_j,p_j})h_{n_j,p_j}\in U.$$

Indeed, if the first $k-1$ elements of the sequences, and thus $g_{k-1} \in U$, are given, choose $a_k$ small enough so that (2.13, 15) are satisfied. Consider

$$M_{k-1} := \limsup_{n\to\infty} \|T_{n,p_k} g_{k-1} - T_{p_k} g_{k-1}\|_Y / \omega(\varphi_{n,p_k}).$$

In case $M_{k-1} < \infty$ there exists an integer $n_k$ with

$$\|T_{n_k,p_k} g_{k-1} - T_{p_k} g_{k-1}\|_Y \leqslant (M_{k-1}+1)\,\omega(\varphi_{n_k,p_k}).$$

Moreover, $n_k$ can be chosen so large that (2.14) is satisfied (cf. (2.6, 10)), as well as

$$\|T_{n_k,p_k} h_{n_k,p_k}\|_Y \geqslant k a_k^{-k} + C_1 \|T_{p_k}\|_{[X,Y]} + (M_{k-1}+1)\, a_k^{-k}$$

(cf. (2.9)). Hence, setting $\delta_k = 1$, the element $g_k$ is well-defined and (cf. (2.7))

$$\|T_{n_k,p_k} g_k - T_{p_k} g_k\|_Y \geqslant a_k^k \omega(\varphi_{n_k,p_k}) \|T_{n_k,p_k} h_{n_k,p_k}\|_Y -$$
$$- a_k^k \omega(\varphi_{n_k,p_k}) \|T_{p_k} h_{n_k,p_k}\|_Y - (M_{k-1}+1)\,\omega(\varphi_{n_k,p_k}) \geqslant k\omega(\varphi_{n_k,p_k}),$$

i.e., (2.16) holds true. In case $M_{k-1} = \infty$, condition (2.16) (as well as (2.14)) of course holds true for $g_k := g_{k-1}$, i.e., $\delta_k := 0$, with suitable $n_k$.

Since $X$ is complete and (cf. (2.2, 7, 13, 14))

$$(2.17)\qquad \sum_{j=k+1}^{\infty} \delta_j a_j^j \omega(\varphi_{n_j,p_j}) \|h_{n_j,p_j}\|_X \leqslant C_1 a_{k+1}^{k+1} \omega(\varphi_{n_{k+1},p_{k+1}}) \sum_{j=0}^{\infty} 2^{-j} < \infty,$$

the element

$$f_\omega := \sum_{j=1}^{\infty} \delta_j a_j^j \omega(\varphi_{n_j,p_j}) h_{n_j,p_j}$$

is well-defined in $X$. Moreover, $f_\omega \in X_\omega^0$ and $f_{\omega_1} \in X_{\omega_1}$, respectively. Indeed, since for each $t \in (0, \varphi_{1,1})$ there exists $k$ such that $\varphi_{n_{k+1},p_{k+1}} \leqslant t < \varphi_{n_k,p_k}$ (note that $\lim_{k\to\infty} \varphi_{n_k,p_k} = 0$ in view of (2.6, 12, 14)), one obtains in case (a) by (2.1, 8, 13, 14 (ii), 17) and finally (2.2, 3) that

$$K(t, f_\omega) \leqslant \|f_\omega - g_k\|_X + t|g_k|_U$$
$$\leqslant \sum_{j=k+1}^{\infty} \delta_j a_j^j \omega(\varphi_{n_j,p_j}) \|h_{n_j,p_j}\|_X + t \sum_{j=1}^{k} \delta_j a_j^j \omega(\varphi_{n_j,p_j}) |h_{n_j,p_j}|_U$$
$$\leqslant 2C_1 a_{k+1}^{k+1} \omega(\varphi_{n_{k+1},p_{k+1}}) + 2tC_2 a_k^k \omega(\varphi_{n_k,p_k}) / \varphi_{n_k,p_k}$$
$$\leqslant (2C_1 2^{-k-1} + 4C_2 2^{-k})\,\omega(t) = o(\omega(t)),$$

i.e., $f_\omega \in X_\omega^0$. In case (b) one obtains analogously

$$K(t, f_{\omega_1}) \leqslant \|f_{\omega_1} - g_k\|_X + t|g_k|_U$$
$$\leqslant 2C_1 \varphi_{n_{k+1},p_{k+1}} + tC_2 \sum_{j=1}^{\infty} a_j^j \leqslant Ct,$$



thus $f_{\omega_1} \in X_{\omega_1}$. Now (2.15–17) deliver

$$\|T_{n_k,p_k} f_\omega - T_{p_k} f_\omega\|_Y \geqslant \|T_{n_k,p_k} g_k - T_{p_k} g_k\|_Y - \|T_{n_k,p_k} - T_{p_k}\|_{[X,Y]} \|f_\omega - g_k\|_X$$
$$\geqslant (k/2)\,\omega(\varphi_{n_k,p_k}).$$

Since for each $p \in \mathbf{N}$ there are infinitely many $k \in \mathbf{N}$ such that $p_k = p$ (cf. (2.12)), this completes the proof.

Let us mention that Thm. 1 (a) subsumes the classical CP as a special case, namely for $\omega = \omega_0 := 1$, $U = X$, $T_p = 0$. Indeed, conditions (1.1, 2) and (2.7, 9) coincide, whereas (2.8) is then trivially satisfied for any $\{\varphi_{n,p}\}$. Note that now $X_\omega^0 = X$.

Dealing with (proper) rates, however, one may even replace large $O$-rates in Thm. 1 (a) by small $o$-ones if the limiting case $\omega_0$ (and $\omega_1$) is excluded.

THEOREM 2. *Let $\{T_{n,p}\}_{n,p\in\mathbf{N}} \subset [X, Y]$, and let $\{\varphi_{n,p}\}_{n,p\in\mathbf{N}}$ satisfy (2.6). Suppose that for each $n, p \in \mathbf{N}$ there exists $h_{n,p} \in U$ such that conditions (2.7, 8) hold true as well as*

$$(2.18)\qquad \limsup_{n\to\infty} \|T_{n,p} h_{n,p}\|_Y \geqslant C_3 > 0.$$

*Then there exists a constant $C_4 > 0$ such that for each modulus $\omega$ satisfying (2.2) as well as (2.10) and*

$$(2.19)\qquad \lim_{t\to 0+} \omega(t) = \omega(0) = 0$$

*there exists $f_\omega \in X_\omega$, independent of $p \in \mathbf{N}$, such that for each $p \in \mathbf{N}$*

$$(2.20)\qquad \limsup_{n\to\infty} \|T_{n,p} f_\omega\|_Y / \omega(\varphi_{n,p}) \geqslant C_4.$$

*In particular,*

$$(2.20)^*\qquad \|T_{n,p} f_\omega\|_Y \neq o(\omega(\varphi_{n,p})) \quad (n\to\infty).$$

Proof. Proceeding as in the proof of Thm. 1 (a), let the sequence $\{p_k\}_{k\in\mathbf{N}}$ be given by (2.12). Then one may construct sequences $\{n_k\}_{k\in\mathbf{N}} \subset \mathbf{N}$, $\{\delta_k\}_{k\in\mathbf{N}} \subset \{0, 1\}$ with $n_1 = \delta_1 = 1$ satisfying for $k \geqslant 2$ conditions (2.14) (of course, (ii) without any $a_j$'s) as well as

$$(2.21)\qquad \omega(\varphi_{n_k,p_k}) \leqslant (1/2)\,\omega(\varphi_{n_{k-1},p_{k-1}}),$$

$$(2.22)\qquad \|T_{n_{k-1},p_{k-1}}\|_{[X,Y]} \leqslant (C_3/20C_1)\,\omega(\varphi_{n_{k-1},p_{k-1}}) / \omega(\varphi_{n_k,p_k}),$$

$$(2.23)\qquad \|T_{n_k,p_k} g_k\|_Y \geqslant (C_3/5)\,\omega(\varphi_{n_k,p_k}),$$

$$g_k := \sum_{j=1}^{k} \delta_j \omega(\varphi_{n_j,p_j}) h_{n_j,p_j} \in U.$$

Indeed, if the first $k-1$ elements of the sequences, and thus $g_{k-1}\in U$, are given, one may consider

$$M_{k-1}:=\limsup_{n\to\infty}\|T_{n,p_k}g_{k-1}\|_Y/\omega(\varphi_{n,p_k}).$$

If $M_{k-1}\leqslant 2C_3/5$, one has for all but a finite number of values of $n$

$$\|T_{n,p_k}g_{k-1}\|_Y/\omega(\varphi_{n,p_k})\leqslant 3C_3/5. \tag{2.24}$$

Therefore, setting $\delta_k:=1$, we may choose $n_k$ (herewith determining $h_{n_k,p_k}$ and $g_k$) large enough to satisfy (2.14, 21, 22, 24) (cf. (2.19)) as well as (cf. (2.18))

$$\|T_{n_k,p_k}h_{n_k,p_k}\|_Y\geqslant 4C_3/5,$$

thus (2.23). If $M_{k-1}>2C_3/5$, take $\delta_k=0$ thus $g_k:=g_{k-1}$. Then, of course, one may find $n_k$ large enough to satisfy (2.23) as well as (2.14, 21, 22).

Since $X$ is complete and (cf. (2.7, 21))

$$\sum_{j=k+1}^{\infty}\delta_j\,\omega(\varphi_{n_j,p_j})\|h_{n_j,p_j}\|_X\leqslant C_1\,\omega(\varphi_{n_{k+1},p_{k+1}})\sum_{j=0}^{\infty}2^{-j}<\infty, \tag{2.25}$$

the element

$$f_\omega:=\sum_{j=1}^{\infty}\delta_j\,\omega(\varphi_{n_j,p_j})h_{n_j,p_j}$$

is well-defined in $X$. Moreover, $f_\omega\in X_\omega$. Indeed, choosing $t,k$ as in the proof of Thm. 1, one analogously obtains (cf. (2.14 (ii), 25))

$$\begin{aligned}K(t,f_\omega)&\leqslant\|f_\omega-g_k\|_X+t|g_k|_U\\&\leqslant 2C_1\,\omega(\varphi_{n_{k+1},p_{k+1}})+2C_2t\,\omega(\varphi_{n_k,p_k})/\varphi_{n_k,p_k}\leqslant(2C_1+4C_2)\,\omega(t).\end{aligned}$$

Now by (2.22, 23, 25)

$$\begin{aligned}\|T_{n_k,p_k}f_\omega\|_Y&\geqslant\|T_{n_k,p_k}g_k\|_Y-\|T_{n_k,p_k}\|_{[X,Y]}\|f_\omega-g_k\|_X\\&\geqslant[C_3/5-C_3/10]\,\omega(\varphi_{n_k,p_k}).\end{aligned}$$

In view of (2.12) this proves (2.20) with $C_4=C_3/10>0$.

Note that the limiting cases $\omega=\omega_0,\omega_1$ have to be excluded in Thm. 2. In fact, this remark already applies to the particular situation of the UBP with rates (case $T_{n,p}=T_n$, cf. [5], [7]).

**3. Condensation for uncountable index families.** While the preceding section was concerned with double sequences of operators we now want to give a first contribution to the case when $p$ varies over an uncountable index set $\mathscr{A}$. To this end we shall use the following lemmas.

LEMMA 1. *Let $\mathscr{A}$ denote a topological space of second category and $\mathscr{N}\subset\mathscr{A}$ a dense subset. Let $\{g_n\}_{n\in N}$ be a sequence of continuous functions on $\mathscr{A}$ such that for each $t\in\mathscr{N}$*

$$\limsup_{n\to\infty}g_n(t)=\infty. \tag{3.1}$$

*Then the set ($\mathscr{N}\subset\mathscr{B}\subset\mathscr{A}$)*

$$\mathscr{B}:=\{t\in\mathscr{A};\ \limsup_{n\to\infty}g_n(t)=\infty\}$$

*is of second category in $\mathscr{A}$.*

LEMMA 2. *Let $\mathscr{A},\mathscr{N},\{g_n\}_{n\in N}$ satisfy the hypotheses of La. 1 with (3.1) replaced by*

$$\limsup_{n\to\infty}g_n(t)\geqslant c_0>0. \tag{3.1)'$$

*Then the set ($\mathscr{N}\subset\mathscr{B}'\subset\mathscr{A}$)*

$$\mathscr{B}':=\{t\in\mathscr{A};\ \limsup_{n\to\infty}g_n(t)\geqslant c_0\}$$

*is of second category in $\mathscr{A}$.*

Proofs. For La. 1 see Orlicz [15]. In fact, the proof of La. 2 is just a copy of that given in [15], but included here for the sake of completeness. Since $g_n$ is continuous on $\mathscr{A}$, the sets ($m,n\in N$)

$$\mathscr{S}_{n,m}:=\{t\in\mathscr{A};\ g_n(t)\leqslant c_0(1-1/m)\}$$

are closed. Thus also the intersections $\mathscr{F}_{k,m}:=\bigcap_{n\geqslant k}\mathscr{S}_{n,m}$ are closed in $\mathscr{A}$. Now observe that

$$\mathscr{A}\setminus\mathscr{B}'=\{t\in\mathscr{A};\ \limsup_{n\to\infty}g_n(t)<c_0\}=\bigcup_{m\in N}\bigcup_{k\in N}\mathscr{F}_{k,m}.$$

Moreover, each set $\mathscr{F}_{k,m}$ is nowhere dense in $\mathscr{A}$. Indeed, suppose there is a set $\mathscr{F}_{k_0,m_0}$ with an inner point $t_0$ in its closure. Then $\mathscr{F}_{k_0,m_0}$ contains an open neighbourhood $\mathscr{V}_0$ since $\mathscr{F}_{k_0,m_0}$ is closed. Of course, $\mathscr{V}_0\subset\mathscr{A}\setminus\mathscr{B}'$, but the density of $\mathscr{N}$ in $\mathscr{A}$ also implies $\mathscr{V}_0\cap\mathscr{N}\neq\emptyset$ and thus $\mathscr{V}_0\cap\mathscr{B}'\neq\emptyset$, a contradiction. Hence $\mathscr{A}\setminus\mathscr{B}'$ is of first category. Since $\mathscr{A}=\mathscr{B}'\cup(\mathscr{A}\setminus\mathscr{B}')$ was assumed to be of second category, $\mathscr{B}'$ must be of second category, too.

Note that La. 1, 2 do not state that $\mathscr{B}$ (or $\mathscr{B}'$) is equal to $\mathscr{A}$. In fact, this cannot be true in general as already mentioned in [15] (see also Sec. 4.1).

Of course, La. 1, 2 may now be used to derive extensions of the results of Sec. 2 to families of operators depending on an uncountable index $p$. Rather than to formulate a most general theorem, let us just state one typical version.

COROLLARY 1. *Let $\mathscr{A}$ denote an interval of $\mathbf{R}$ (:= the real line with the natural topology), and let $\mathscr{N}$ be the set of rational numbers in $\mathscr{A}$. For $n\in N$, $t\in\mathscr{A}$ let $T_{n,t}\in[X,Y]$, and let $\varphi_{n,t}$ satisfy* (2.6) *such that for all*

*$n \in \mathbf{N}$, $t \in \mathcal{N}$ conditions* (2.7, 8, 18) *hold true. Suppose that*

$$g_n(t) := \|T_{n,t} f\|_Y / \omega(\varphi_{n,t})$$

*is a continuous function on $\mathcal{A}$ for each $f \in X$. Then there exists a constant $C_4 > 0$ such that for each modulus $\omega$ satisfying* (2.2, 10, 19) *there exists $f_\omega \in X_\omega$, independent of $t$, with*

$$\limsup_{n\to\infty} \|T_{n,t} f_\omega\|_Y / \omega(\varphi_{n,t}) \geqslant C_4$$

*simultaneously for all $t$ in a set $\mathcal{B}'$ of second category in $\mathcal{A}$, containing $\mathcal{N}$.*

## 4. Applications.

**4.1.** *Fourier partial sums.* Let $C_{2\pi}$ denote the space of continuous, $2\pi$-periodic functions with the usual sup-norm and $C_{2\pi}^{(1)}$ the subspace of continuously differentiable functions (with seminorm $\|f'\|_C$). The corresponding $K$-functional turns out to be equivalent to the usual modulus of continuity of functions, i.e., if

$$\omega(t,f) := \sup_{|h|\leqslant t} \|f(x+h) - f(x)\|_C,$$

then there exist constants $c_1, c_2 > 0$, independent of $f \in C_{2\pi}$, $t \geqslant 0$, such that (cf. [4], p. 192 f)

$$c_1 \omega(t,f) \leqslant K(t,f; C_{2\pi}, C_{2\pi}^{(1)}) \leqslant c_2 \omega(t,f). \tag{4.1}$$

For the Fourier partial sums

$$(S_n f)(x) := \sum_{|k|\leqslant n} f^\wedge(k) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_n(u)\, du,$$

$$f^\wedge(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u) e^{-iku}\, du, \qquad D_n(u) := \sum_{|k|\leqslant n} e^{iku},$$

one has the well-known (uniform) direct estimate

$$|(S_n f)(x) - f(x)| \leqslant C\omega(n^{-1}, f) \log n \qquad (x \in \mathbf{R}).$$

Then Thm. 2 and La. 2 (or Cor. 1) give

COROLLARY 2. *For each modulus $\omega$ satisfying* (2.2, 10, 19) *there exists $f_\omega \in C_{2\pi}$ such that*

$$\omega(t, f_\omega) = O(\omega(t)) \qquad (t \to 0+), \tag{4.2}$$

*but on the other hand*

$$\limsup_{n\to\infty} |(S_n f_\omega)(x) - f_\omega(x)| / \omega(1/n) \log n \geqslant c_0 > 0 \tag{4.3}$$

*simultaneously for all $x$ in a dense set of second category in $\mathbf{R}$.*



Proof. Let us first check the conditions of Thm. 2 for the spaces $X = C_{2\pi}$, $Y = C$ (: = the complex plane), $U = C_{2\pi}^{(1)}$, and the bounded linear functionals $T_{n,x} f := [(S_n f)(x) - f(x)]/\log n$. Since for the functionals $S_{n,x} f := (S_n f)(x)$ one has $\|S_{n,x}\|_{[C_{2\pi}, C]} = \|D_n\|_1 > c \log n$, independent of $x$, there are elements $f_{n,x} \in C_{2\pi}$ with $\|f_{n,x}\|_C = 1$ and $|S_{n,x} f_{n,x}| \geqslant c' \log n$. Now choose $h_{n,x} = V_n f_{n,x}$, where $V_n := (1/n) \sum_{k=n+1}^{2n} S_k$ are the standard delayed means of de La Vallée Poussin. Then $h_{n,x}$ is a trigonometric polynomial of degree $2n$ satisfying (2.7, 8) with $\varphi_{n,x} = 1/n$, independent of $x$, as a consequence of the classical Bernstein inequality. Condition (2.18) is fulfilled since for sufficiently large $n$

$$\begin{aligned} |T_{n,x} h_{n,x}| &= |S_{n,x} V_n f_{n,x} - (V_n f_{n,x})(x)| / \log n \\ &\geqslant [|S_{n,x} f_{n,x}| - C_1] / \log n \geqslant c'' > 0. \end{aligned}$$

Thus an application of Thm. 2 in connection with (4.1) gives the existence of $f_\omega$ satisfying (4.2,3) but only for countable many $x \in \mathbf{R}$, e.g., for rational $x$. Since $|T_{n,x} f_\omega|$ is a continuous function of $x \in \mathbf{R}$, application of La. 2 states that (4.3) moreover holds true on a dense set of second category in $\mathbf{R}$.

It is interesting to consider the case $\omega(t) \sim (\log(1/t))^{-1}$. Then (4.3) states that $\limsup_{n\to\infty} |(S_n f_\omega)(x) - f_\omega(x)| \geqslant c_0 > 0$ on a dense set of second category, while the famous theorem of Carleson states the convergence almost everywhere on $\mathbf{R}$. This fact does not only show (as already mentioned in Sec. 3) that in general $\mathcal{B}, \mathcal{B}' \neq \mathcal{A}$ in La. 1, 2, but that even a CP with rates is not possible for an arbitrary (second) index set without additional assumptions.

In connection with the development of the various kinds of UBP's and CP's, the Fourier partial sums have always been used as *the* test for the applicability of the functional analytical principles under consideration. So there is a lot of background material. Rather than to be complete, let us only mention that the present treatment should be compared with the classical ones, given in [2] without rates and in [13] with rates, but without condensation. On the other hand, there are even results on the (non) convergence with rates almost everywhere (see [16], [17]), which of course cannot be covered by the present abstract approach.

**4.2.** *Bernstein polynomials.* Let $C[0,1]$ denote the space of continuous functions on $[0,1]$ with the usual sup-norm and $C^{(2)}[0,1]$ the subspace of twice continuously differentiable functions (with seminorm $\|f''\|_C$). Again the corresponding $K$-functional is equivalent to the usual second modulus of continuity of functions, i.e., if

$$\omega^*(t,f) := \sup_{|h|\leqslant t} \sup_{h\leqslant x\leqslant 1-h} |f(x-h) - 2f(x) + f(x+h)|,$$

there exist constants $c_1, c_2 > 0$ such that

$$c_1\omega^*(t,f) \leqslant K(t^2, f;\ C[0,1], C^{(2)}[0,1]) \leqslant c_2\omega^*(t,f). \tag{4.4}$$

For the Bernstein polynomials ($x \in [0,1]$, $n \in \mathbf{N}$)

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k f(k/n)$$

one has the well-known direct estimate (cf. [3])

$$|B_n(f;x) - f(x)| \leqslant C\,\omega^*(\varphi_{n,x}^{1/2}, f),$$

where $\varphi_{n,x} := x(1-x)/n$. Now an application of the general results of the previous sections deliver

Corollary 3. *For each modulus $\omega$ satisfying* (2.2, 10, 19) *there exists $f_\omega \in C[0,1]$ such that*

$$\omega^*(t, f_\omega) = O\big(\omega(t^2)\big) \quad (t \to 0+), \tag{4.5}$$

*but on the other hand*

$$\limsup_{n\to\infty} |B_n(f_\omega; x) - f_\omega(x)|/\omega(\varphi_{n,x}) \geqslant c_0 > 0 \tag{4.6}$$

*simultaneously for all $x$ in a dense set of second category in* $[0,1]$.

Proof. We check the conditions of Thm. 2 for the spaces $X = C[0,1]$, $Y = \mathbf{R}$, $U = C^{(2)}[0,1]$, and the bounded linear functionals $T_{n,x}f := B_n(f;x) - f(x)$. For $x \in (0,1)$ consider the functions ($\xi_{n,x} \in [0,1]$)

$$\begin{aligned} h_{n,x}(u) &= 2\sin^2\big((x-u)/2\varphi_{n,x}^{1/2}\big) = 1 - \cos\big((x-u)/\varphi_{n,x}^{1/2}\big) \\ &= (x-u)^2/2\varphi_{n,x} - (1/4!)(x-u)^4 h_{n,x}^{(\mathrm{IV})}(\xi_{n,x}). \end{aligned}$$

Of course, $h_{n,x}$ satisfies (2.7, 8) (with $\varphi_{n,x}$). Moreover, (2.18) follows immediately, since in view of the positivity of the Bernstein polynomials

$$\begin{aligned} |B_n(h_{n,x};x) - h_{n,x}(x)| &= B_n(h_{n,x};x) \\ &\geqslant (1/2\varphi_{n,x}) B_n\big((x-u)^2;x\big) - (1/4!)\varphi_{n,x}^{-2} B_n\big((x-u)^4;x\big) \\ &= 1/2 - (1/4!)\{3 + 1/nx(1-x) - 6/n\} \geqslant 1/3, \end{aligned}$$

if $n \geqslant 1/x(1-x)$. Thus Thm. 2 delivers the existence of an element $f_\omega \in C[0,1]$ such that (4.5, 6) hold true for all rational $x \in (0,1)$. In view of the continuity of $\varphi_{n,x}^{-1}$ on $(0,1)$ an application of La. 2 completes the proof.

Let us mention that for $\omega(t) = t^\beta$, $0 < \beta \leqslant 1$, even a Bernstein-type inverse theorem is valid: It states that

$$|B_n(f;x) - f(x)| \leqslant C\big(x(1-x)/n\big)^\beta \Rightarrow \omega^*(t,f) = O(t^{2\beta}).$$

For details concerning this result see [3], [14].



**4.3.** *Lagrange interpolation.* Let $L_n f$ denote the Lagrange interpolation polynomials for $f \in C[0,1]$ with respect to the triangular matrix of knots $\{0 \leqslant x_{1n} < x_{2n} < \ldots < x_{nn} \leqslant 1\}_{n\in\mathbf{N}}$, i.e.,

$$L_n(f;x) := \sum_{j=1}^{n} f(x_{jn}) l_{jn}(x), \qquad l_{jn}(x) := \prod_{\substack{k=1\\ k\neq j}}^{n} \frac{x - x_{kn}}{x_{jn} - x_{kn}} \qquad (0 \leqslant x \leqslant 1).$$

Setting

$$\lambda_n(x) := \sum_{j=1}^{n} |l_{jn}(x)|, \qquad \lambda_n := \max_{0\leqslant x\leqslant 1} \lambda_n(x),$$

one has for almost every $x \in [0,1]$ (cf. [8])

$$\limsup_{n\to\infty} \lambda_n(x) = \infty. \tag{4.7}$$

Thus for every Lagrange interpolation process the following divergence properties hold true.

Corollary 4. *For each modulus $\omega$ satisfying* (2.2, 10) *there exists $f_\omega \in C[0,1]$ such that*

$$\omega^*(t, f_\omega) = o\big(\omega(t^2)\big) \quad (t \to 0+) \tag{4.8(a)}$$

*as well as*

$$\limsup_{n\to\infty} |L_n(f_\omega;x) - f_\omega(x)|/\omega\big((n^2\lambda_n)^{-2}\big) = \infty \tag{4.9}$$

*simultaneously for all $x$ in a dense set of second category in* $[0,1]$. *For $\omega = \omega_1$ condition* (4.9) *holds true for a function $f_\omega \in C[0,1]$ such that*

$$\omega^*(t, f_\omega) = O(t^2) \quad (t \to 0+). \tag{4.8(b)}$$

Proof. We first check the conditions of Thm. 1 for the spaces $X = C[0,1]$, $Y = \mathbf{R}$, $U = C^{(2)}[0,1]$, and the bounded linear functionals $T_{n,x}f := L_n(f;x)$, $T_x f = f(x)$, where $x$ varies over a dense, countable set $\mathcal{N} \subset [0,1]$ of indices for which (4.7) holds true. Since $\lambda_n(x) = \|T_{n,x}\|_{[C,\mathbf{R}]}$, there exist functions $f_{n,x} \in C[0,1]$ such that $\|f_{n,x}\|_C \leqslant 1$ and $\limsup_{n\to\infty} |L_n(f_{n,x};x)| = \infty$ for each $x \in \mathcal{N}$. Choosing now

$$h_{n,x}(u) := \begin{cases} f_{n,x}(x_{1n}) & \text{for} \quad 0 \leqslant u \leqslant x_{1n}, \\ f_{n,x}(x_{jn}) + [f_{n,x}(x_{j+1,n}) - f_{n,x}(x_{jn})]\, h\big((u - x_{jn})/(x_{j+1,n} - x_{jn})\big) \\ \qquad\qquad \text{for} \quad x_{jn} \leqslant u \leqslant x_{j+1,n}, \ 1 \leqslant j < n, \\ f_{n,x}(x_{nn}) & \text{for} \quad x_{nn} \leqslant u \leqslant 1, \end{cases}$$

where $h$ denotes some infinitely often differentiable function on $\mathbf{R}$ with

$$h(u) \begin{cases} = 0 & \text{for} \quad u \leqslant 0, \\ \in (0,1) & \text{for} \quad 0 < u < 1, \\ = 1 & \text{for} \quad u \geqslant 1, \end{cases}$$

one obtains (2.7–9) with $\varphi_{n,x} = 1/(n^2\lambda_n)^2$. In fact,

$$\|h_{n,x}\|_C \leqslant \|f_{n,x}\|_C \leqslant 1, \quad \|h''_{n,x}\|_C \leqslant 2\,\|h''\|/d_n^2$$

upon using (cf. [9])

$$d_n := \min_{1\leqslant j<n} (x_{j+1,n} - x_{jn}) > 1/n^2\lambda_n.$$

Moreover, $\limsup_{n\to\infty} |T_{n,x}h_{n,x}| = \infty$ for each $x \in \mathcal{N}$ since $L_n(h_{n,x}; u) = L_n(f_{n,x}; u)$. Thus an application of Thm. 1 (a), (b) delivers the existence of $f_\omega$ satisfying (4.8 (a), (b)) and (4.9) for all $x \in \mathcal{N}$. Since, moreover,

$$|L_n(f_\omega; x) - f_\omega(x)|/\omega\big((n^2\lambda_n)^{-2}\big)$$

is a continuous function of $x$ on $\mathcal{A} := [0, 1]$, La. 1 proves (4.9) completely.

For a result stating the existence of functions $f_{\omega,x}$, depending upon the individual $x \in [0, 1]$ (cf. (1.5)), see [12] (and also [7] for the treatment within the frame of general UBP's with rates). On the other hand, it was shown in [10] (see also [19]) that there exists a continuous function $f_0$ such that $\limsup_{n\to\infty} |L_n(f_0; x)| = \infty$ even for almost all $x$. Of course, results of this type are again beyond the scope of our functional analytical approach.

**4.4.** *Numerical quadrature.* For $f \in C[0, 1]$, $x \in [0, 1]$ consider the trapezoidal rule for the approximate integration of the indefinite integral $\int_0^x f(u)\,du$, i.e.,

$$Q_{n,x}f := h\{f(0)/2 + \sum_{k=1}^{n-1} f(kh) + f(x)/2\}, \qquad h = x/n.$$

Then one has the well-known direct estimate ($f \in C[0, 1]$, $x \in [0, 1]$)

$$|Q_{n,x}f - \int_0^x f(u)\,du| \leqslant Cx\,\omega^*(x/n, f).$$

An application of the general results of Sec. 2, 3 now shows that this estimate is indeed sharp in the following sense:

COROLLARY 5. *For each modulus $\omega$ satisfying* (2.2, 10, 19) *there exists $f_\omega \in C[0, 1]$ such that*

$$\omega^*(t, f_\omega) = O\big(\omega(t^2)\big) \qquad (t \to 0+), \tag{4.10}$$

*but on the other hand*

$$\limsup_{n\to\infty} \Big|Q_{n,x}f_\omega - \int_0^x f_\omega(u)\,du\Big| \big/ x\omega\big((x/n)^2\big) \geqslant c > 0 \tag{4.11}$$

*simultaneously for all $x$ in a dense set of second category in* $[0, 1]$.



Proof. For the spaces $X = C[0, 1]$, $Y = \mathbf{R}$, $U = C^{(2)}[0, 1]$, and the bounded linear functionals $T_{n,x}f := x^{-1}[Q_{n,x}f - \int_0^x f(u)\,du]$, $x \in (0, 1]$, Thm. 2 delivers (4.10, 11) for all rational $x \in (0, 1]$ so that La. 2 completes the argument. Indeed, the functions $h_{n,x}(u) = \sin^2(2\pi nu/x)$ satisfy (2.7, 8) with $\varphi_{n,x} = (x/n)^2$ as well as (2.18) since

$$|T_{n,x}h_{n,x}| = x^{-1}\int_0^x \sin^2(2\pi nu/x)\,du = n^{-1}\int_0^n \sin^2(2\pi u)\,du = \int_0^1 \sin^2(2\pi u)\,du = 1/2.$$

Note that $h_{n,x}$ is chosen in such a way that it is positive and vanishes on the knots of the rule.

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# On the extension of continuous linear maps in function spaces and the splitness of Dolbeaut complexes of holomorphic Banach bundles

by

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**Abstract.** The paper investigates the extension of continuous linear maps with values in the spaces of sections of coherent analytic sheaves over analytic spaces. It is shown that the space $H^0(X, \mathcal{S})$, where $\mathcal{S}$ is a coherent analytic sheaf over a paracompact analytic space $X$ has the extension property with respect to the class of $s$-nuclear spaces if and only if it is isomorphic to $C^A$ for some set $A$. We also investigate the existence of continuous linear projections of the space $C_\xi^\infty(R(X))$ onto $\mathcal{O}_\xi(X)$, where $R(X)$ is the regular part of $X$ and $\xi$ is a holomorphic Banach bundle over $X$. The splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds is considered. We prove that on complex manifolds which are increasing unions of open Stein sets these complexes split only at positive dimensions.

**Introduction.** In the present paper we consider extensions of continuous linear maps with values in some function spaces of complex analysis and the splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds. These problems have been investigated by several authors ([6], [8]). The paper contains three sections.

In § 1 we prove that the space $H^0(X, \mathcal{S})$ has the extension property with respect to the class of $s$-nuclear spaces if and only if it is isomorphic to $C^A$ for some set $A$.

Section § 2 is devoted to the study of the existence of continuous linear projections of $C_\xi^\infty(R(X))$ onto $\mathcal{O}_\xi(X)$. It is shown that when $X$ is Stein such a projection exists if and only if $X$ is discrete.

In § 3 we investigate the splitness of Dolbeaut complexes of holomorphic Banach bundles over complex manifolds. We prove that on complex manifolds which are increasing unions of open Stein sets these complexes split only at positive dimensions. Let us note that the splitness of Dolbeaut complexes of holomorphic vector bundles over Stein manifolds has been established by Palamodov ([8]).