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Finite dimensional projection constants

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Abstract. We derive a formula for the relative projection constant of a k-dimensional subspace of a finite dimensional Banach space which is better than $\sqrt{k}$. Various cases of the optimality and non-optimality of the formula are studied, using a combinatorial reformulation. Similar estimates are given for reflections instead of projections.

Let X be a closed subspace of a Banach space Y. The relative projection constant of X in Y is given by

$$\lambda(X, Y) := \inf\{\|P\| \mid P: Y \to Y \text{ is a projection onto } X\},$$

the absolute projection constant of X by

$$\lambda(X) := \sup\{\lambda(X, Y) \mid Y \text{ a Banach space containing } X \text{ as a subspace}\}.$$

We are mainly interested in cases where X and Y are finite dimensional. For convenience in these instances, their respective dimensions will be indicated by lower indices. Thus $X_k \subseteq Y_n$ means that X_k is a k-dimensional subspace of an n-dimensional space Y_n. Denote for $k \leqslant n$

$$\lambda(k, n) := \sup\{\lambda(X_k, Y_n) \mid X_k \subseteq Y_n\},$$

$$\lambda(k) := \sup\{\lambda(k, n) \mid n \in \boldsymbol{N}\},$$

$$f(k, n) := \sqrt{k}\left(\sqrt{k}/n + \sqrt{(n-1)(n-k)}/n\right).$$

If statements on these constants only hold in the real or complex case, we indicate this by additional superscripts $\boldsymbol{R}$ or $\boldsymbol{C}$,

It is well known that $\lambda(k) \leqslant \sqrt{k}$. In the finite-dimensional case this can be improved.

THEOREM 1. $\lambda(k, n) \leqslant f(k, n) \leqslant \sqrt{k}\left(1 - (\sqrt{k}-1)^2/2n\right)$. *Thus for large subspaces* $X_k \subseteq Y_n$, $k = cn$, $c > 0$,

$$\lambda(X_k, Y_n) \leqslant \sqrt{1-c}\,\sqrt{k} + c.$$

It is unknown whether there are spaces X_k such that $\lambda(X_k)/\sqrt{k} \to 1$. A positive answer to this question would follow if there are sequences of

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spaces $X_k \subseteq Y_n$ with

$$\lambda(X_k, Y_n) = f(k, n), \qquad k \to \infty, \ k/n \to 0.$$

A natural candidate for Y_n seems to be l_n^∞ because then $\lambda(X_k, Y_n) = \lambda(X_k)$. The following characterization holds:

THEOREM 2. *Let $1 \leqslant k < n$. The following are equivalent:*

(1) *There exists a subspace $X_k \subseteq l_n^\infty$ such that $\lambda(X_k) = f(k, n)$.*

(2) *There exists an operator $T\colon l_n^\infty \to l_n^\infty$ with nuclear norm $\nu(T) = 1$ and eigenvalues $\lambda_1(T) = \ldots = \lambda_k(T) = f(k, n)/k$ and $\lambda_{k+1}(T) = \ldots = \lambda_n(T) = \big(1-f(k, n)\big)/(n-k)$ (thus one eigenvalue being k-fold, the other $(n-k)$-fold).*

(3) *There exists an $n \times n$ matrix $A = (a_{ij})$ with $A^2 = I$ and $a_{ii} = 2k/n - 1$ $(i = 1, \ldots n)$, $|a_{ij}| = 2/n \sqrt{k(n-k)/(n-1)}$ $(i, j = 1, \ldots, n;\ i \neq j)$.*

If (1)–(3) hold, the eigenspace X_k of A associated to the eigenvalue $+1$ has dimension k and projection constant $f(k, n)$. $P := 1/2\,(I + A)$ is a projection onto X_k of minimal norm; given X_k for $k \neq 1$, P is unique. For $k \neq 1$, any such P has the form

$$p_{ii} = k/n \qquad (i = 1, \ldots, n),$$

$$|p_{ij}| = 1/n \sqrt{k(n-k)/(n-1)} \qquad (i, j = 1, \ldots, n;\ i \neq j).$$

We will later discuss various cases of (k, n) for which matrices A as in (3) can or cannot be constructed. Although we have not been able to solve the question whether $\lambda(X_k)/\sqrt{k} \to 1$ is possible, we show that there are spaces with

$$\lambda(X_k)/\sqrt{k} \to \sqrt{4/5} \approx .89$$

which is the largest known value so far (Y. Gordon [1] constructed spaces X_k with $\lambda(X_k)/\sqrt{k} \to (2-\sqrt{2/\pi})^{-1} \approx .83$).

Of course, there can be other spaces Y_n and $X_k \subseteq Y_n$ with $\lambda(X_k, Y_n) = f(k, n)$. A natural example is given in the following proposition which has the additional feature that, given n, k is such that $f(k, n) = \max_{1<l<n} f(l, n) = (1+\sqrt{n})/2$; $k = (n+\sqrt{n})/2$.

PROPOSITION 1. *Let $n = N^2$ and $Y_n = L(l_N^\infty)$ the space of operators on l_N^∞ and X_k = subspace of selfadjoint matrices. Then $k = (n+\sqrt{n})/2$ and*

$$\lambda(X_k, Y_n) = f(k, n) = (1+N)/2.$$

Theorem 1 and some approximation method yields a different proof of the following result of Y. Gordon [1]:

PROPOSITION 2. *We have $\lambda^{\mathbf{R}}(2) < 2-\varepsilon$, $\varepsilon \geqslant 10^{-4}$.*

Remark. $\lambda^{\mathbf{R}}(2)$ is conjectured to be $4/3 = f(2, 3)$ which is the projection constant of the 2-dimensional space whose unit ball is the hexagon.

We now turn to the proofs of the previous results. The first lemma is basically known, cf. e.g. [3].

LEMMA 1. *Let X and Y with $X \subseteq Y$ be finite dimensional. Then*

$$\lambda(X, Y) = \sup\{|\mathrm{tr}(T\colon X \to X)| \mid T\colon Y \to Y \text{ with } \nu(T) = 1,\ T(X) \subseteq X\}.$$

Proof. We only have to show the inequality "$\leqslant$" since the other one is immediate. Since X and Y are finite dimensional, there exists a projection $P_0\colon Y \to X \subseteq Y$ onto X of minimal norm, $\|P_0\| = \lambda(X, Y)$. Consider

$$\mathscr{B} = \{S \in L(Y) \mid \|S\| < \|P_0\|\}$$

and

$$\mathscr{P} = \Big\{P \in L(Y) \Big| P = P_0 + \sum_{i=1}^n x_i^* \otimes x_i,\ n \in \mathbf{N},\ x_i^* \in X^\perp \subseteq Y^*,\ x_i \in X\Big\}.$$

Then $\mathscr{B} \cap \mathscr{P} = \emptyset$ since $\mathscr{P}$ consists of projections. Moreover, $\mathscr{B}$ and $\mathscr{P}$ are convex sets in $L(Y)$ which can be separated. Thus by the trace duality there is $T \in L(Y)$ such that

$$\mathrm{Re}\big(\mathrm{tr}(TS)\big) \leqslant \|P_0\| \leqslant \mathrm{Re}\big(\mathrm{tr}(TP)\big), \qquad S \in \mathscr{B},\ P \in \mathscr{P}$$

which implies $\|P_0\| = \mathrm{tr}(TP_0)$ and $\nu(T) = \sup|\mathrm{tr}(TS)|/\|S\| = 1$. To prove that $T(X) \subseteq X$, i.e. $\langle x^*, Tx\rangle = 0$ for all $x^* \in X^\perp$ and $x \in X$ we take $P = P_0 + x^* \otimes x$. Then

$$\|P_0\| \leqslant \mathrm{Re}\big(\mathrm{tr}(TP_0)\big) + \mathrm{tr}\big(T(x^* \otimes x)\big) = \|P_0\| + \mathrm{Re}\,\langle x^*, Tx\rangle.$$

Hence $\mathrm{Re}\langle x^*, Tx\rangle \geqslant 0$ for all $x^* \in X^\perp$, $x \in X$. Since X, $X^\perp$ are linear spaces, this yields $\langle x^*, Tx\rangle = 0$.

LEMMA 2. $1 < r < \infty$, $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$ *and* $Z_r = (\mathbf{K}^n, \|\cdot\|_{Z_r})$ *where*

$$\|(\xi_i)_{i=1}^n\|_{Z_r} := \max\Big(\Big|\sum_{i=1}^n \xi_i\Big|, \|(\xi_i)_{i=1}^n\|_r\Big).$$

Then the dual norm is given by

$$\|(\mu_i)_{i=1}^n\|_{Z_r^*} = \inf_{t \in \mathbf{K}}\{|t| + \|(\mu_i - t)_{i=1}^n\|_{r'}\}, \qquad 1/r + 1/r' = 1.$$

Proof. The map $\varphi\colon Z_r \to (l_r \oplus \mathbf{K})_\infty$, $(\xi_i)_{i=1}^n \mapsto \big((\xi_i)_{i=1}^n, \sum_{i=1}^n \xi_i\big)$ is an isometric imbedding. Hence $\varphi^*\colon (l_{r'} \oplus \mathbf{K})_1 \to Z_r^*$, $\big((\lambda_i)_{i=1}^n, t\big) \mapsto (\lambda_i + t)_{i=1}^n$ is a quotient map and

$$\|(\mu_i)_{i=1}^n\|_{Z_r^*} = \inf_{t \in \mathbf{K}}\Big\{\big\|\big((\lambda_i)_{i=1}^n, t\big)\big\|_{(l_{r'} \oplus \mathbf{K})_1} \Big| \mu_i = \lambda_i + t,\ i = 1, \ldots, n\Big\}$$

$$= \inf_{t \in \mathbf{K}}\{|t| + \|(\mu_i - t)_{i=1}^n\|_{r'}\}.$$

Proof of Theorem 1. Let $X_k \subseteq Y_n$. By Lemma 1, it suffices to show $|\mathrm{tr}(T\colon X_k \to X_k)| \leqslant f(k, n)$ for all $T\colon Y_n \to Y_n$ with $\nu(T) = 1$ and $T(X_k) \subseteq X_k$. Let $\lambda_1(T), \ldots, \lambda_n(T)$ denote the eigenvalues of T (counted according to their multiplicity); k of them (WLoG $\lambda_1(T), \ldots, \lambda_k(T)$) are the eigenvalues of $T|_{X_k}\colon X_k \to X_k$. Then $\|(\lambda_i(T))_{i=1}^n\|_{Z_2} \leqslant 1$ since

$$|\mathrm{tr}(T)| = \Big|\sum_{i=1}^n \lambda_i(T)\Big| \leqslant \nu(T) = 1 \quad \text{and} \quad \big\|\big(\lambda_i(T)\big)_{i=1}^n\big\|_2 \leqslant \nu(T),$$

cf. [3]. For the convenience of the reader, here is a direct argument for the last inequality: Let

$$T = \sum_j \delta_j x_j^* \otimes x_j, \qquad x_j^* \in Y_n^*,\ x_j \in Y_n$$

with $\|x_j^*\| = \|x_j\| = 1$, $\sum_j |\delta_j| \leqslant 1+\varepsilon$. Then T factors as $T = SR$ where

$$R\colon Y_n \to l_2, \quad S\colon l_2 \to Y_n, \quad Rx = \big(\sqrt{\delta_j}\, x_j^*(x)\big)_j, \quad S(\xi_j)_j = \sum_j \sqrt{\delta_j}\, \xi_j x_j.$$

It is easily checked that $\tilde{T} := RS$ has the same eigenvalues as T, with the possible exception of zero, cf. Pietsch [5], Chap. 27.3, and that the Hilbert–Schmidt norm hs of $\tilde{T}$ is less than $1+\varepsilon$, $hs(\tilde{T}) \leqslant 1+\varepsilon$. Thus

$$\big\|\big(\lambda_i(T)\big)_{i=1}^n\big\|_2 = \big\|\big(\lambda_i(\tilde{T})\big)_{i=1}^n\big\|_2 \leqslant hs(\tilde{T}) \leqslant 1+\varepsilon.$$

Hence

$$|\mathrm{tr}(T\colon X_k \to X_k)| = \Big|\sum_{i=1}^k \lambda_i(T)\Big| \leqslant \Big|\sum_{i=1}^k \lambda_i(T)\Big| \,/\, \|(\lambda_i)_{i=1}^n\|_{Z_2}$$

$$\leqslant \|(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})\|_{Z_2^*}$$

$$= \inf_{t \in K} \big\{|t| + \big(k|1-t|^2 + (n-k)|t|^2\big)^{1/2}\big\},$$

using Lemma 2. The inf is attained for $t_0 = k/n - 1/n\ \sqrt{k(n-k)/(n-1)}$, its value at t_0 turns out (after a slight calculation) to be just $f(k, n)$.

A modification of this proof yields

Corollary 1. *Let $1 < p < \infty$ and $1/r := |1/2 - 1/p|$. Then there is $c > 0$ such that for any $1 < k < n$ and any $X_k \subseteq l_n^p$,*

$$\lambda(X_k, l_n^p) \leqslant n^{1/r}\big(1 - c(k/n)^{2/r}\big).$$

Proof. By Pisier [6], $\big\|\big(\lambda_i(T)\big)_{i=1}^n\big\|_{r'} \leqslant \nu(T\colon l_n^p \to l_n^p)$, $1/r + 1/r' = 1$. The same argument as before, with $Y_n = l_n^p$, yields

$$\lambda(X_k, l_n^p) \leqslant \|(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})\|_{Z_{r'}^*}$$

$$= \inf_{t \in K} \big\{|t| + \big(k|1-t|^r + (n-k)|t|^r\big)^{1/r}\big\}$$

which, because of $r \geqslant 2$, is

$$\leqslant \inf_{t \in K} \big\{|t| + \big(k^{2/r}|1-t|^2 + (n-k)^{2/r}|t|^2\big)^{1/2}\big\}.$$

The infimum is attained for $t_0 = 1/A\ \big(k^{2/r} - k^{1/r}(n-k)^{1/r} \,/\, \sqrt{A-1}\big)$ where $A := k^{2/r} + (n-k)^{2/r}$, as its value one finds after some calculation

$$= k^{1/r}\big\{1/A\big((n-k)^{1/r}\sqrt{A-1} + k^{1/r}\big)\big\}$$

$$\leqslant k^{1/r}\{1 - (k^{1/r}-1)^2/2A\}$$

from where corollary 1 follows.

For the proof of Theorem 2 we need

Lemma 3. *Let $(\lambda_i)_{i=1}^n \in Z_2$ with $\|(\lambda_i)_{i=1}^n\|_{Z_2} = 1$. Then $\sum_{i=1}^n \lambda_i = f(k, n)$ if and only if $\lambda_1 = \ldots = \lambda_k = f(k, n)/k$ and $\lambda_{k+1} = \ldots = \lambda_n = \big(1 - f(k, n)\big)/(n-k)$.*

Proof. It is easily seen that $\lambda_i \in \mathbf{R}$ is necessary for $\sum_{i=1}^n \lambda_i = f(k, n)$ to hold. Moreover, $\sum_{i=1}^n \lambda_i = 1$ (if $\sum_{i=1}^n \lambda_i < 1$, one could enlarge the positive λ_i's and negative λ_i's leaving $\|(\lambda_j)_{j=1}^n\|_2 \leqslant 1$ and enlarge thus $\sum_{j=1}^k \lambda_j$). If $\sum_{i=1}^k \lambda_i = f(k, n)$, we get by Hölder's inequality

$$f(k, n)^2/k = 1/k\Big(\sum_{i=1}^k \lambda_i\Big)^2 \leqslant \sum_{i=1}^k |\lambda_i|^2,$$

$$\sum_{i=k+1}^n |\lambda_i|^2 = 1 - \sum_{i=1}^k |\lambda_i|^2 \leqslant 1 - f(k, n)^2/k = 1/(n-k)\big(1 - f(k, n)\big)^2$$

$$= 1/(n-k)\Big(\sum_{i=k+1}^n \lambda_i\Big)^2 \leqslant \sum_{i=k+1}^n |\lambda_i|^2.$$

We used the functional equation $f(k, n)^2 - 2k/n \cdot f(k, n) - k(n-k-1)/n = 0$. Since equality holds in the above inequalities, we must have $\lambda_1 = \ldots = \lambda_k$ and $\lambda_{k+1} = \ldots = \lambda_n$. Thus $\lambda_1 = f(k, n)/k$ and $\lambda_n = \big(\sum_{i=1}^n \lambda_i - \sum_{i=1}^k \lambda_i\big)/(n-k) = \big(1 - f(k, n)\big)/(n-k)$.

Lemma 4. *Let $T \in L(l_n^\infty)$ with $\|(\lambda_i(T))_{i=1}^n\|_2 = \nu(T) = 1$ where $\lambda_i(T) \in \mathbf{R}$. Then $t_{ij}t_{ji} \in \mathbf{R}^+$ and $|t_{ij}| = |t_{ji}|$ for all $i, j = 1, \ldots, n$.*

Proof. The assumptions imply

$$1 = \nu(T) \geqslant \mathrm{tr}(T^2)/\|T\| = \sum_{i=1}^n \lambda_i(T)^2/\|T\| = 1/\|T\|, \qquad \|T\| = \nu(T) = 1.$$

The operator norm of T is $\|T\| = \sup_i \sum_j |t_{ij}|$, the nuclear norm of T $\nu(T) = \sum_j \sup_i |t_{ij}|$, (the second formula follows from the first by the trace duality). Hence there is $i_0 \in \{1, \ldots, n\}$ such that $|t_{i_0 j}| = \sup_i |t_{ij}|$ for all $j = 1, \ldots, n$ (the i_0 for which the sup in $\|T\|$ is attained). Moreover,

$$1 = \operatorname{tr}(T^2) = \sum_{i,j} t_{ij} t_{ji} \leqslant \sum_{i,j} |t_{i_0 j}|\ |t_{i_0 i}| = \Big(\sum_i |t_{i_0 i}|\Big)^2 = 1.$$

This equality implies $t_{ij} t_{ji} \in \boldsymbol{R}^+$ and $|t_{ij}| = |t_{i_0 j}| = |t_{jj}|$ for all $i, j \in \{1, \ldots, n\}$.

Proof of Theorem 2. The proof of Theorem 1 shows that there is $X_k \subseteq l_n^\infty$ such that $\lambda(X_k) = f(k, n)$ if and only if there is $T\colon l_n^\infty \to l_n^\infty$ with $\nu(T) = 1$ and $\|(\lambda_i(T))_{i=1}^n\|_{l_2} = f(k, n)$, where $\lambda_i(T)$ are again the eigenvalues of T. By Lemma 3, this is equivalent to (2). Hence (1) and (2) are equivalent. The implication (3) $\Rightarrow$ (2) is easy: If $A = (a_{ij})_{i,j=1}^n$ is given as in (3) of Theorem 2, let $T = (\lambda_1 + \lambda_n)/2 I + (\lambda_1 - \lambda_n)/2 A$, where $\lambda_1 = f(k, n)/k$, $\lambda_n = \big(1 - f(k, n)\big)/(n-k)$. Then

$$\operatorname{tr}(A) = \sum_{i=1}^n a_{ii} = 2k - n = \sum_{i=1}^n \lambda_i(A).$$

However $\lambda_i(A) \in \{\pm 1\}$ since $A^2 = I$. Thus the eigenvalue $+1$ of A is k-fold and -1 is $(n-k)$-fold. Hence T has a k-fold eigenvalue λ_1 and an $(n-k)$-fold eigenvalue λ_n. Moreover,

$$\nu(T) = \sum_j \sup_i |t_{ij}| = 1$$

as seen by calculation using the given values for a_{ii} and $|a_{ij}|$.

The main step is the implication (2) $\Rightarrow$ (3). Assume T is given as in (2). Since T has only two different eigenvalues $\lambda_1 := \lambda_1(T)$ and $\lambda_n := \lambda_n(T)$, it satisfies the minimal equation

$$(T - \lambda_1 I)(T - \lambda_n I) = 0$$

which is equivalent to

$$T = (\lambda_1 + \lambda_n)/2 I + (\lambda_1 - \lambda_n)/2 A \quad \text{where} \quad A^2 = I.$$

A is restricted by the fact that $\nu(T) = \operatorname{tr}(T) = \|(\lambda_i(T))_{i=1}^n\|_2 = 1$ (the latter equalities follow by easy calculation from the given values of the eigenvalues of T). Let

$$\gamma := (\lambda_1 + \lambda_n)/2 = 1/n + (n-2k)/2n \sqrt{(n-1)/k(n-k)},$$

$$\delta := (\lambda_1 - \lambda_n)/2 = 1/2 \sqrt{(n-1)/k(n-k)}.$$

By Lemma 4, $|a_{ij}| =: \delta_j$ depends only on j for $i \in \{1, \ldots, n\}$, $i \neq j$ and $|t_{ij}| = \delta \delta_j$. Since $\nu(T) = \operatorname{tr}(T) = 1$,

$$\sum_j \delta \delta_j = \sum_j |t_{jj}| = \sum_j |\gamma + \delta a_{jj}| = 1 = \sum_j (\gamma + \delta a_{jj}).$$

Hence

$$\sum_{j=1}^n \delta_j = \delta^{-1} = 2\sqrt{k(n-k)/(n-1)}$$

and

$$a_{jj} = \delta_j - \gamma/\delta = (2k/n - 1) + \Big(\delta_j - 1/n \sum_{k=1}^n \delta_k\Big).$$

Our aim is to show that all δ_j's are the same $(= 1/n)$, i.e. that the last term cancels. For notational convenience, we just prove $\delta_1 = 1/n \sum_{k=1}^n \delta_k$. By Lemma 4, $a_{1j} a_{j1} \in \boldsymbol{R}^+$. Since $A^2 = I$,

$$1 = (A^2)_{11} = a_{11}^2 + \delta_1(\delta_2 + \ldots + \delta_n) = a_{11}^2 + \delta_1\Big(\sum_1^n \delta_k\Big) - \delta_1^2,$$

$$1 = \Big[(2k/n - 1) + \Big(\delta_1 - 1/n \sum_1^n \delta_k\Big)\Big]^2 + \delta_1\Big(\sum_1^n \delta_k\Big) - \delta_1^2$$

$$= \Big[(2k/n - 1) - 1/n \sum_1^n \delta_k\Big]^2 + \delta_1\Big[(n-2)/n \sum_1^n \delta_k + 2(2k/n - 1)\Big],$$

$$\delta_1 = \Big(\sum_{k=1}^n \delta_k\Big) \frac{1 - \big[(2k/n - 1) - 1/n \sum_1^n \delta_k\big]^2}{\big((n-2)/n\big)\big(\sum_1^n \delta_k\big)^2 + 2(2k/n - 1)\big(\sum_1^n \delta_k\big)}$$

$$= \Big(\sum_{k=1}^n \delta_k\Big) \frac{4\big((n-2)/n^2\big)k(n-k)/(n-1) + \big((4k-2n)/n^2\big)\big(\sum_1^n \delta_k\big)}{4\big((n-2)/n\big)k(n-k)/(n-1) + \big((4k-2n)/n\big)\big(\sum_1^n \delta_k\big)}$$

$$= 1/n\Big(\sum_{k=1}^n \delta_k\Big).$$

The last calculation is possible only for $k \neq 1$ (then the denominator is non-zero). For $k = 1$, the existence of A with the required properties is easy. This shows $\delta_1 = \ldots = \delta_n = 1/n$ $(k \neq 1)$, $a_{jj} = 2k/n - 1$ and $|a_{ij}| = (\delta n)^{-1}$ for $i \neq j$.

The previous arguments and the proofs of Theorem 1 and Lemma 1 also show that, if (1)–(3) hold, the k-dimensional eigenspace X_k associated with the eigenvalue $+1$ of A which is the same as the k-dimensional eigenspace associated with the eigenvalue $\lambda_1 = f(k, n)/k$ of T has projection constant $f(k, n)$. Given A, let $P := 1/2\,(I+A)$. An immediate calculation shows $\|P\|_{L(l_n^\infty)} = f(k, n)$. Thus P is a projection of minimal norm onto X_k. To show that a minimal projection P onto a given subspace $X_k \subseteq l_n^\infty$ with $\lambda(X_k) = f(k, n)$ is uniquely determined, note that the sets

$$\mathscr{T} = \{T \in L(l_n^\infty) \mid \operatorname{tr}(TP) = \|P\|\},$$

$$\mathscr{A} = \{A \in L(l_n^\infty) \mid T = \gamma I + \delta A \in \mathscr{T}\}$$

are convex. Hence the set

$$\{A \in \mathscr{A} \mid A \text{ satisfies (3) of Theorem 2}\}$$

is convex. This can happen only if it is reduced to a point, i.e. A is unique. Thus T and P are unique, in view of the one-to-one correspondence of T and A for $k \neq 1$.

The form of P follows from the form of A and $P = 1/2\,(I+A)$.

Remarks. (i) For $k = 1$, the form of P clearly can be different: Choose any $x, a \in \boldsymbol{K}^n$ with $\|x\|_\infty = \|a\|_1 = 1$ and $\langle a, x\rangle = 1$. Then $P = a \otimes x$ is a minimal projection (of norm 1) onto $\operatorname{span}[x] \subseteq l_n^\infty$.

(ii) The conditions on T and A imply that $T = T^*$, $A = A^*$.

(iii) If the matrix A of (3) works for the index pair (k, n), the matrix $-A$ works for $(n-k, n)$ and vice-versa. Nevertheless $f(k, n) \neq f(n-k, n)$ for $k \neq n/2$. Actually, fixing n, $f(k, n)$ attains its maximum at $k = 1/2(n+\sqrt{n})$ (for n being a square number).

The problem of whether there are $X_k \subseteq l_n^\infty$ with $\lambda(X_k) = f(k, n)$ is thus equivalent to the following

Normalized Problem. Given (k, n), is there a matrix $A = (a_{ij})_{i,j=1}^n$ with $A^2 = (m^2+n-1)I_n$, $a_{ii} = m$, $|a_{ij}| = 1$ for $i \neq j$, where $m = (k-n/2) \times \sqrt{(n-1)/k(n-k)}$?

We now consider those cases of (k, n) where answers are known to us.

(a) $k = 1$ and $n-1$, $f(1, n) = 1$ and $f(n-1, n) = 2(1-1/n)$ (hyperplanes). In this case $\pm m = (n-1)/2$, the matrices $\pm A$ exist (take $a_{ii} = m$, $a_{ij} = -1$ for $i \neq j$). For $n = 3$, this yields $f(2, 3) = 4/3$, attained by the hexagonal unit ball

$$\Big\{x \in l_3^\infty \mid \sum_1^3 x_i = 0,\ \|x\|_\infty \leqslant 1\Big\}.$$

(b) $k = n/2$, $f(n/2, n) = 1 + \sqrt{n-1}/2$. Here $m = 0$. The corresponding matrices A, "symmetric conference matrices", have been studied in Hadamard matrix theory. In the *real* case, a sufficient condition for the existence of A is $n = p^r + 1 \equiv 2 \pmod 4$, p being a prime number. A necessary condition is $n = a^2+b^2+1$ for $a, b \in \boldsymbol{Z}$, if n is of the form $n \equiv 2 \pmod 4$. Thus such A's exist e.g. for $n = 2, 6, 10, 14, 18, 26, \ldots$ but not for $n = 22, 34, \ldots$ In the *complex* case, matrices A also exist if $n = 2^N$ or $n = p^r + 1 \equiv 0 \pmod 4$. For $n = 2^N$, they can be given inductively by

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix},$$

$$A_{2^{N+1}} = \begin{bmatrix} A_{2^N} & T_{2^N} \\ T_{2^N} & A_{2^N} \end{bmatrix}, \quad T_{2^{N+1}} = T_2 \otimes T_{2^N}, \quad N \geqslant 1.$$

In particular, there is $X_2 \subseteq \boldsymbol{C}^4$ such that

$$\lambda^{\boldsymbol{C}}(X_2) = f(2, 4) = (1+\sqrt{3})/2 > 4/3,$$

4/3 being the conjectured value of $\sup_{X_2} \lambda^{\boldsymbol{R}}(X_2)$.

For the other facts mentioned, we refer to J. Sebery-Wallis [7].

(c) $k = (n \pm \sqrt{n})/2$, $f((n-\sqrt{n})/2, n) = (1+\sqrt{n})/2 - 1/\sqrt{n}$, $f((n+\sqrt{n})/2, n) = (1+\sqrt{n})/2$. Here $m = \pm 1$. In the real case, the corresponding matrices exist e.g. for $n = 4^N$, defined inductively by

$$A_4 = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \quad A_{4^{N+1}} = A_4 \otimes A_{4^N},$$

as shown already by A. Sobczyk [8] in 1938. In the complex case, they exist for any square $n = N^2$: To see this, start with an $N \times N$ matrix B with $BB^* = NI_N$ which has entries of modulus 1. Let

$$a_{N(i-1)+l,\,N(j-1)+m} = \bar{b}_{i,m} b_{j,l}, \qquad 1 \leqslant i, j, l, m \leqslant N.$$

Then A is an $n \times n$ matrix with 1's on the diagonal and entries of modulus 1 such that $A^* = A$. It suffices to show that the rows of A are orthogonal in the complex sense,

$$\sum_{j,m=1}^N \bar{b}_{im} b_{jl} b_{i'm} \bar{b}_{jl'} = \Big(\sum_{j=1}^N b_{jl} \bar{b}_{jl'}\Big)\Big(\sum_{m=1}^N \bar{b}_{im} b_{i'm}\Big) = \delta_{ll'} \delta_{ii'}$$

is the inner product of the $N(i-1)+l^{\text{th}}$ and the $N(i'-1)+l'^{\text{th}}$ rows.

(d) It can be shown by calculation that a matrix A does not exist for $k = 2$, $n = 5$ (neither in the real nor the complex case). Thus $\lambda(X_2) < f(2, 5)$ for all $X_2 \subseteq l_5^\infty$.

(e) C. Lam and J. Seberry [4] showed by methods of combinatorial design theory that the required matrices exist in the real case for $k = (20u+11)(16u+9)$, $n = 20(5u+3)(16u+9)$ corresponding to the values $m = 30u+19$, for infinitely many $u \in \boldsymbol{N}$. This means $k/n \to 1/5$ and yields spaces $X_k \subseteq l_n^\infty$ with

$$\lambda(X_k)/\sqrt{k} \to \sqrt{4/5}.$$

Concerning cases of $X_k \subseteq Y_n$ with $\lambda(X_k, Y_n) = f(k, n)$ where $Y_n \neq l_n^\infty$, we now give the

Proof of Proposition 1. The map $P\colon L(l_N^\infty) \mapsto L(l_N^\infty)$, $S \to 1/2(S+S^*)$ is a projection of norm $\|P\| \leqslant (1+\sqrt{n})/2 = (1+N)/2$ onto X_k. Thus it suffices to prove $\lambda(X_k, L(l_N^\infty)) \geqslant (1+N)/2$. To do so, it is sufficient to show that $T\colon L(l_N^\infty) \to L(l_N^\infty)$, $S \to S^*$ has nuclear norm $\leqslant N$, since this yields by Lemma 1

$$\lambda(X_k, L(l_N^\infty)) \geqslant \operatorname{tr}(T|_{X_k})/\nu(T) \geqslant k/N = (1+N)/2.$$

Identifying matrices with its collection of rows, we consider the following natural factorization of T (with $\tilde{T}$ induced by T)

$$T\colon L(l_N^\infty) = \underbrace{(l_N^1 \oplus \ldots \oplus l_N^1)_\infty}_{N\text{-fold}} \xrightarrow{I} (l_N^\infty \oplus \ldots \oplus l_N^\infty)_1 \xrightarrow{\tilde{T}} (l_N^1 \oplus \ldots \oplus l_N^1)_\infty = L(l_N^\infty).$$

Then $\|\tilde{T}\| \leqslant 1$ since

$$\|\tilde{T}(S)\|_{L(l_N^\infty)} = \sup_i \sum_j |s_{ij}^*| \leqslant \sum_i \sup_j |s_{ij}| = \|S\|_{(l_N^\infty \oplus \ldots \oplus l_N^\infty)_1}.$$

Hence $\nu(T) \leqslant \nu(I) \leqslant N\,\nu(\mathrm{Id}\colon l_N^1 \to l_N^\infty)$ by the ideal properties of the nuclear norm. But $\nu(\mathrm{Id}\colon l_N^1 \to l_N^\infty) = 1$. Let $B\colon l_N^\infty \to l_N^1$ with $\|B\| = 1$. For any $\varepsilon_i \in \{+1, -1\}$ one gets $\left|\sum_{i,j} b_{ij}\varepsilon_i\varepsilon_j\right| \leqslant 1$. Averaging over all $\varepsilon_i, \varepsilon_j$ yields

$$|\operatorname{tr}(B)| = \left|\sum_{i=1}^n b_{ii}\right| \leqslant 1.$$

Thus $\nu(\mathrm{Id}\colon l_N^1 \to l_N^\infty) = \sup\{|\operatorname{tr}(B)| \mid \|B\colon l_N^\infty \to l_N^1\| = 1\} \leqslant 1$ (the last argument was mentioned to us by A. Pełczyński), and the proof of $\nu(T) \leqslant N$ is completed.

Example. Whereas in the cases of $n = N^2$, $Y_n = l_n^\infty$ or $Y_n = L(l_N^\infty)$, the dimension k, for which $X_k \subseteq Y_n$ attains the maximal possible value of $\lambda(X_k, Y_n)$, is uniquely determined as $k = (n+\sqrt{n})/2$, this is no longer true for other spaces Y_n. Consider e.g. $Y_n = L(l_N^2)$ where always for $X_k \subseteq Y_n$

$$\lambda(X_k, Y_n) \leqslant 1/2\big(1+d(Y_n, l_n^2)\big) \leqslant (1+\sqrt{N})/2 = (1+n^{1/4})/2.$$

This value is attained for subspaces with different dimensions k if $n = N^2$, $N = 4^M$: First of all l_N^∞ imbeds isometrically into $L(l_N^2)$ (diagonal maps)

and contains X_k with $k = (N+\sqrt{N})/2 = (n^{1/2}+n^{1/4})/2$ and $\lambda(X_k) = \lambda(X_k; L(l_N^2)) = (1+n^{1/4})/2$ by (c) of the examples of matrices A. Consider next the $4^M \times 4^M$ matrices $A_{4^M} = (a_{ij})$ constructed there and let

$$\tilde{X}_{\tilde{k}} := \{S \in L(l_N^2) \mid s_{ij} = 0 \text{ for all } (i,j) \text{ with } a_{ij} = -1\}.$$

Then $\dim \tilde{X}_{\tilde{k}} =$ number of $+1$'s in the matrices A_{4^M} which is $\tilde{k} = (N^2+N^{3/2})/2 = (n+n^{3/4})/2$. This space also satisfies

$$\lambda(\tilde{X}_{\tilde{k}}, L(l_N^2)) = (1+n^{1/4})/2.$$

To give the idea of the proof of this fact, let $T\colon L(l_N^2) \to L(l_N^2)$ be given by $(s_{ij}) \to (a_{ij}s_{ij})$. By some averaging method and the trace duality, one can show $\nu(T) \leqslant N^{3/2}$ (actually $= N^{3/2}$), depending only on the fact that A_{4^M} is a Hadamard matrix. Hence

$$\lambda(\tilde{X}_{\tilde{k}}, L(l_N^2)) \geqslant \operatorname{tr}(T\colon \tilde{X}_{\tilde{k}} \to \tilde{X}_{\tilde{k}})/\nu(T) \geqslant \tilde{k}/N^{3/2} = (1+n^{1/4})/2.$$

If there would be Hadamard matrices with a larger number of $+1$'s than in the above A_{4^M}'s, the same estimate would give a contradiction to $\lambda(X_k, L(l_N^2)) \leqslant (1+n^{1/4})/2$ for all $X_k \subseteq L(l_N^2)$. Thus we have

Corollary 2. *For any $N \times N$-matrix $A = (a_{ij})$ with $AA^t = NI_N$ and $a_{ij} \in \{+1, -1\}$, at least $(N^2-N^{3/2})/2$ and at most $(N^2+N^{3/2})/2$ of its entries are $+1$ (or -1).*

For $X_k \subseteq Y_n$ with n fixed, equality in $\lambda(X_k, Y_n) \leqslant (1+\sqrt{n})/2$ could be attained only for $k = (n+\sqrt{n})/2$. In the situation of $Y_n = l_n^p$ of Corollary 1, a better estimate can be shown:

Lemma 5. *Let $1 < p < \infty$ and $1/r := |1/2-1/p|$. Then for any $X_k \subseteq l_n^p$ $\lambda(X_k, l_n^p) \leqslant (1+n^{1/r})/2$ where equality could possibly be only attained for $k = (n+n^{1/r'})/2$ (requiring this to be an integer).*

Proof. As seen in the proofs of Theorem 1 and Corollary 1,

$$\lambda(X_k, l_n^p) \leqslant \sup\Big\{\sum_{i=1}^k \lambda_i \;\Big|\; \sum_{i=1}^n \lambda_i = 1, \|(\lambda_i)_{i=1}^n\|_{r'} = 1\Big\}.$$

Letting $x = \sum_{i=1}^k \lambda_i$, we have

$$(n-k)^{r'-1}x^{r'} + k^{r'-1}(x-1)^{r'}$$
$$\leqslant (n-k)^{r'-1}k^{r'/r}\Big(\sum_{i=1}^k |\lambda_i|^{r'}\Big) + k^{r'-1}(n-k)^{r'/r}\Big(\sum_{i=k+1}^n |\lambda_i|^{r'}\Big)$$
$$= [(n-k)k]^{r'-1}.$$

Thus $f_k(x) = x^{r'}/k^{r'-1} + (x-1)^{r'}/(n-k)^{r'-1} \leqslant 1$. Since f_k is increasing in x,

$$\lambda(X_k, l_n^p) \leqslant x_0$$

where x_0 is the unique solution of $f_k(x_0) = 1$. We claim that for fixed $n \in \boldsymbol{N}$

$$\sup\{x_0 \in \boldsymbol{R}^+ |\ 1 \leqslant k \leqslant n,\ f_k(x_0) = 1\} \leqslant (1+n^{1/r})/2,$$

which is attained at $k = (n+n^{1/r'})/2$, if this is an integer. This can be seen by treating k as a real variable and differentiating with respect to k.

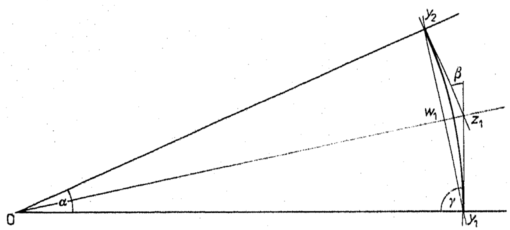
The estimate $\lambda(X_k, l_n^p) \leqslant (1+n^{1/r})/2$ has been given already by Sobczyk [8].

We still have to give the

Proof of Proposition 2. We will approximate the dual unit ball of a given real 2-dimensional space X_2 by a polygonal unit ball being the dual ball of a space $\tilde{X}_2$ and then use

$$\lambda(X_2) \leqslant \lambda(\tilde{X}_2)\, d(X_2, \tilde{X}_2).$$

By John's theorem [2] we may assume that the unit ball B_Y of $Y := X_2^*$ satisfies $B_2 \subseteq B_Y \subseteq \sqrt{2} B_2$ where B_2 is the euclidean unit ball. Moreover, B_Y may be assumed to be smooth (by approximation). Choose $y_1 \in Y$ with $\|y_1\|_Y = 1$. Next choose $y \in Y$ with $\|y\|_Y = 1$, $y \neq y_1$ in the positive direction around zero starting from y_1. Let $t(y)$ and $t(y_1)$ be the tangents to B_Y at y and y_1 and $\beta(y)$ be the angle between these tangents and z be their point of intersection. Let $\alpha(y)$ be the angle between the lines $\overrightarrow{0y_1}$ and $\overrightarrow{0y}$ and γ the angle between $\overrightarrow{0y_1}$ and $\overrightarrow{y_1 z}$. Then $\pi/4 \leqslant \gamma \leqslant 3\pi/4$. Since $\alpha(y)$ and $\beta(y)$ depends continuously on y with $\|y\|_Y = 1$, there is a unique $y_2 \in Y$ with $\|y_2\|_Y = 1$ and $\min(\alpha(y_2), \beta(y_2)) = 2\pi/n$, for a fixed given $n \in \boldsymbol{N}$ to be determined later. The corresponding point z will be called z_1.



Continuing in this way, we find points $y_1, \ldots, y_k$ of norm $\|y_i\|_Y = 1$ s.t. $\alpha(y_{i+1}, y_i) \leqslant 2\pi/n$, $\beta(y_{i+1}, y_i) \leqslant 2\pi/n$. At most n points are needed to "travel half way around the circle"; we denote them by $y_1, \ldots, y_m$, $m \leqslant n$. Let $\tilde{Y}$ be the space whose unit ball is the absolutely convex hull of the $y_1, \ldots, y_m$ and $\tilde{X}_2 := \tilde{Y}^*$. Since $\operatorname{ext} B_{\tilde{Y}} = \{\pm y_i |\ i = 1, \ldots, m\}$, there is

a canonical isometric imbedding $\tilde{Y} \subseteq l_m^\infty \subseteq l_n^\infty$. By Theorem 1,

$$\lambda(\tilde{Y}) \leqslant \sqrt{2}(1-c/n), \qquad c = 3/2 - \sqrt{2}.$$

Let w_1 be the point of intersection of $\overrightarrow{0z_1}$ and $\overrightarrow{y_1 y_2}$ and define similarly w_i. Then the Banach–Mazur distance can be estimated by

$$d(X_2, \tilde{X}_2) = d(Y, \tilde{Y}) \leqslant \max_i \|z_i\|_2 / \|w_i\|_2.$$

Some elementary plane geometry shows that this ratio can be bounded by

$$d(X_2, \tilde{X}_2) \leqslant 1 + \alpha\beta/2 \leqslant 1 + 2\pi^2/n^2,$$

using that the angle $\pi/4 \leqslant \gamma \leqslant 3\pi/4$ does not "degenerate" to 0 or π. Hence

$$\lambda(X_2) \leqslant (1+2\pi^2/n^2)(1-c/n)\sqrt{2}$$

which for $n = 450$ yields $\lambda(X_2) \leqslant \sqrt{2} - \varepsilon$, $\varepsilon = 1.4 \cdot 10^{-4}$.

The involutions A of (3) of Theorem 2 are connected to the problem of minimal reflections. An operator $R\colon Y \to Y$ is called a *reflection* about $X \subseteq Y$ iff $R = 2P - I$, where $P\colon Y \to X \subseteq Y$ is a projection onto X. Let

$$\mu(X, Y) = \inf\{\|R\| \mid R\colon Y \to Y \text{ is a reflection about } X\},$$

$$\mu(k, n) = \sup\{\mu(X_k, Y_n) |\ X_k \subseteq Y_n\},$$

$$g(k, n) = \begin{cases} 1/n\left(|n-2k| + 2\sqrt{k(n-k)(n-1)}\right) & \text{if } |n-2k| \geqslant \sqrt{n}, \\ \sqrt{n} & \text{if } |n-2k| < \sqrt{n}. \end{cases}$$

The result corresponding to Theorem 1 in the case of reflections is

PROPOSITION 3. *We have* $\mu(k, n) \leqslant g(k, n)$.

Proof. Let $X_k \subseteq Y_n$ and $R_0\colon Y_n \to Y_n$ be a reflection about X_k of minimal norm. A similar argument as in the proof of Lemma 1 shows

$$\mu(X_k, Y_n) = \sup\{|\operatorname{tr}(TR_0)| \mid T\colon Y_n \to Y_n,\ T(X_k) \subseteq X_k,\ \nu(T) = 1\}.$$

Since $R_0 = 2P_0 - I$ for some projection P_0 onto X, $TR_0 = 2TP_0 - T$. Thus if $\lambda_1(T), \ldots, \lambda_1(T)$ are the eigenvalues of T where $\lambda_1(T), \ldots, \lambda_k(T)$ are those with respect to X_k, we get

$$\mu(X_k, Y_n) = \sup\Big\{\Big|\sum_{i=1}^{k} \lambda_i(T) - \sum_{i=k+1}^{n} \lambda_i(T)\Big| \Big|$$

$$T\colon Y_n \to Y_n,\ T(X_k) \subseteq X_k,\ \nu(T) = 1\Big\}$$

$$\leqslant \|(\underbrace{1, \ldots, 1}_{k}, \underbrace{-1, \ldots, -1}_{n-k})\|_{Z_2^*}$$

using the same notation as in Lemma 2. Lemma 2 yields

$$\mu(X_k, Y_n) \leqslant \inf_{t\in K}\{|t|+(k|1-t|^2+(n-k)|1+t|^2)^{1/2}\} =: \inf_{t\in K} h(t).$$

The infimum of h is attained at some $t\in \boldsymbol{R}$; differentiation shows:

$$\begin{aligned} &\text{at } t_0 = 0 && \text{if } |n-2k|\leqslant \sqrt{n},\\ &\text{at } t_+ = (2k-n)/n-2/n\sqrt{k(n-k)/(n-1)}>0 && \text{if } 2k-n>\sqrt{n},\\ &\text{at } t_- = (2k-n)/n+2/n\sqrt{k(n-k)/(n-1)}<0 && \text{if } n-2k>\sqrt{n}.\end{aligned}$$

We remark that $|(n-2k)/n|>2/n\sqrt{k(n-k)/(n-1)}$ iff $|n-2k|>\sqrt{n}$. In the first case $h(0)=\sqrt{n}$, in the second or third case

$$h(t_\pm) = 1/n\big(|2k-n|+2\sqrt{k(n-k)(n-1)}\big).$$

To prove an analogue of Theorem 2, we need a result corresponding to Lemma 3.

LEMMA 6. *Let $(\lambda_i)_{i=1}^n \in Z_2$ with $\|(\lambda_i)_{i=1}^n\|_{Z_2}=1$. The following are equivalent*:

(1) $$\sum_{i=1}^k \lambda_i - \sum_{i=k+1}^n \lambda_i = g(k,n).$$

(2)
$$\lambda_1=\ldots=\lambda_k = 1/\sqrt{n},\quad \lambda_{k+1}=\ldots=\lambda_n=-1/\sqrt{n}\quad \textit{if}\quad |2k-n|\leqslant\sqrt{n},$$

$$\lambda_1=\ldots=\lambda_k = f(k,n)/k,\quad \lambda_{k+1}=\ldots=\lambda_n=\big(1-f(k,n)\big)/(n-k)\quad \textit{if}\quad 2k-n>\sqrt{n},$$

$$\lambda_1=\ldots=\lambda_k=\big(f(n-k,n)-1\big)/k,$$
$$\lambda_{k+1}=\ldots=\lambda_n=-f(n-k,n)/(n-k)\quad \textit{if}\quad n-2k>\sqrt{n}.$$

Proof. (2) implies (1) since $g(k,n)=2f(k,n)-1$ if $2k-n>\sqrt{n}$. If (1) holds and $|2k-n|\leqslant\sqrt{n}$,

$$\sqrt{n} = g(k,n)\leqslant \sum_{i=1}^n|\lambda_i|\leqslant\sqrt{n}\Big(\sum_{i=1}^n|\lambda_i|^2\Big)^{1/2}\leqslant\sqrt{n}$$

yields that $|\lambda_i|=1/\sqrt{n}$ for all i, the first k being positive and the others negative. If $|2k-n|>\sqrt{n}$, for reasons of symmetry we may assume $2k-n>\sqrt{n}$. In this case, denoting $g:=g(k,n)$

$$h(x) := 1/k\big((g+x)/2\big)^2+1/(n-k)\big((g-x)/2\big)^2$$

is decreasing for $x\in[-1,1]$ since $h'(x)=\big((n-2k)g+nx\big)/\big(2k(n-k)\big)<0$ for $|x|\leqslant 1$. Thus $h(x)\geqslant h(1)$. Calculation shows $h(1)=1$. Since $\|(\lambda_i)_{i=1}^n\|_{Z_2}=1$, $x:=\sum_{i=1}^n\lambda_i$ satisfies $|x|\leqslant 1$. $\sum_{i=1}^k\lambda_i-\sum_{i=k+1}^n\lambda_i=g$ implies $\sum_{i=1}^n\lambda_i=(g+x)/2$, and by Hölder's inequality

$$\sum_{i=k+1}^n|\lambda_i|^2\leqslant 1-\sum_{i=1}^k|\lambda_i|^2\leqslant 1-1/k\Big(\sum_{i=1}^k\lambda_i\Big)^2 = 1-1/k\big((g+x)/2\big)^2.$$

Using $h(x)\geqslant 1$ we find

$$\sum_{i=k+1}^n|\lambda_i|^2\leqslant 1/(n-k)\big((g-x)/2\big)^2 = 1/(n-k)\Big(\sum_{i=k+1}^n\lambda_i\Big)^2\leqslant\sum_{i=k+1}^n|\lambda_i|^2.$$

Thus the previous inequalities are equalities. This yields $\lambda_1=\ldots=\lambda_k$, $\lambda_{k+1}=\ldots=\lambda_n$ and $x=\sum_{i=1}^n\lambda_i=1$ which easily implies $\lambda_1=f(k,n)/k$, $\lambda_n=\big(1-f(k,n)\big)/(n-k)$ using again $g(k,n)=2f(k,n)-1$.

PROPOSITION 4. *Let $1\leqslant k<n$. The following are equivalent*:

(1) *There exists a subspace $X_k\subseteq l_n^\infty$ such that $\mu(X_k,l_n^\infty)=g(k,n)$.*

(2) *There exists an operator $T\colon l_n^\infty\to l_n^\infty$ with nuclear norm $\nu(T)=1$ and eigenvalues equal to the values $(\lambda_i)_{i=1}^n$ in (2) of Lemma 6.*

(3) *If $|2k-n|\geqslant\sqrt{n}$, there is a matrix $A=(a_{ij})$ with $A^2=I$ and $a_{ii}=2k/n-1$ $(i=1,\ldots,n)$, $|a_{ij}|=2/n\sqrt{k(n-k)/(n-1)}$ $(i,j=1,\ldots,n,\ i\neq j)$.*

If $|2k-n|<\sqrt{n}$, there is a matrix $A=(a_{ij})$ with $A^2=I$ and $|a_{ij}|=1/\sqrt{n}$ $(i,j=1,\ldots,n)$ and $\sum_{i=1}^n a_{ii}=2k-n$.

If (3) holds, the eigenspace X_k corresponding to the eigenvalue $+1$ of A has dimension k and satisfies $\mu(X_k,l_n^\infty)=g(k,n)$; A itself is a reflection of minimal norm.

Proof. The equivalence of (1) and (2) follows directly from the proof of Proposition 3 and from Lemma 6. For $2k-n\geqslant\sqrt{n}$, the values of the $\lambda_i=\lambda_i(T)$ are the same as in Theorem 2; thus (2)$\Leftrightarrow$(3) follows from there. In the case $n-2k\geqslant\sqrt{n}$, just interchange k with $n-k$ and (T,A) with $(-T,-A)$. To prove (2)$\Leftrightarrow$(3) in the remaining case $|2k-n|<\sqrt{n}$, note that (3) implies (2) letting $T:=1/\sqrt{n}A$, since then $\nu(T)=\sum_j\sup_i|t_{ij}|=1$, $\lambda_i(T)\in\{+1/\sqrt{n},-1/\sqrt{n}\}$ since $A^2=I$. Moreover, k of the eigenvalues have to be $+1/\sqrt{n}$, the rest $-1/\sqrt{n}$ to ensure that $\sum_i\lambda_i(T)=\operatorname{tr}(T)=1/\sqrt{n}\operatorname{tr}(A)=(2k-n)/\sqrt{n}$. If (2) holds, let $A:=\sqrt{n}\,T$. Since $\lambda_i(A)$

$\in \{\pm 1\}$, $A^2 = I_n$. Lemma 4 yields $|a_{ii}| = |a_{ij}| =: \delta_i$ for all $i, j \in \{1, \ldots, n\}$, i.e. constancy of the absolute values on the columns. The condition $(A^2)_{ii} = 1$ gives

$$\delta_i\Big(\sum_{j=1}^{n} \delta_j\Big) = 1, \quad i = 1, \ldots, n.$$

Hence $\delta_1 = \ldots = \delta_n = 1/\sqrt{n}$. Thus $|a_{ij}| = 1/\sqrt{n}$ for all i and j. Moreover,

$$\sum_{i=1}^{n} a_{ii} = \operatorname{tr}(A) = \sqrt{n}\operatorname{tr}(T) = \sqrt{n}\sum_{i=1}^{n}\lambda_i(T) = 2k-n.$$

The last statements follow from the one-to-one correspondence of T and A and the fact that (3) implies $\|A\| = g(k, n)$.

Remarks. (i) Thus for $|2k-n| \geqslant \sqrt{n}$, the problem of existence of subspaces of l_n^∞ with worst possible reflection constants is the same as the one for projection constants; the combinatorial matrices needed in (3) of Theorem 2 are in fact relections of minimal norm about "worst complemented and worst reflected" subspaces.

(ii) The case $|2k-n| < \sqrt{n}$ is different from the projection case, since here $|\operatorname{tr}(T)| \leqslant 1$ is automatically satisfied and $\operatorname{tr}(T) = 1$ can no longer be guaranteed, in fact $|\operatorname{tr}(T)| = |(2k-n)/\sqrt{n}| < 1$. In the $|2k-n| < \sqrt{n}$ — real case, $|2k-n|\sqrt{n} \in \boldsymbol{N}$ is necessary for the existence of A.

We now discuss a few *examples* which follow from the previous results.

(a) $k = n-1$, $n \geqslant 4$: $g(k, n) = 3-4/n$ is attained by some $X_k \subseteq l_n^\infty$.

(b) $k = (n \pm \sqrt{n})/2$, $n = 4^N$ in the real case or $n = N^2$ in the complex case: $g(k, n) = \sqrt{n}$ is attained by some $X_k \subseteq l_n^\infty$.

(c) $k = n/2$, $n = 2^N$: $g(k, n) = \sqrt{n}$ is best possible. One can take for A e.g. a (correctly scaled) Walsh-matrix. This also solves the case $k = n-1$, $n = 2$.

(d) For $k = n-1$, $n = 3$, $g(k, n) = \sqrt{3}$ is *not* attained in the real case for some $X_2 \subseteq l_3^\infty$ since $\sqrt{3} \notin \boldsymbol{N}$.

In fact, one can show $\mu(X_2, l_3^\infty) \leqslant 5/3 < \sqrt{3}$ by some ad-hoc considerations. A natural guess seems to be $\mu(2, 3) = 5/3$ (attained by the space with hexagonal unit ball).

(e) In the complex case, for $n = N^2$ and $|n-2k| \leqslant \sqrt{n}$, $g(k, n) = \sqrt{n}$ is attained: There is $X_k \subseteq l_n^\infty$ with $\mu(X_k, l_n^\infty) = \sqrt{n}$. To prove this, we construct a matrix A as required in (3):

Let B be an $N \times N$-matrix with $BB^* = B^*B = I_N$ and $|b_{ij}| = 1/\sqrt{N}$, e.g. $B = 1/\sqrt{N}(w^{ij})_{1 \leqslant i, j \leqslant N}$ where w is a primitive N^{th} root of unity. Let

$$c_{ij} := \begin{cases} 1 & \text{if } i \neq j \text{ or } i = j \text{ and } 1 \leqslant i \leqslant k-(n-\sqrt{n})/2, \\ -1 & \text{if } i = j \text{ and } k-(n-\sqrt{n})/2 < i \leqslant N. \end{cases}$$

Define A by

$$a_{N(i-1)+l,\, N(j-1)+m} = c_{ij}\bar{b}_{im} b_{jl}, \quad 1 \leqslant i, j, l, m \leqslant N.$$

Then $A = A^*$, $|a_{\nu,\mu}| = 1/\sqrt{n}$ and

$$\operatorname{tr}(A) = \sum_{i=1}^{N} c_{ii} \sum_{l=1}^{N} \bar{b}_{il} b_{il} = \sum_{i=1}^{N} c_{ii} = 2k-n.$$

Moreover, $A^2 = I_n$ since any two rows are orthogonal: the inner product of the $N(i-1)+l^{\text{th}}$ and $N(i'-1)+l'^{\text{th}}$ row is

$$\sum_{j=1}^{N} c_{ij} c_{i'j} b_{jl} \bar{b}_{jl'} \cdot \sum_{m=1}^{N} \bar{b}_{im} b_{i'm}.$$

If $i \neq i'$, the second sum is zero, if $i = i'$, the inner product is

$$\sum_{j=1}^{N} c_{ij}^2 b_{jl} \bar{b}_{jl'} = \delta_{ll'}.$$

The same construction applies in the real case if $n = N^2$ and N is an index for which a Hadamard matrix exists, e.g. $N = 2^M$ or $N = p^M+1 \equiv 2 \pmod 4$, p a prime.

(f) Let $1 < p < \infty$ and $1/r = |1/2 - 1/p|$. For a subspace $X_k \subseteq l_n^p$ we get by similar considerations as before that

$$\mu(X_k, l_n^p) = \sup\Big\{\Big|\sum_{i=1}^{k}\lambda_i(T) - \sum_{i=k+1}^{n}\lambda_i(T)\Big| \Big| \; T\colon l_n^p \to l_n^p,\ \nu(T) = 1,\ T(X_k) \subseteq X_k\Big\}$$
$$\leqslant \inf_t \big\{|t| + \big(k(1-t)^r + (n-k)(1+t)^r\big)^{1/r}\big\} \leqslant n^{1/r}.$$

For k with $|n-2k| \leqslant \sqrt{n}$ there are $X_k \subseteq l_n^p$ such that equality $\mu(X_k, l_n^p) = n^{1/r}$ is attained: Let A be a matrix as in (e), X_k be the (k-dimensional) image of $P = 1/2(I+A)$ and $T = n^{-1/r'}A$. Then $\nu(T\colon l_n^p \to l_n^p) \leqslant 1$ since $|t_{ij}| = n^{-(1+\min(1/p, 1/p'))}$. Since $\lambda_j(T) = n^{-1/r'}$, $1 \leqslant j \leqslant k$ and $\lambda_j(T) = n^{-1/r'}$, $k+1 \leqslant j \leqslant n$ we get

$$\mu(X_k, l_n^p) \geqslant kn^{-1/r'} - (n-k)(-n^{-1/r'}) = n^{1/r}.$$

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