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Projections onto gradient fields and L^p-estimates for degenerated elliptic operators

by

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Abstract. Let $L^m(\mathbf{R}^N, \mathbf{R}^N)$ be the space of all vector-valued functions $f: \mathbf{R}^N \to \mathbf{R}^N$, which are integrable with the power $m \geqslant 2$. Consider the subspace $D^m(\mathbf{R}^N)$ of all functions which are the gradients of scalar functions on $\mathbf{R}^N$. We study the closest point projection $P_m: L^m(\mathbf{R}^N, \mathbf{R}^N) \to D^m(\mathbf{R}^N)$. The main result of the paper is the inequality $\|P_m f\|_p \leqslant A_p \|f\|_p$, for any $p \geqslant m$. In the proof an inequality of Fefferman and Stein is used. As an application of the methods presented we give some regularity results on PDE's and quasiconformal mappings. In particular, we get a stronger version of the Gehring theorem on L^p-integrability of first derivatives of quasiconformal mappings.

Introduction and statement of the results. The main objects of this paper are the Lebesgue spaces $L^m(\mathbf{R}^N, \mathbf{R}^N)$, $1 \leqslant m < \infty$, of mappings f from $\mathbf{R}^N$ to $\mathbf{R}^N$ with the standard norm

$$\|f\|_m = \Big(\int |f(x)|^m dx\Big)^{1/m}$$

and their subspaces $D^m(\mathbf{R}^N)$ of gradient fields, i.e. of vector-functions of the form $f = \nabla u$, where ∇ is the gradient operator acting on locally integrable functions u for which we can define $f \in L^m(\mathbf{R}^N, \mathbf{R}^N)$ such that

$$\int \langle f, \varphi \rangle = -\int u \operatorname{div} \varphi$$

for any test function $\varphi \in C_0^\infty(\mathbf{R}^N, \mathbf{R}^N)$. Hereafter $\langle , \rangle$ is reserved for the scalar product in $\mathbf{R}^N$.

Our main results concern the L^p-estimates, $p \geqslant m$, for a projection

$$P_m: L^m(\mathbf{R}^N, \mathbf{R}^N) \to D^m(\mathbf{R}^N), \qquad m \geqslant 2.$$

The interest in bounding such projections is motivated by a number of applications we give to problems of regularity in PDE and quasiconformal mappings.

Let us first consider $m = 2$; then $L^2(\mathbf{R}^N, \mathbf{R}^N)$ and $D^2(\mathbf{R}^N)$ are Hilbert spaces and the orthogonal projection $P: L^2(\mathbf{R}^N, \mathbf{R}^N) \to D^2(\mathbf{R}^N)$ is linear. Therefore, for any $f \in L^2(\mathbf{R}^N, \mathbf{R}^N)$ we have $Pf = \nabla u$ with an u which minimizes the Dirichlet integral

$$\inf_v \Big(\int |f-\nabla v|^2\Big)^{1/2} = \Big(\int |f-\nabla u|^2\Big)^{1/2} = \|Pf-f\|_2;$$

thus u can be found from the Euler equation

$$\operatorname{div}\nabla u = \operatorname{div} f. \tag{0.1}$$

The standard method of the Fourier transform yields an explicit formula

$$Pf = -R\langle R, f\rangle, \tag{0.2}$$

where for $R = (R_1, \ldots, R_N)$, $f = (f^1, \ldots, f^N)$ we denote $\langle R, f\rangle = \sum_i R_i f^i$ and R_i stand for the Riesz transforms

$$R_i h(x) = c_N \int \frac{(x_i - y_i)\, h(y)\, dy}{|x-y|^{N+1}} \quad (i = 1, \ldots, N). \tag{0.3}$$

Since R_i are L^p-bounded for $1 < p < \infty$ and are isometries in $L^2(\mathbf{R}^N)$, it follows that

PROPOSITION 1. *If f is in $L^2(\mathbf{R}^N, \mathbf{R}^N)$ and in $L^p(\mathbf{R}^N, \mathbf{R}^N)$ for some $1 < p < \infty$, so is Pf. Moreover, for some $C_p = C_p(N)$ we have*

$$\|Pf\|_p \leqslant C_p \|f\|_p, \qquad C_2 = 1. \tag{0.4}$$

From now on m will be a fixed real number $\geqslant 2$. The projection $P_m\colon L^m(\mathbf{R}^N, \mathbf{R}^N) \to D^m(\mathbf{R}^N)$ is defined by $P_m f = \nabla u$, where u is a locally integrable function which minimizes the integral

$$\inf_v \Big(\int |f-\nabla v|^m\Big)^{1/m} = \Big(\int |f-\nabla u|^m\Big)^{1/m} = \|P_m f - f\|_m. \tag{0.5}$$

In contrast to the previous case the operator P_m, $m > 2$, is non-linear and the corresponding Euler equation

$$\operatorname{div}|\nabla u - f|^{m-2}(\nabla u - f) = 0 \tag{0.6}$$

is not uniformly elliptic; thus the study of P_m is much more complicated and requires deeper arguments from PDE theory.

We shall prove a straightforward generalization of Proposition 1, namely

THEOREM 1. *If f is in $L^m(\mathbf{R}^N, \mathbf{R}^N)$ and in $L^p(\mathbf{R}^N, \mathbf{R}^N)$, $p \geqslant m$, so is $P_m f$. Moreover,*

$$\|P_m f\|_p \leqslant B_p \|f\|_p \tag{0.7}$$

for any f and some constant $B_p = B_p(m, N)$ independent of f.

For a full analogy it remains to have (0.7) for $p > m-1$, which we conjecture to be true but are unable to prove. However, for the applications which we have in mind, the stronger result is not essential; our main interest is in the estimates for large p.

In the paper we shall be mostly concerned with a somewhat more general equation

$$\operatorname{div}|\nabla u|^{m-2}\nabla u = \operatorname{div} f, \tag{0.8}$$

which can be recognized as the Euler equation for the variational integral functional

$$I(v) = \int |\nabla v|^m - m\langle f, \nabla v\rangle$$

defined for $\nabla v \in D^m(\mathbf{R}^N)$ with given f from $L^{m/(m-1)}(\mathbf{R}^N, \mathbf{R}^N)$. As usual, (0.8) is understood in the distributional sense, i.e. the equality

$$\int |\nabla u|^{m-2}\langle \nabla u, \nabla \eta\rangle = \int \langle f, \nabla\eta\rangle \tag{0.9}$$

must hold for any test function $\eta \in C_0^\infty(\mathbf{R}^N)$, whence for those with $\nabla\eta \in D^m(\mathbf{R}^N)$ as well. We immediately observe that (0.9) for $\eta = u$ and the Hölder inequality yield

$$\int |\nabla u|^m \leqslant \int |f|^{m/(m-1)}. \tag{0.10}$$

We shall generalize this as follows:

THEOREM 2. *Let f be in $L^{m/(m-1)}(\mathbf{R}^N, \mathbf{R}^N)$ and in $L^{p/(m-1)}(\mathbf{R}^N, \mathbf{R}^N)$ for some $p \geqslant m$ while u is the solution of (0.8) with ∇u in $D^m(\mathbf{R}^N)$. Then ∇u is in $D^p(\mathbf{R}^N)$ and*

$$\|\nabla u\|_p^{m-1} \leqslant A_p \|f\|_{p/(m-1)}, \tag{0.11}$$

where $A_p = A_p(m, N)$ are constants independent of f ($A_m = 1$).

This theorem will be utilized in the proof of Theorem 1 and two other ones from PDE and quasiconformal mappings. Let us formulate applications to PDE.

Take a domain Ω in $\mathbf{R}^N$ and a measurable map $G\colon \Omega \to \mathrm{GL}(N, \mathbf{R})$ from Ω into the general linear group. Suppose that both conditions:

(0.12) $G(x)$ is symmetric

and

(0.13) There are constants $0 < a \leqslant b < \infty$ such that

$$a^2|\xi|^2 \leqslant \langle G(x)\xi, \xi\rangle \leqslant b^2|\xi|^2, \qquad \xi \in \mathbf{R}^N,$$

hold for almost all x in Ω.

We intend to apply Theorem 2 to a regularity problem for the equation

$$\operatorname{div}\langle G(x)\nabla u, \nabla u\rangle^{(m-2)/2} G(x)\nabla u = 0 \tag{0.14}$$

which arises at several places of analysis: let us just mention the capacity theory and quasiconformal theory ($m = N$); it is also a typical example in the theory of degenerated elliptic equations. As before, the equation is understood in the distributional sense and the solutions u are sought in the Sobolev space $W^1_m(\Omega)$; thus they are stationary points of the functional

$$J(u) = \int \langle G(x)\nabla u, \nabla u\rangle^m dx.$$

The results which follow depend on the "G-coefficients" of (0.14), essentially on the characteristic

$$K_G = \operatorname*{ess\,sup}_{x\in\Omega} \big(1+|G(x)-E|\big)^{m/2}, \qquad K_G \geqslant 1 \tag{0.15}$$

alone. Here E stands for the identity matrix and $|A|$ is the norm of A. Later we shall show that

$$\big|\langle G(x)\xi, \xi\rangle^{(m-2)/2} G(x)\xi - |\xi|^{m-2}\xi\big| \leqslant (K_G-1)|\xi|^{m-1}, \tag{0.16}$$

which entitles us to interpret K_G-1 as a distance measure between the operators (0.8) and (0.14).

THEOREM 3. *Let $p \geqslant m$. Suppose that K_G-1 is such that*

$$(K_G-1)A_p < 1,$$

where A_p has been determined in Theorem 2. Then any solution of equation (0.14) belongs to $W^1_{p,\mathrm{loc}}(\Omega)$. Moreover, for any concentric balls $B_r(x_0) \subset B_R(x_0) \subseteq \Omega$, $R > r$ the following apriori estimate holds:

$$\Big(\bar{\int_{B_r}} |\nabla u|^p\Big)^{1/p} \leqslant C_p \Big(\bar{\int_{B_R}} |\nabla u|^m\Big)^{1/m}, \tag{0.17}$$

where $C_p = C_p(m, N, R/r)$ is a constant independent of u and the integral mean value $(\operatorname{mes} A)^{-1}\int_A$ has been denoted by a barred integral $\bar{\int_A}$.

L^p-estimates of this kind are considered in PDE. Taking into account the results given by N. G. Meyers and A. Elcrat [2], we find that the solutions of equation (0.14) belong to $W^1_{p,\mathrm{loc}}(\Omega)$ for some $p > m$. Meyers and Elcrat have shown, making use of the well-known Gehring lemma [4] that no regularity assumptions on G are necessary for this result. In this paper we shall see that the exponent p may be arbitrarily large as soon as $G(x)$ is close to a given continuous matrix-function. It should be mentioned that Gehring's lemma does not work for large p even for the case of smooth G.

In connection with quasiregular mappings we shall also investigate the regularity problem.

DEFINITION 1. Assume $f\colon \Omega\to \mathbf{R}^N$, $f = (f^1, \ldots, f^N)$; then f is said to be *quasiregular* if $f \in W^1_{N,\mathrm{loc}}(\Omega)$ and the following inequality holds:

$$|Df(x)|^N \leqslant \mathscr{K} J(x, f) \tag{0.18}$$

for almost all $x \in \Omega$, where $Df = \left(\dfrac{\partial f^i}{\partial x_j}\right)$ is the Jacobi matrix of f, $J(x,f)$ is the Jacobian of f, and $\mathscr{K}$ is a constant.

According to the well-known definition for the matrix-function, inequality (0.18) can be reduced to the Beltrami equation

$$D^*f(x)Df(x) = J(x,f)^{2/N} G(x), \tag{0.19}$$

where G is defined by

$$G(x) = \begin{cases} \dfrac{D^*f(x)Df(x)}{J(x,f)^{2/N}} & \text{if } J(x,f) \neq 0, \\ E & \text{if } J(x,f) = 0. \end{cases}$$

Here the matrix transposed to Df is denoted by D^*f. In view of (0.18) we observe that G satisfies conditions (0.12), (0.13) and $\det G(x) = 1$ for almost all x in Ω. Let $0 < \beta_1(x) \leqslant \beta_2(x) \leqslant \ldots \leqslant \beta_N(x) < \infty$ denote the eigenvalues of $G(x)$. Then the number

$$K_f = \operatorname*{ess\,sup}_{x\in\Omega} \frac{\beta_N(x)}{\beta_1(x)} \geqslant 1 \tag{0.20}$$

will be called the *dilatation* of f. Moreover, if $K_f \leqslant K$, then the map f is called *K-quasiregular*.

From the Beltrami equation one can derive the following differential equation of second order (see [1] and [6]):

$$\operatorname{div}\langle G^{-1}(x)\nabla u, \nabla u\rangle^{(N-2)/2} G^{-1}(x)\nabla u = 0, \tag{0.21}$$

which is satisfied by each component $u = f^i$, $i = 1, \ldots, N$ of the map f. The dilatation K_f and the characteristic $K_{G^{-1}}$ of the inverse matrix G^{-1} are related by

$$\sqrt{K_f} \leqslant K_{G^{-1}} \leqslant \sqrt{K_f^{N-1}}. \tag{0.22}$$

The proof of the above inequalities will be given in Section 6. As a consequence of Theorem 3 we immediately obtain the following regularity properties of quasiregular mappings.

THEOREM 4. *Any K-quasiregular map $f\colon \Omega\to\mathbf{R}^N$ belongs to the Sobolev space $W^1_{p,\mathrm{loc}}(\Omega)$ whenever $p \geqslant N$ and*

$$(K^{(N-1)/2}-1)A_p < 1, \tag{0.23}$$

where A_p is determined in Theorem 2. Moreover, the following local estimate holds:

(0.24) $$\Big(\overline{\int\limits_{B_r}} |Df(x)|^p\,dx\Big)^{1/p} \leqslant C_p(N, K, R/r)\Big(\overline{\int\limits_{B_R}} J(x, f)\,dx\Big)^{1/N}$$

for any concentric balls $B_r(x_0) \subset B_R(x_0) \subseteq \Omega$, $R > r$.

Such estimates for quasiregular mappings have been investigated by J. Rešetnjak [7]. He has made use of some essential properties of quasiconformal mappings, for instance the stability theorem and also some specific generalization of the John–Nirenberg lemma on BMO spaces. Therefore, the methods which he has introduced are not useful for differential equations.

By Theorem 4 one can give some explanation of the Gehring conjecture, which proposes an explicit formula for the exponent $p = p_N(K)$.

The idea and the results of this paper may easily be extended for differential operators of a more general form. For example one can obtain L^p-estimates for a large class of monotone operators of the type

(0.25) $$L(u) = \operatorname{div} A(x, u, \nabla u),$$

where $A = A(x, u, \xi)$ is a given vector-function satisfying the usual regularity and growth conditions. We shall not give these generalizations here since they would involve extremely long expressions.

1. Preliminary facts. In order to clarify the proofs we collect here some elementary inequalities, estimates of maximal functions, a Poincaré-type inequality and a property of the Riesz potential.

The inequality

(1.1) $$|A^2 - c| \leqslant |B^2 - c| + \frac{\beta+1}{\beta}|A - B|^2 + \beta A^2$$

holds for arbitrary real numbers A, B, c and any positive β.

Let $m \geqslant 2$ and $\xi, \zeta \in \boldsymbol{R}^N$. Then

(1.2) $$2\langle|\xi|^{m-2}\xi - |\zeta|^{m-2}\zeta, \xi - \zeta\rangle \geqslant (|\xi|^{m-2} + |\zeta|^{m-2})|\xi - \zeta|^2,$$

(1.3) $$(|\xi|^{m-2} + |\zeta|^{m-2})|\xi - \zeta|^2 \geqslant 2m^{-2}(|\xi|^{m/2} - |\zeta|^{m/2})^2,$$

(1.4) $$2^{m-2}(|\xi|^{m-2} + |\zeta|^{m-2})|\xi - \zeta|^2 \geqslant |\xi - \zeta|^m,$$

(1.5) $$2^{m-1}\langle|\xi|^{m-2}\xi - |\zeta|^{m-2}\zeta, \xi - \zeta\rangle \geqslant |\xi - \zeta|^m,$$

(1.6) $$\big||\xi|^m - |\zeta|^m\big| \leqslant (m/2)(|\xi|^{m-1} + |\zeta|^{m-1})|\xi - \zeta|,$$

(1.7) $$(|\xi| + |\zeta|)^{m-1} \leqslant (1+\theta)^{m-2}|\xi|^{m-1} + (1+1/\theta)^{m-2}|\zeta|^{m-1}, \qquad \theta > 0,$$

(1.8) $$\big||\xi - \zeta|^{m-2}(\xi - \zeta) - |\xi|^{m-2}\xi\big| \leqslant \big((1+\theta)^{m-2} - 1\big)|\xi|^{m-1} + (1+1/\theta)^{m-2}|\zeta|^{m-1}.$$

Conditions (0.12) and (0.13) imply

(1.9) $$\langle\langle G\xi, \xi\rangle^{(m-2)/2}G\xi - \langle G\zeta, \zeta\rangle^{(m-2)/2}G\zeta, \xi - \zeta\rangle \geqslant (a^m/2)(|\xi|^{m-2} + |\zeta|^{m-2})|\xi - \zeta|^2.$$

These inequalities are rather elementary, and so we omit the proofs. As promised in Introduction, we give the proof of inequality (0.16). For any measurable map $G\colon \Omega \to \mathrm{GL}(N, \boldsymbol{R})$ and almost all $x \in \Omega$ we have

$$\begin{aligned} &\big|\langle G(x)\xi, \xi\rangle^{(m-2)/2}G(x)\xi - |\xi|^{m-2}\xi\big| \\ &= \Big|\int_0^1 \frac{d}{dt}\langle(E + t(G-E))\xi, \xi\rangle^{(m-2)/2}(E + t(G-E))\xi\,dt\Big| \\ &\leqslant \int_0^1 \frac{m}{2}|G - E|(1 + t|G - E|)^{(m-2)/2}|\xi|^{m-1}\,dt \\ &= \big[(1 + |G(x) - E|)^{m/2} - 1\big]|\xi|^{m-1} \leqslant (K_G - 1)|\xi|^{m-1}. \end{aligned}$$

Now we recall some results on maximal operators in L^p spaces. Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$. The Hardy–Littlewood maximal function is defined by

$$Mf(x) = \sup_{r>0} \overline{\int\limits_{B_r(x)}} |f(y)|\,dy,$$

where $B_r(x)$ is the ball of radius r and centre x.

Lemma 1 (see E. Stein [8]). *If $f \in L^p(\boldsymbol{R}^N)$, $1 < p \leqslant \infty$, then*

(1.10) $$\|f\|_p \leqslant \|Mf\|_p \leqslant 2\Big(\frac{5^N p}{p-1}\Big)^{1/p}\|f\|_p.$$

Another operator we use is the maximal operator of C. Fefferman and E. M. Stein [3].

(1.11) $$f^\natural(x) = \sup_{r>0} \overline{\int\limits_{B_r(x)}} |f(y) - f_{B_r}|\,dy,$$

where $f_B = \overline{\int\limits_B} f(y)\,dy$ is the integral mean value over a (measurable) subset $B \subset \boldsymbol{R}^N$. Notice that in the original definition of $f^\natural(x)$ the supremum in (1.11) is taken over all the balls which contain the point x, not necessarily as a centre. But in view of the inequality

$$\sup_B\Big\{\overline{\int\limits_B}|f(y) - f_B|\,dy;\ x \in B\text{ — a ball}\Big\} \leqslant 2^{N+1}\sup_{r>0}\overline{\int\limits_{B_r(x)}}|f(y) - f_{B_r}|\,dy$$

both definitions are essentially equivalent. The last inequality follows from the lemma:

LEMMA 2. *For any measurable subset $B \subset \mathbf{R}^N$ and any $f \in L^1(B)$ we have*

$$\fint_B |f(y)-f_B|\,dy \leqslant 2 \fint_B |f(y)-c|\,dy \tag{1.12}$$

for arbitrary real c.

LEMMA 3 (Fefferman and Stein [3]). *For any $1 \leqslant p < \infty$*

$$\|Mf\|_p \leqslant B_p \|f^\natural\|_p, \qquad B_p \leqslant (C_N)^p. \tag{1.13}$$

Next we present two lemmas on Sobolev spaces.

LEMMA 4 (Sobolev–Poincaré inequality). *Let $B \subset \mathbf{R}^N$ be a ball of radius ϱ and let u be a function from the Sobolev space $W_q^1(B)$, $1 \leqslant q < N$. Then*

$$\Big(\fint_B |u(y)-u_B|^{Nq/(N-q)}\,dy\Big)^{(N-q)/Nq} \leqslant C_N(q)\,\varrho \Big(\fint_B |\nabla u(y)|^q\,dy\Big)^{1/q}. \tag{1.14}$$

LEMMA 5. *Take $1 \leqslant s < \infty$ and $u \in W_s^1(B)$. Then*

$$\Big(\fint_B |u(y)-u_B|^{Ns/(N-1)}\,dy\Big)^{(N-1)/sN} \leqslant C_N(s)\,\varrho \Big(\fint_B |\nabla u(y)|^s\,dy\Big)^{1/s}. \tag{1.15}$$

This lemma follows from (1.14), where we put $q = sN/(N+s-1) < N$ and apply the Hölder inequality to the right hand integral.

We end this section with an estimate of Riesz potentials. Take a function h. The problem now is to find a vector field H such that

$$\operatorname{div} H = h. \tag{1.16}$$

If $h \in C_0(\mathbf{R}^N)$, we solve the Poisson equation

$$\operatorname{div} \nabla\psi = h. \tag{1.17}$$

Its solution is given by the Newtonian potential

$$\psi(x) = -\frac{1}{(N-2)\sigma_N}\int \frac{h(y)\,dy}{|x-y|^{N-2}}, \qquad N>2,$$

where σ_N is the measure of the unit sphere in $\mathbf{R}^N$. The gradient $\nabla\psi$ is expressed by means of the Riesz potential of order 1, namely

$$\nabla\psi(x) = \frac{1}{\sigma_N}\int \frac{(x-y)\,h(y)\,dy}{|x-y|^N}. \tag{1.18}$$

LEMMA 6. *Let r be such that $r > N/(N-1)$ and $h \in L^{rN/(r+N)}(\mathbf{R}^N)$; then $\nabla\psi$ belongs to $D^r(\mathbf{R}^N)$ and*

$$\|\nabla\psi\|_r \leqslant C_N(r)\,\|h\|_{rN/(r+N)}. \tag{1.19}$$

For the proof we recommend E. Stein [8].

From this result we get an estimate under additional assumptions on the function h.

LEMMA 7. *Let $s>1$, $q > \max(1, sN/(s+N))$, $N>2$. Suppose that $h \in L^q(\mathbf{R}^N)$, $\operatorname{supp} h \subset B_\varrho(x_0)$, $\int h(y)\,dy = 0$. Then formula* (1.18) *defines a vector field $H = \nabla\psi \in L^s(\mathbf{R}^N, \mathbf{R}^N)$ which satisfies*

$$\operatorname{div} H = h, \tag{1.20}$$

$$\|H\|_s \leqslant C_N(s,q)\,\varrho^{1+N/s}\Big(\fint_{B_\varrho(x_0)} |h(y)|^q\,dy\Big)^{1/q}, \tag{1.21}$$

where $C_N(s,q)$ depends neither on h nor on the ball $B_\varrho(x_0)$.

Proof. Since $\int h(y)\,dy = 0$, we have for $|x| > \varrho$

$$|H(x)| = \frac{1}{\sigma^N}\left|\int_{B_\varrho}\Big(\frac{x-y}{|x-y|^N} - \frac{x}{|x|^N}\Big)h(y)\,dy\right| \leqslant \frac{C_N\varrho}{(|x|-\varrho)^N}\int_{B_\varrho}|h(y)|\,dy$$

and by the Hölder inequality

$$|H(x)| \leqslant C_N \frac{\varrho^{1+N}}{(|x|-\varrho)^N}\Big(\fint_{B_\varrho}|h(y)|^q\,dy\Big)^{1/q}.$$

Therefore

$$\int_{|x|\geqslant 2\varrho}|H(x)|^s\,dx \leqslant C_N\varrho^{s+sN}\Big(\fint_{B_\varrho}|h|^q\Big)^{s/q}\int_{|x|\geqslant 2\varrho}\frac{dx}{(|x|-\varrho)^{sN}} \tag{1.22}$$

$$\leqslant C_N(s)\,\varrho^{s+N}\Big(\fint_{B_\varrho}|h|^q\Big)^{s/q}.$$

It suffices to estimate the integral $\int_{|x|<2\varrho}|H(x)|^s\,dx$. To do this we take

$$r = \begin{cases}\max\big(s, (N+1)/(N-1)\big) & \text{if } q \geqslant N-1/2,\\ qN/(N-q) & \text{if } q < N-1/2.\end{cases}$$

One can check that $r \geqslant s$, $r > N/(N-1)$, $rN/(r+N) \leqslant q$. By the Hölder inequality and Lemma 6 it follows that

$$\int_{|x|<2\varrho} |H(x)|^s dx \leqslant C_N \varrho^{N-sN/r} \|H\|_r^s \leqslant C_N(s,q) \varrho^{N-sN/r} \|h\|^s_{rN/(r+N)}$$
$$\leqslant C_N(s,q)\varrho^{s+N}\Big(\overline{\int_{B_\varrho}} |h|^{rN/(r+N)}\Big)^{s(r+N)/rN}$$
$$\leqslant C_N(s,q)\varrho^{s+N}\Big(\overline{\int_{B_\varrho}} |h|^q\Big)^{s/q}.$$

This together with (1.22) implies (1.21) and ends the proof.

2. Some regularity theorems. Some deep regularity results on degenerated elliptic equations are of principal importance for what follows.

LEMMA 8. *Let* $f \in C_0^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ *and* $G\colon \Omega \to \mathrm{GL}(N, \boldsymbol{R})$ *be a* C^∞ *map which satisfies* (0.12), (0.13). *Suppose that* u *solves one of the equations*

(a) $\quad \operatorname{div}|\nabla u|^{m-2}\nabla u = \operatorname{div} f, \qquad \nabla u \in L^m(\boldsymbol{R}^N, \boldsymbol{R}^N),$

(b) $\quad \operatorname{div}|\nabla u - f|^{m-2}(\nabla u - f) = 0, \qquad \nabla u \in L^m(\boldsymbol{R}^N, \boldsymbol{R}^N),$

(c) $\quad \operatorname{div}\langle G\nabla u, \nabla u\rangle^{(m-2)/2} G\nabla u = 0, \qquad \nabla u \in L^m(\Omega, \boldsymbol{R}^N).$

Then ∇u *is locally bounded, i.e.* $\nabla u \in L^\infty_{\mathrm{loc}}(\cdot, \boldsymbol{R}^N)$.

One can recognize this lemma in the theory of quasiregular variational equations, which have been investigated mainly by Ladyzenskaya and Uralt'seva (see [5], Chapter V). Actually ∇u is Hölder continuous but we do not need such a stronger property. The most difficult regularity result we use here is the following one (see [5], Theorem 7.1, p. 414).

If $w \in W^1_m(\Omega)$ *satisfies the homogeneous equation* $\operatorname{div}|\nabla w|^{m-2}\nabla w = 0$ *in an open subset* Ω, *then* ∇w *is Hölder continuous with exponent* $\alpha = \alpha_N(m)$ *independent of* w *and the domain* Ω. *Moreover, for any compact subset* $\Omega' \subseteq \Omega$ *the Hölder norm*

$$\|\nabla w\|_{\Omega'}^{(\alpha)} = \sup_{\Omega'}|\nabla w| + \sup_{\substack{x,y\in\Omega'\\ x\neq y}} \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^\alpha}$$

is dominated by the Lebesgue norm $\big(\int_\Omega |\nabla w(y)|^m dy\big)^{1/m}$.

By dimension analysis this implies

LEMMA 9. *Let* $B_r = B_r(x_0)$ *be the open ball of radius* r *and centre* $x_0 \in \boldsymbol{R}^N$ *and* $w \in W^1_m(B_r)$ *such that*

(2.1) $$\operatorname{div}|\nabla w|^{m-2}\nabla w = 0.$$

Then

(2.2) $$\sup_{B_{r/2}}|\nabla w| \leqslant C_N(m)\Big(\overline{\int_{B_r}} |\nabla w(y)|^m dy\Big)^{1/m}$$

and for any $\varrho \leqslant r/2$

(2.3) $$\sup_{x\in B_\varrho(x_0)} |\nabla w(x) - \nabla w(x_0)| \leqslant C_N(m)(\varrho/r)^\alpha \Big(\overline{\int_{B_r}} |\nabla w(y)|^m dy\Big)^{1/m},$$

where $\alpha = \alpha_N(m)$ *is a positive exponent independent of* w, r, *and* ϱ.

We derive from this result the following complement of Lemma 8.

COROLLARY 1. *Assume that either* (a) *or* (b) *in Lemma* 8 *holds. Then* ∇u *belongs to* $L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$ *for any* $p \geqslant m$.

Proof. The case $p = m$ is included in the definition of u. So, let $p > m$. Choose the number R so large that $f(x) = 0$ for $|x| \geqslant R$. Then u satisfies the homogeneous equation

$$\operatorname{div}|\nabla u|^{m-2}\nabla u = 0 \quad \text{in} \quad |x| > R.$$

Take into consideration the ball $B_r(x)$, where $r = |x| - R$. Then inequality (2.2) implies

$$|\nabla u(x)| \leqslant C_N(m)\Big(\overline{\int_{B_r(x)}} |\nabla u(y)|^m dy\Big)^{1/m} \leqslant \frac{C_N(m)}{(|x|-R)^{N/m}} \|\nabla u\|_m.$$

Hence for any $p > m$ we obtain

$$\int_{|x|>2R} |\nabla u(x)|^p dx < \infty.$$

Since ∇u is locally bounded, $\nabla u \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$ as we declared.

3. Proof of Theorem 2. The case $p = m$ has been considered in Introduction. Therefore we assume $p > m$. First we shall show estimate (0.11) under the condition $\nabla u \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$. Take the ball $B_R = B_R(x_0)$; then u belongs to $W^1_m(B_R)$ and we may solve the Dirichlet problem

(3.1) $$\begin{cases} \operatorname{div}|\nabla w|^{m-2}\nabla w = 0, \\ w - u \in \mathring{W}^1_m(B_R), \end{cases}$$

where $\mathring{W}^1_m(B_R)$ denotes the completion of $C_0^\infty(B_R)$ in the norm of $W^1_m(B_R)$. Put the test function $\eta = w - u$ into the weak forms of equations (3.1) and (0.8), obtaining, respectively,

$$\int_{B_R} |\nabla w|^{m-2}\langle\nabla w, \nabla\eta\rangle = 0, \qquad \int_{B_R} |\nabla u|^{m-2}\langle\nabla u, \nabla\eta\rangle = \int_{B_R}\langle f, \nabla\eta\rangle.$$

Then we get

$$\int_{B_R} \langle|\nabla u|^{m-2}\nabla u - |\nabla w|^{m-2}\nabla w, \nabla u - \nabla w\rangle = \int_{B_R}\langle f, \nabla u - \nabla w\rangle.$$

We estimate from below the left-hand integral by using (1.2). To the right-hand integral we apply the Hölder inequality and (1.4)

$$\int_{B_R} (|\nabla u|^{m-1}+|\nabla w|^{m-2})|\nabla u-\nabla w|^2 \leqslant 2\int_{B_R} |f||\nabla u-\nabla w|$$
$$\leqslant 2\Big(\int_{B_R} |f|^{m/(m-1)}\Big)^{(m-1)/m}\Big(\int_{B_R} |\nabla u-\nabla w|^m\Big)^{1/m}$$
$$\leqslant 4^{(m-1)/m}\Big(\int_{B_R}|f|^{m/(m-1)}\Big)^{(m-1)/m}\Big(\int_{B_R}(|\nabla u|^{m-2}+|\nabla w|^{m-2})|\nabla u-\nabla w|^2\Big)^{1/m}.$$

Hence

$$\int_{B_R}(|\nabla u|^{m-2}+|\nabla w|^{m-2})|\nabla u-\nabla w|^2\leqslant 4\int_{B_R}|f|^{m/(m-1)}.$$

According to (1.3) the last inequality implies

$$\int_{B_r}(|\nabla u|^{m/2}-|\nabla w|^{m/2})^2\leqslant 2m^2\int_{B_R}|f|^{m/(m-1)}\quad \text{for any } r\leqslant R.$$

Now, we apply inequality (1.1) with $A=|\nabla u|^{m/2}$, $B=|\nabla w|^{m/2}$, an arbitrary c, and $\beta>0$.

$$(3.2)\quad \int_{B_r}\big||\nabla u|^m-c\big|\leqslant\int_{B_r}\big||\nabla w|^m-c\big|+\beta\int_{B_r}|\nabla u|^m+\frac{\beta+1}{\beta}\int_{B_r}(|\nabla u|^{m/2}-$$
$$-|\nabla w|^{m/2})^2\leqslant\int_{B_r}\big||\nabla w|^m-c\big|+\beta\int_{B_r}|\nabla u|^m+\frac{2(\beta+1)m^2}{\beta}\int_{B_R}|f|^{m/(m-1)}.$$

One can replace c by any integrable function. In particular, if $c=|\nabla u|^m+$ $+|\nabla w|^m$, $\beta=1$, and $r=R$, then

$$(3.3)\quad \int_{B_R}|\nabla w|^m\leqslant 2\int_{B_R}|\nabla u|^m+4m^2\int_{B_R}|f|^{m/(m-1)}.$$

On the other hand, if $r=\varrho\leqslant R/2$, $c=c_0=|\nabla w(x_0)|^m$ and β still remains arbitrary positive number, then

$$(3.4)\quad \int_{B_\varrho}\big||\nabla u|^m-c_0\big|$$
$$\leqslant\int_{B_\varrho}\big||\nabla w(x)|^m-|\nabla w(x_0)|^m\big|dx+\beta\int_{B_\varrho}|\nabla u|^m+\frac{2(\beta+1)m^2}{\beta}\int_{B_R}|f|^{m/(m-1)}.$$

By Lemma 9 and inequalities (3.3), (1.6) we get

$$\big||\nabla w(x)|^m-|\nabla w(x_0)|^{m'}\big|$$
$$\leqslant(m/2)\big(|\nabla w(x)|^{m-1}+|\nabla w(x_0)|^{m-1}\big)\,|\nabla w(x)-\nabla w(x_0)|$$
$$\leqslant mC_N^2(m)(\varrho/R)^\alpha\,\bar{\int_{B_R}}|\nabla w|^m\leqslant C(m,N)(\varrho/R)^\alpha\Big[\bar{\int_{B_R}}|\nabla u|^m+\bar{\int_{B_R}}|f|^{m/(m-1)}\Big].$$

Put this into (3.4) and use the definition of the Hardy–Littlewood maximal function

$$\bar{\int_{B_\varrho}}\big||\nabla u|^m-c_0\big|\leqslant[\beta+C(m,N)(\varrho/R)^\alpha]M|\nabla u|^m+$$
$$+\Big[C(m,N)(\varrho/R)^\alpha+\frac{2(\beta+1)m^2}{\beta}(R/\varrho)^N\Big]M|f|^{m/(m-1)}.$$

This inequality holds for any ϱ, any $R\geqslant 2\varrho$, and any positive β. Take $R=\varrho\beta^{-1/\alpha}$ with $\beta\leqslant 2^{-\alpha}$ and write $U=|\nabla u|^m$, $F=|f|^{m/(m-1)}$. Then, using inequality (1.12), we obtain

$$\bar{\int_{B_\varrho(x_0)}}|U(x)-c_0|\,dx\leqslant C(m,N)\beta MU(x_0)+\frac{C(m,N)}{\beta^{1+N/\alpha}}MF(x_0).$$

In view of Lemma 2 we conclude that the maximal function $U^\natural(x_0)$ satisfies the inequality

$$(3.5)\quad U^\natural(x_0)\leqslant C(m,N)\beta MU_0(x_0)+\frac{C(m,N)}{\beta^{1+N/\alpha}}MF(x_0),$$

where we still have freedom to choose the parameter $\beta\leqslant 2^{-\alpha}$. Now, we use Lemma 3 as expected. For any $s\geqslant 1$ this gives

$$\|MU\|_s\leqslant\beta C(m,N)B_s\|MU\|_s+\frac{C(m,N)B_s}{\beta^{1+N/\alpha}}\|MF\|_s.$$

At last take β so small that $\beta C(m,N)B_s\leqslant 1/2$. Then

$$(3.6)\quad \|MU\|_s\leqslant C(m,N)B_s^{2+N/\alpha}\|MF\|_s.$$

According to Lemma 1

$$(3.7)\quad \|U\|_s\leqslant 2C(m,N)\Big(\frac{5^Ns}{s-1}\Big)^{1/s}B_s^{2+N/\alpha}\|F\|_s\quad\text{whenever}\quad 1<s<\infty.$$

For $s=p/m$ we have

$$(3.8)\quad \|\nabla u\|_p^{m-1}\leqslant A_p(m,N)\|f\|_{p/(m-1)},$$

where

(3.9)
$$A_p(m, N) = C(m, N)\left(\frac{p}{p-m}\right)^{(m-1)/p} B_{p/m}^{(m-1)(2a+N)/ma} \leqslant [C_N(m)]^p \frac{1}{p-m}.$$

Here B_s is the constant of Lemma 3.

This completes the proof of inequality (0.11). Recall that we have assumed $\nabla u \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$. To eliminate this extra assumption we use an approximation argument.

Let f_j be a sequence of $C_0^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ mappings which converges to f in $L^{p/(m-1)}(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and in $L^{m/(m-1)}(\boldsymbol{R}^N, \boldsymbol{R}^N)$. Define the sequence u_j as solutions to the equations
$$\operatorname{div} |\nabla u_j|^{m-2} \nabla u_j = \operatorname{div} f_j.$$

By Corollary 1, ∇u_j belongs to $L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$. This fact allows us to write inequality (3.8) for the solutions u_j, $j = 1, 2, \ldots$

(3.10)
$$\|\nabla u_j\|_p^{m-1} \leqslant A_p(m, N) \|f_j\|_{p/(m-1)}.$$

On the other hand, the sequence ∇u_j converges to ∇u in $L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$. In fact, for any test function η such that $\nabla\eta \in L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$ we have
$$\int \langle |\nabla u_j|^{m-2} \nabla u_j - |\nabla u|^{m-2} \nabla u, \nabla\eta \rangle = \int \langle f_j - f, \nabla\eta \rangle.$$

Put $\eta = u_j - u$ and apply inequality (1.5); then
$$\int |\nabla u_j - \nabla u|^m \leqslant 2^{m-1} \int |f_j - f|\, |\nabla u_j - \nabla u|$$
$$\leqslant 2^{m-1} \Big(\int |f_j - f|^{m/(m-1)}\Big)^{(m-1)/m} \Big(\int |\nabla u_j - \nabla u|^m\Big)^{1/m}.$$

Here we have used the Hölder inequality. Therefore
$$\int |\nabla u_j - \nabla u|^m \leqslant 2^m \int |f_j - f|^{m/(m-1)} \to 0.$$

What has been done implies that $\nabla u \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and inequality (0.11) is satisfied (as the limit case of (3.10)). In this way we complete the proof of Theorem 2.

Remark. Observe that $A_p = A_p(m, N)$ is unbounded when p comes near to m. It is unexpected on account of $A_m = 1$. This abnormal situation appeared when we used Lemma 1 in passing from (3.6) to (3.7). Since we are interested mostly in large exponents, we do not discuss the question how to make the constant A_p better. There are many reasons for expecting that inequality (0.11) remains true for any $m-1 < p < \infty$ as it does in the linear case, i.e. for $m = 2$. In this case we have two methods at our disposal. One of them has been presented in the Introduction, when we

were making use of the Calderón–Zygmund theorem for the Riesz transform. Another way is to adopt duality arguments. Unfortunately neither method works for non-linear operators. We shall derive from the Theorem the following auxiliary lemma:

LEMMA 10. *Suppose that we are given* $p > m$, $G: \boldsymbol{R}^N \to \mathrm{GL}(N, \boldsymbol{R})$ *which satisfies* (0.12) *and* (0.13), $g \in L^m(\boldsymbol{R}^N, \boldsymbol{R}^N) \cap L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$,
$$H \in L^{m/(m-1)}(\boldsymbol{R}^N, \boldsymbol{R}^N) \cap L^{p/(m-1)}(\boldsymbol{R}^N, \boldsymbol{R}^N)$$
and a function u *which solves the equation*

(3.11)
$$\operatorname{div} \langle G(\nabla u - g), \nabla u - g\rangle^{(m-2)/2} G(\nabla u - g) = \operatorname{div} H.$$

If $\nabla u \in D^m(\boldsymbol{R}^N) \cap D^p(\boldsymbol{R}^N)$, *then for any positive* θ *we have the inequality*

(3.12)
$$\|\nabla u\|_p^{m-1} \leqslant [(1+\theta)^{m-2} K_G - 1] A_p \|\nabla u\|_p^{m-1} + (1+1/\theta)^{m-2} K_G A_p \|g\|_p^{m-1} + A_p \|H\|_{p/(m-1)},$$

where $A_p = A_p(m, N)$ *is the same constant as in Theorem* 2.

Proof. For the proof we formally transform equation (3.11) into the following form

(3.13)
$$\operatorname{div} |\nabla u|^{m-2} \nabla u = \operatorname{div}(\Phi + H),$$

where
$$-\Phi = \langle G(\nabla u - g), \nabla u - g\rangle^{(m-2)/2} G(\nabla u - g) - |\nabla u - g|^{m-2}(\nabla - g) + |\nabla u - g|^{m-2}(\nabla u - g) - |\nabla u|^{m-2} \nabla u.$$

Then by (0.16) and (1.8) we get the estimation
$$|\Phi| \leqslant (K_G - 1)|\nabla u - g|^{m-1} + [(1+\theta)^{m-2} - 1]|\nabla u|^{m-1} + (1+1/\theta)^{m-2}|g|^{m-1}.$$

The first term of the right sum can be estimated by using inequality (1.7). This leads to

(3.14)
$$|\Phi| \leqslant [(1+\theta)^{m-2} K_G - 1]|\nabla u|^{m-1} + (1+1/\theta)^{m-2} K_G |g|^{m-1}.$$

Now, apply inequality (0.11) to (3.13)
$$\|\nabla u\|_p^{m-1} \leqslant A_p \|\Phi\|_{p/(m-1)} + A_p \|H\|_{p/(m-1)}.$$

This together with (3.14) implies (3.12)

4. Proof of Theorem 1. First we prove inequality (0.7) when $Pf \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$. Observe that equation (0.6) is a special case of (3.11) when $G = E$, $H = 0$, $g = f$. In this case inequality (3.12) reads

$$\|Pf\|_p^{m-1} \leqslant [(1+\theta)^{m-2}-1]A_p\|Pf\|_p^{m-1}+(1+1/\theta)^{m-2}A_p\|f\|_p^{m-1}.$$

Now it is time to fix the free parameter θ. Let us take

$$\theta = \left((m-1)2^{m-1}A_p\right)^{-1} < 1;$$

then by elementary computations we simply get

$$(1+\theta)^{m-2}-1 \leqslant (m-2)2^{m-3}\theta \leqslant 1/2A_p \quad \text{and} \quad (1+1/\theta)^{m-2} \leqslant \tfrac{1}{2}4^{m^2-m}A_p^{m-2}.$$

These inequalities imply that

$$\|Pf\|_p^{m-1} \leqslant \tfrac{1}{2}\|Pf\|_p^{m-1}+\tfrac{1}{2}4^{m^2-m}A_p^{m-1}\|f\|_p^{m-1}.$$

Hence

$$\|Pf\|_p \leqslant 4^m A_p\|f\|_p. \tag{4.1}$$

Therefore inequality (0.7) holds for $B_p = 4^m A_p$.

To complete the proof of Theorem 1 we again appeal to an approximation argument. We construct a sequence $f_j \in C_0^\infty(\boldsymbol{R}^N, \boldsymbol{R}^N)$ which converges in $L^m(\boldsymbol{R}^N, \boldsymbol{R}^N) \cap L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$ to the given f. From Corollary 1 we see that $Pf_j \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and then we may write inequality (4.1) for f_j:

$$\|Pf_j\|_p \leqslant 4^m A_p\|f_j\|_p. \tag{4.2}$$

As in the previous section we prove that the Pf_j's tend to Pf in the space $L^m(\boldsymbol{R}^N, \boldsymbol{R}^N)$ and conclude that $Pf \in L^p(\boldsymbol{R}^N, \boldsymbol{R}^N)$. In particular, inequality (4.1) is true for the map f not neccesarily smooth.

5. Proof of Theorem 3. We begin by reducing the problem to the case where G is a smooth matrix-function. Take a measurable $G\colon \Omega \to \mathrm{GL}(N, \boldsymbol{R})$ and extend it as the identity matrix outside Ω. Now, approximate G by smooth matrices $G_j = G*\varphi_j$, where the convolution is taken with C^∞ functions φ_j which are non-negative and $\int \varphi_j(x)\,dx = 1$, $\operatorname{supp}\varphi_j \subset B_{1/j}(0)$. The maps $G_j\colon \boldsymbol{R}^N \to \mathrm{GL}(N, \boldsymbol{R})$ are smooth and converge to G almost everywhere. One can check that such approximation gives

$$a^2|\xi|^2 \leqslant \langle G_j(x)\xi, \xi\rangle \leqslant b^2|\xi|^2 \quad \text{and} \quad K_{G_j} \leqslant K_G.$$

The function u can be approximated in $W_m^1(\Omega)$ by a sequence of more regular functions, namely by the solutions u_j of the following Dirichlet problem:

$$\begin{cases} \operatorname{div}\langle G_j(x)\nabla u_j, \nabla u_j\rangle^{(m-2)/2}G_j(x)\nabla u_j = 0, \\ u_j - u \in \overset{\circ}{W}{}_m^1(\Omega). \end{cases} \tag{5.1}$$

By Lemma 8 the gradients ∇u_j are locally bounded. In particular, they belong to $L_{\mathrm{loc}}^p(\Omega, \boldsymbol{R}^N)$ for any $p \geqslant 1$. Suppose for a moment that we have proved inequality (0.17) for the functions u_j. This implies that $\nabla u \in L_{\mathrm{loc}}^p(\Omega, \boldsymbol{R}^N)$ with p announced in Theorem 3 and u must satisfy (0.17) as well.

Therefore we may assume, without loss of generality, that G is smooth. In such a case ∇u is locally bounded. This property ensures that the integral formulas below make sense. That is why we wanted to have it.

The proof of inequality (0.17) is based on induction with respect to p. We shall show that if (0.17) holds for some exponent $p' \geqslant m$, then it also holds for any p such that $m < p \leqslant \dfrac{N}{N-1}p'$ provided that $(K_G-1)A_p < 1$.

Fix concentric balls $B_r \subset B_R \subseteq \Omega$, $R > r$ and take a function φ such that

$$0 \leqslant \varphi \leqslant 1, \quad \varphi = 1 \ \text{on} \ B_r, \quad |\nabla\varphi| \leqslant \frac{C(N)}{R-r} \quad \text{and} \quad \varphi \in C_0^\infty(B_\varrho), \tag{5.2}$$

where $\varrho = (r+R)/2 < R$. Since (0.14) and (0.17) are invariant under the replacement of u by any additive constant, we may assume that

$$\int_{B_\varrho} u(y)\,dy = 0. \tag{5.3}$$

Multiply both sides of (0.14) by φ^{m-1}

$$\begin{aligned}\operatorname{div}\langle G(\nabla\varphi u - u\nabla\varphi), \nabla\varphi u - u\nabla\varphi\rangle^{(m-2)/2}G(\nabla\varphi u - u\nabla\varphi) \\ = \langle G\nabla u, \nabla u\rangle^{(m-2)/2}\langle G\nabla u, \nabla\varphi^{m-1}\rangle.\end{aligned} \tag{5.4}$$

For simplicity we introduce the following auxiliary functions:

$$v = \varphi u, \quad g = u\nabla\varphi, \quad h = \langle G\nabla u, \nabla u\rangle^{(m-2)/2}\langle G\nabla u, \nabla\varphi^{m-1}\rangle. \tag{5.5}$$

All of them have a compact support in the ball B_ϱ. Thus the equation

$$\operatorname{div}\langle G(\nabla v - g), \nabla v - g\rangle^{(m-2)/2}G(\nabla v - g) = h \tag{5.6}$$

may be considered in the whole space $\boldsymbol{R}^N$ with arbitrarily extended $G = G(x)$ outside Ω. We fix the extension by letting G be the identity matrix in $\boldsymbol{R}^N \setminus \Omega$. In order to make use of Lemma 10 we express the right-hand side of (5.6) in the divergence form

$$h = \operatorname{div}H, \tag{5.7}$$

where H is the vector field defined by formula (1.18). Then for any $p > m$ and any positive θ we have

$$(5.8)\quad \|\nabla v\|_p^{m-1} \leqslant [(1+\theta)^{m-2}K_G - 1]A_p\|\nabla v\|_p^{m-1} + (1+1/\theta)^{m-2}K_G A_p\|g\|_p^{m-1} + A_p\|H\|_{p/(m-1)}.$$

According to the assumption $(K_G-1)A_p < 1$ one can choose θ so small that $[(1+\theta)^{m-2}K_G - 1]A_p < 1$. For such θ inequality (5.8) takes the form

$$(5.9)\quad \|\nabla v\|_p^{m-1} \leqslant C(m, N, p, K_G)(\|g\|_p^{m-1} + \|H\|_{p/(m-1)}),$$

where $C(m, N, p, K_G)$ does not depend on the balls $B_r \subset B_\varrho \subset B_R$. Now, we analyse all the terms in (5.9).

$$(5.10)\quad \|\nabla v\|_p = \Big(\int_{B_\varrho} |\nabla v|^p\Big)^{1/p} \geqslant \Big(\int_{B_r} |\nabla u|^p\Big)^{1/p}.$$

By (5.2), the Hölder inequality, and the assumption $p \leqslant p'N/(N-1)$ we get:

$$\|g\|_p \leqslant C_N \varrho^{N/p}(R-r)^{-1}\Big(\overline{\int_{B_\varrho}} |u|^p\Big)^{1/p} \leqslant C_N \varrho^{N/p}(R-r)^{-1}\Big(\overline{\int_{B_\varrho}} |u|^{p'N/(N-1)}\Big)^{(N-1)/p'N}.$$

We prolong these inequalities applying (5.3) and Lemma 5:

$$(5.11)\quad \|g\|_p \leqslant C_N R^{1+N/p}(R-r)^{-1}\Big(\overline{\int_{B_\varrho}} |\nabla u|^{p'}\Big)^{1/p'}.$$

In order to estimate $\|H\|_{p/(m-1)}$ we shall check that $\int h(y)\,dy = 0$. By the definition of h we have

$$\int h(y)\,dy = \int \langle G\nabla u, \nabla u\rangle^{(m-2)/2}\langle G\nabla u, \nabla \varphi^{m-1}\rangle.$$

The last integral vanishes because the identity

$$\int \langle G\nabla u, \nabla u\rangle^{(m-2)/2}\langle G\nabla u, \nabla \eta\rangle = 0$$

holds for any test function $\eta \in C_0^\infty(\Omega)$. This equation is nothing but the weak form of (0.14).

This allows us to use Lemma 7 with $s = p/(m-1)$ and $q = p'/(m-1)$. Observe that in view of the assumption $m < p \leqslant \frac{N}{N-1}p'$ we have $q > \max\left(1, \frac{sN}{s+N}\right)$. Therefore we obtain

$$\|H\|_{p/(m-1)} \leqslant C_N(p, m)\,\varrho^{1+N(m-1)/p}\Big(\overline{\int_{B_\varrho}} |h|^{p'/(m-1)}\Big)^{(m-1)/p'}.$$

It follows from the definition of h' and φ that $|h| \leqslant C_N(m)(R-r)^{-1}|\nabla u|^{m-1}$. Hence we get the following estimate:

$$(5.12)\quad \|H\|_{p/(m-1)} \leqslant \frac{C_N R^{1+N(m-1)/p}}{R-r}\Big(\overline{\int_{B_\varrho}} |\nabla u|^{p'}\Big)^{(m-1)/p'}.$$

Inequalities (5.9), (5.10), (5.11), (5.12) all together yield

$$\Big(\int_{B_r} |\nabla u|^p\Big)^{(m-1)/p} \leqslant C_N(m, p, K_G)\left[\frac{R^{(m-1)(1+N/p)}}{(R-r)^{m-1}} + \frac{R^{1+(m-1)N/p}}{R-r}\right]\Big(\overline{\int_{B_\varrho}} |\nabla u|^{p'}\Big)^{(m-1)/p'}.$$

Finally

$$(5.13)\quad \Big(\overline{\int_{B_r}} |\nabla u|^p\Big)^{1/p} \leqslant C(m, N, p, K_G, R/r)\Big(\overline{\int_{B_R}} |\nabla u|^{p'}\Big)^{1/p'},$$

where $\varrho = (r+R)/2$ as in (5.2).

Now, we make use of the induction hypothesis, which gives

$$\Big(\overline{\int_{B_\varrho}} |\nabla u|^{p'}\Big)^{1/p'} \leqslant C(m, N, p', K_G, R/\varrho)\Big(\overline{\int_{B_R}} |\nabla u|^m\Big)^{1/m}.$$

This and (5.13) show that inequality (0.17) is satisfied. This ends the proof of the theorem.

6. Proof of Theorem 4. Theorem 4 is a simple consequence of Theorem 3. Let $f\colon \Omega \to \mathbf{R}^N$ be a K-quasiconformal mapping. Then each component $u = f^i$, $i = 1, 2, \ldots, N$, satisfies (0.21). If we assume that $(K^{(N-1)/2}-1)A_p$ is less than 1, then on account of (0.22) we also have $(K_{G^{-1}}-1)A_p < 1$. By Theorem 3 we conclude that

$$\Big(\overline{\int_{B_r}} |\nabla u|^p\Big)^{1/p} \leqslant C_p(N, K, R/r)\Big(\overline{\int_{B_R}} |\nabla u|^N\Big)^{1/N}.$$

Inequality (0.23) now immediately follows from (0.18).

To complete the paper we prove inequality (0.22). The eigenvalues of the inverse matrix $G^{-1}(x)$ are equal to

$$\frac{1}{\beta_N(x)} \leqslant \frac{1}{\beta_{N-1}(x)} \leqslant \ldots \leqslant \frac{1}{\beta_1(x)},$$

where $\beta_1(x) \leqslant \beta_2(x) \leqslant \ldots \leqslant \beta_N(x)$ are eigenvalues of $G(x)$. Since $\det G(x) = 1$, we have $\beta_1\beta_2\ldots\beta_N = 1$ and consequently

$$\beta_1^{N-1}\beta_N \leqslant 1 \leqslant \beta_N^{N-1}\beta_1.$$

Therefore

$$(6.1)\quad \left(1+|G^{-1}(x)-E|\right)^{N/2} = \max\left(\beta_1^{-N/2}, (2-1/\beta_N)^{N/2}\right) \geqslant \beta_1^{-N/2} \geqslant \sqrt{\frac{\beta_N(x)}{\beta_1(x)}}.$$

On the other hand

$$\beta_1^{-N/2} \leqslant \sqrt{\left(\frac{\beta_N}{\beta_1}\right)^{N-1}}$$

and

$$(2-1/\beta_N)^{N/2} \leqslant \beta_N^{N/2} \leqslant \sqrt{\left(\frac{\beta_N}{\beta_1}\right)^{N-1}}.$$

The last two inequalities give

$$(6.2)\quad \left(1+|G_1^{-1}(x)-E|\right)^{N/2} \leqslant \sqrt{\left(\frac{\beta_N(x)}{\beta_1(x)}\right)^{N-1}}.$$

Since inequalities (6.1) and (6.2) hold for almost all $x \in \Omega$, we conclude by the definitions of $K_{G^{-1}}$ and K_f that

$$K_{G^{-1}} \geqslant \sqrt{K_f} \quad \text{and} \quad K_{G^{-1}} \leqslant \sqrt{K_f^{N-1}},$$

which is nothing but (0.22).

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On convergence in the Mikusiński operational calculus

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Abstract. A new description of the convergence of type I′ in the field of Mikusiński operators is given in terms of some family of functionals on the space L of locally integrable functions on $[0, \infty)$. As a consequence, sequential completeness of $\mathscr{F}$ and characterizations of boundedness and precompactness in L, $\mathscr{F}$ and in some subalgebra $\mathscr{F}_0$ of $\mathscr{F}$ are obtained. In particular, it is shown that a set A in $\mathscr{F}$ is precompact if and only if A is bounded (with respect to type I′ convergence).

1. Introduction. The field $\mathscr{F}$ of Mikusiński operators, considered in [7], has various applications, and is interesting also from theoretical point of view. A convergence used in the Mikusiński operational calculus, called type I convergence, is not topological (see [2], [9]). In spite of this, it is sensible to consider completeness with respect to type I convergence. In fact, we can define Cauchy sequences in every abelian group endowed with a convergence. We shall give two definitions (see [8], [5]).

Let X be an abelian group with a convergence G. A sequence $\{x_n\}$ ($x_n \in X$) is called

(i) *P-Cauchy* if $x_{p_{n+1}} - x_{p_n} \to 0$ in G as $n \to \infty$ for every increasing sequence $\{p_n\}$ of positive integers;

(ii) *Q-Cauchy* if $x_{p_n} - x_{q_n} \to 0$ in G as $n \to \infty$ for every pair of increasing sequences $\{p_n\}$ and $\{q_n\}$ of positive integers.

An abelian group X with a convergence G is called *P-complete* (or *Q-complete*) if every P-Cauchy (Q-Cauchy) sequence is convergent in G.

Of course, each P-complete group is also Q-complete but not conversely. The converse implication holds if the convergence G satisfies the Urysohn condition and, additionally, some other natural conditions (see [8]).

Professor J. Mikusiński has posed the problem of P-completeness and Q-completeness of the field $\mathscr{F}$ equipped with type I convergence.

In this paper (Section 9) we shall show that $\mathscr{F}$ with type I convergence (which does not satisfy the Urysohn condition) is Q-complete. The problem of P-completeness of $\mathscr{F}$ (wiht type I convergence) remains open.