Norm inequalities relating singular integrals and the maximal function

by

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Abstract. We prove that if the weighted $L^p$ norms ($1 < p < \infty$) of the Riesz transforms are bounded by the weighted $L^p$ norm of the maximal function, then the weight function satisfies the $G_\theta$ condition of B. Muckenhoupt. Conversely we show that if the weight function satisfies the $G_\theta$ condition for some $\theta > p$, then the weighted $L^p$ norm of any standard singular integral is bounded by the weighted $L^p$ norm of the maximal function.

§ 1. Introduction. We consider the problem of characterizing the non-negative weights $\omega$ for which $(1 < p < \infty)$

$$\int |f|^p \omega \leq C \int |f|^p \omega$$

for all appropriate $f$ where $T_f = K f$ is a singular integral in $\mathbb{R}^n$ with kernel $K$ satisfying the standard conditions

(i) $|K|_\infty \leq C,$

(ii) $|K|_1 \leq C |\omega|^{-1},$

(iii) $|K(x) - K(x+y)| \leq C |y|^{n-1}$ for $|y| \leq |x|/2.$

R. Coifman and C. Fefferman have shown ([1]; Theorem XIII) that (1) holds for $1 < p < \infty$ provided the weight $\omega$ satisfies the $A_{\infty}$ condition. B. Muckenhoupt has shown ([1]; Theorem 2.1) that in the case when $T$ is the Hilbert transform, inequality (1) does not imply that $\omega$ satisfies the $A_{\infty}$ condition. He has derived ([7]; Theorem 1.2) the following necessary condition for (1) (with $T$ the Hilbert transform) which has been conjectured to be sufficient.

$$(C_\theta)$$

There are positive constants $C_0$, $c$ such that

$$\int_E \omega \leq C((|E|/|Q|)^c) \int_{|E|} \omega$$

whenever $E$ is a subset of a cube $Q \subset \mathbb{R}^n$. 

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Here \( |E| \) denotes the Lebesgue measure of \( E \) and \( M \) is the maximal operator defined by
\[
Mf(x) = \sup_{x \in Q \subset E} \frac{1}{|Q|} \int_Q |f|.
\]

Our first result is that if (1) holds for the Riesz transforms, then the weight \( w \) satisfies the \( C_p \) condition. The one dimensional case of this result was obtained by B. Muckenhoupt ([7], Theorem 1.2). Our second result is that if \( w \) satisfies the \( C_p \) condition for some \( q \geq p \), then (1) holds. The question of whether or not \( C_p \) implies (1) remains open. We now state these results precisely. Throughout this paper \( Q \) will denote a cube in \( \mathbb{R}^n \) with sides parallel to the co-ordinate planes and for \( R > 0 \), \( RQ \) denotes the cube concentric with \( Q \) having diameter \( R \) times that of \( Q \). Finally, the letter \( C \) will be used to denote a positive constant not necessarily the same at each occurrence.

**Theorem A.** Let \( 1 < p < \infty \). If the weight \( w \) satisfies
\[
\int \log^p (|f|w^{1-p}) |f|w 
\]
where \( \delta_j \) denotes the \( j \)-th Riesz transform (formally \( \delta_j f(x) = \int \frac{x_j}{|x|^{n+1}} f(x) \)), then \( w \) satisfies the \( C_p \) condition.

**Theorem B.** Let \( 1 < p < q < \infty \). If \( w \) satisfies the \( C_p \) condition, then (1) holds for all singular integrals with kernel satisfying (i), (ii), and (iii) above.

**An Application.** We give sufficient conditions on a pair of weights \((w, e)\) in order that
\[
\int |Tf|^p w \leq C \int |f|^p w
\]
for all singular integrals \( T \) as above. Recall that the pair of weights \((w, e)\) satisfies inequality (3) with \( T \) replaced by the maximal function \( M \) if and only if (8)
\[
\int |M(\frac{1}{q} e^{1-\nu})|p w \leq C \int e^{1-\nu} w
\]
for all cubes \( Q \).

Thus if the weight pair \((w, e)\) satisfies (4) and if \( w \) satisfies the condition \( C_{p+\epsilon} \), for some \( \epsilon > 0 \), then inequality (3) holds. We remark that \( C_p \) weights, unlike \( A_p \) weights, can vanish on open sets.

**§ 2. Proof of Theorem A.** We first give an alternate description of the \( C_p \) condition due to B. Muckenhoupt ([7]).

**Lemma 1. (Muckenhoupt).** The weight \( w \) satisfies the \( C_p \) condition if (and trivially only if) there is \( C < \infty \) such that
\[
|E|_w \leq C \frac{1}{(1 + \log(|E|/|E|_w))^{1/n}} \int_M |M_{wQ}|^p w
\]
whenever \( E \subset Q \), a cube. Here \( |E|_w = \frac{1}{w} \int_E w \).

The case \( n = 1 \) of this lemma is contained in [7] and the proof given there extends to \( n > 1 \) with minor modifications which we sketch in an appendix below. In any event one can verify that all arguments using the \( C_p \) condition in this paper hold just as well using (5) as the definition.

**Proof of Theorem A.** The key step here is the observation that \( \log Mf \) is in \( \text{BMO} \) if \( Mf \) is finite a.e. ([21]; p. 643). Suppose \( E \subset Q \) a cube and set
\[
f = \log^{1/n} (|Q|/|E|) M_{wE}.
\]
Simple computations show that there is a constant \( C \) independent of \( Q \) and \( E \) such that
\[
\int f = |Q|^{1/n} \int f \leq C,
\]
and
\[
\|f\|_{\text{BMO}} = \sup_{Q \subset E} \int f |f|^{-1} |f - f_Q| < C.
\]

From (8) and the duality of \( L^1 \) and \( \text{BMO} \) ([3]; Theorem 3) we obtain
\[
f = f_Q + \sum_{j=1}^\infty |R_j f_j|
\]
where \( |f_j| \leq C, 0 < j < \infty \). Let \( g_j = \sum_{j=1}^\infty |R_j f_j\) for \( 1 < j < \infty \). Here \( 2Q \) denotes the cube concentric with \( Q \) and twice the side length; \( 2Q' \) denotes its complement. Let \( x \) be the centre of \( Q \) and set \( A_j = |R_j f_j(x)| \). Then for \( \epsilon < Q \) we have by property (iii)
\[
|\int_{Q'} |R_j f_j(x)| dx \leq C \int_{2Q} |h_j(y)| (|x - e|^{-1} |y - e|^{-n+\epsilon}) dy \leq C \quad (\epsilon < Q)
\]
and thus also
\[
|\sum_{j=1}^\infty A_j| \leq C \frac{1}{|Q|} \int \sum_{j=1}^\infty |R_j | h_j + C \frac{1}{|Q|} \int |f| + \sum_{j=1}^\infty |R_j f_j| + C
\]
since \( f = f_Q + \sum_{j=1}^\infty |R_j f_j| + \sum_{j=1}^\infty |R_j f_j| \). However,
\[
\frac{1}{|Q|} \int |R_j f_j| \leq \int \frac{1}{|Q|} \int |R_j f_j|^2 \leq \frac{1}{|Q|} \int |f|^{2/n} \leq C
\]
by Hölder's inequality, the \( L^1 \) boundedness of the Riesz transforms and the boundedness of the \( f_j \). Combining this with (7), (11) and \( \|f\|_{\text{BMO}} \leq C \) we obtain.
\[ \left| \sum_{j=1}^{n} A_j \right| \leq C \text{ and (10) now yields} \]
\[ \left| f - \sum_{j=1}^{n} E_j g_j \right| \leq C \text{ on } Q. \]

From this and equation (9) we have
\[ \sum_{j=1}^{n} |E_j g_j| \geq \log \left( \frac{|Q|}{|E|} \right) - C \text{ a.e. on } E \]
and from (2) we now obtain
\[ |E| \log \left( \frac{|Q|}{|E|} \right) - C \leq C \sum_{j=1}^{n} |E_j g_j|^p w \leq C \sum_{j=1}^{n} |M g_j|^p w \leq C |M X_Q|^p w \]
which is (5). Lemma 1 now completes the proof of Theorem A.

\[ \text{§ 3. Proof of Theorem B.} \]

We begin with a variation of the Whitney covering lemma used in [3].

**Whitney Covering Lemma.** Given \( E \ni 1 \), there is \( C = C(E, n) \) such that if \( \Omega \) open in \( E^n \), then \( \Omega = \bigcup Q_j \), where the \( Q_j \) are disjoint cubes satisfying
\[ \begin{align*}
& (i) \quad \delta \leq \frac{\text{dist}(Q_j, Q_j')}{\text{diam } Q_j} \leq 15 \delta, \\
& (ii) \quad \sum_{Q_j \ni \omega} \leq C \delta. 
\end{align*} \]

**Proof.** Conclusion (ii) is a consequence of (i) and a geometric packing argument ([3]; p. 16). Conclusion (i) in turn can be established easily by standard arguments — see for example [6]; Theorem 2.1.

In attempting to prove Theorem B by the methods of R. Coifman and C. Fefferman in [1], we will be led via the \( A_\infty \) condition to consideration of integrals of the form \( \int |T f|^p w \) where \( \{Q_j\} \) is a Whitney covering of the open set \( \{T f \geq \lambda\} \) (\( T \) is the maximal operator associated to \( T \) — see Lemma 2 below). We thus begin by investigating the operator \( M_{\rho, \gamma} \) defined below in terms of Marcinkiewicz integrals.

**Definition.** Let \( 1 < p, q < \infty \) and suppose \( f : \mathbb{R}^n \to [0, \infty) \) is lower semicontinuous. Let \( Q_0 \ni f > 2^p \) and define
\[ \left( M_{\rho, \gamma} f(x) \right)^p = \sum_{E \ni x} \frac{\rho(E)^{p(n-1)}}{\rho(E)^n + |x - y|^2} dy \]
where \( \rho(y, E) \) denotes the distance from \( y \) to the set \( E \).

Fix \( E \ni 1 \) and let \( \Omega_k = \bigcup Q_j^k \) be as in the Whitney covering lemma. Then
\[ M_{\rho, \gamma} f(x) = \sum_{J \ni x} \rho(J)^{p(n-1)} \frac{\rho(J)^n + |x - y|^2} \int_{Q_j^k} dy \]
in the sense that the ratio of the right and left sides is bounded between two positive constants depending only on \( E \) (and not on \( x \)). We use only this latter expression for \( M_{\rho, \gamma} f \) in the sequel.

**Lemma 2.** Suppose \( 1 < p < q < \infty \) and that \( w \) satisfies the \( C_q \) condition. Let
\[ T^w f(x) = \sup_{t < \epsilon < \epsilon < \infty} \left| \int_{Q(t)} K(y) f(x - y) dy \right| \]
where \( K \) is a kernel satisfying (i), (ii), and (iii) of § 1. Then for all \( f \) with compact support we have
\[ \int \left( M_{\rho, \gamma} (T^w f)^p w \right)^{1/p} \leq C \left[ \int |T^w f|^p w + \int |f|^p w \right] \]

The proof of Lemma 2 is fairly long and will be postponed to § 4. We remark that Lemma 2 may fail when \( p = q \) even for weights \( w \) satisfying the \( A_\infty \) condition. For example when \( p = q = 2 \), let \( f \) be the characteristic function of the unit interval in \( R \). The \( L^p \) transform of \( f \) is given by \( T f = \chi \), hence \( \int \chi \frac{\rho(J)^n + |x - y|^2} \int_{Q_j^k} dy \]
where \( \rho \) is a kernel satisfying (i), (ii), and (iii) of § 1. Then for all \( f \) with compact support we have
\[ \int \left( M_{\rho, \gamma} (T^w f)^p w \right)^{1/p} \leq C \left[ \int |T^w f|^p w + \int |f|^p w \right] \]

The proof of Lemma 2 is fairly long and will be postponed to § 4. We remark that Lemma 2 may fail when \( p = q \) even for weights \( w \) satisfying the \( A_\infty \) condition. For example when \( p = q = 2 \), let \( f \) be the characteristic function of the unit interval in \( R \). The \( L^p \) transform of \( f \) is given by \( T f = \chi \), hence \( \int \chi \frac{\rho(J)^n + |x - y|^2} \int_{Q_j^k} dy \]
since property (ii) of the kernel $K$ shows that $T^* f \leq CMf$ outside $2Q$. If in addition $f$ is bounded, then (9); see 6.2, p. 58) $\int e^{2\pi i n \cdot \xi} d\xi < \infty$ for some $a > 0$ and thus $\| x \| e^{2\pi i \lambda} \leq C e^{-\lambda/2Q}$ for $\lambda > 0$. Applying the $C_2$ condition to this latter inequality and integrating we obtain

$$\int |T^* f|^p w \leq C \int |M_{Q^2}|^p w \leq C \int |Mf|^p w < \infty$$

since $q > p$ and $\sup \| f \| < Q$. Thus (1) holds for bounded $f$ with compact support and a simple limiting argument proves the general case. Indeed, if $\int |Mf|^p w < \infty$ then $f$ is locally integrable and so $T^* f \leq \lim inf \| T^* f \|_w$ where $f_\delta(x) = f(\delta x)$ if $|x|, |f(x)| \leq R$ and 0 otherwise. An application of Fatou's lemma now completes the proof of Theorem B.

§ 4. Proof of Lemma 2. We begin with two preliminary lemmas. The first is a variant of Lemma 5.1 in [7].

**Lemma 3.** Suppose $w$ satisfies the $C_2$ condition, $1 < q < \infty$. Then for all $\varepsilon > 0$, there is $C(\varepsilon) < \infty$ so that whenever $(Q_i)_i$ is a collection of disjoint subcubes of a cube $Q$, then

$$\int \sum_{i \in J} |M_{Q_i}|^p w \leq C(\varepsilon)\|Q_i\|_w + \varepsilon \int |M_{Q_i}|^p w$$

for all $R \geq 2$. Consequently,

$$\int \sum_{i \in J} |M_{Q_i}|^p w \leq C \int |M_{Q_i}|^p w$$

Proof. A classical estimate for the Marcinkiewicz integral (see [4]; Theorem 1 (3)) shows that $|E_1| \leq C e^{-\alpha |Q|}$ for $\lambda > 0$ where $\alpha$ is some positive constant and $E_1 = \left\{ |f| \leq f \right\}$. Since $\sum |M_{Q_i}|^p w$ is bounded outside $2Q$, the $C_2$ condition implies $|E_1| \leq C e^{-\alpha |Q|}$ for $\lambda > 0$ sufficiently large and this in turn yields

$$\int \sum_{i \in J} |M_{Q_i}|^p w \leq C e^{-\alpha |Q|} \int |M_{Q_i}|^p w$$

Choosing $\lambda$ so large that $C e^{-\alpha |Q|} \leq \delta$ we obtain the conclusion of Lemma 3 with $C = \lambda$.

**Lemma 4.** Suppose $1 < q < \infty$ and that $w$ satisfies the $C_4$ condition. Then for all compactly supported $f$

$$\int |M_{p,n}(Mf)|^p w \leq C \int |Mf|^p w$$

Proof. Let $Q_b = (Mf > 2^b) = \bigcup Q_i^b$ be as in the Whitney covering lemma with $R = 10$. Let $N$ be a positive integer (to be chosen later) and fix a Whitney cube $Q_i^b$. We now claim

$$\int \sum_{i \in J} |Q_i^b|^{p-N} \leq C 2^{-N} |Q_i^b|^{p-N}$$

where $C$ depends only on the dimension $n$. Indeed, let $g = f_{x_o}M^{N-N}$ and $h = f - g$. Property (ii) of the Whitney covering lemma shows by a standard argument (see e.g. [9], p. 19) that $Mh(x) \leq C 2^{N-N}$ for $x$ in $5Q^b_i$. Now $Mf \leq Mh + Mh$ and thus for $N$ so large that $2^{-N-N} \leq 1/2$, we have

$$\int \sum_{i \in J} |Q_i^b|^{p-N} \leq \int \left| (Mg > (1/2) |Q_i^b|) \right|$$

$$\leq C 2^{-N} \int |g| = C 2^{-N} \int |f|$$

since $M$ is weak type $1,1$.

$$\leq C 2^{-N} \left( C 2^{N-N} \int |Q_i^b|^{p-N} \right)$$

by (i) of the Whitney lemma

which proves (16).

Now let $S(\delta) = 2^{2b} \sum_{i \in J} |M_{x_o}Q_i^b|^{p-N}w$ and $S(\delta; N, b) = 2^{2b} \sum_{i \in J} |M_{x}Q_i^b|^{p-N}w$. Since $Q_i^b \cap Q_j^b = \emptyset$ implies $Q_i^b \subset 5Q^b_j$, we have

$$S(\delta; N, b) \leq 2^{2b} \sum_{i \in J} |M_{x}Q_i^b|^{p-N}w$$

$$= \int + \int = I + II$$

for $N$ large.

By (14) of Lemma 3

$$I \leq C(\varepsilon) 2^{2b} \int 10Q_i^{b-N}w + \varepsilon 2^{2b} \int |M_{Q_i^b}|^{p-N}w$$

where $\varepsilon > 0$ is at our disposal. Simple estimates on $M_{Q_i^b}$ show that if $x_o^{b-N}$ denotes the centre of $Q_i^b$.
Thus for \( N \) large
\[
S(k) \leq \sum_{j \in H} S(k; j; N, i) 
\leq C(\delta) 2^{\alpha p} \left[ \int \sum_{j \in H} \chi_{Q_i^1}^{1=-N} w \right] w + [\alpha^{2^p} + C2^{Np-\alpha}] S(k-N) 
\leq C \cdot 2\delta^p \cdot \Omega_{k-N} w + (1/2) S(k-N) 
\]
for \( N \) sufficiently large and \( \delta \) sufficiently small upon appealing to property (ii) (with \( R = 10 \)) of the Whitney covering lemma. Thus with \( S_M = \sum_{k \in M} S(k) \), we have
\[
S_M \leq (1/2) S_M + C \int |M|^{1/p} w \quad \text{for all } M. 
\]

Recall now that \( f \) has compact support, say \( \text{supp} f \subset Q \) a cube. Let \( 2^L < |Q|^{-1} \int |f| \leq 2^{L+1} \). Then \( \Omega_{k} \subset 2Q \) for \( k \geq L+1 \) and (15) of Lemma 3 shows that
\[
\sum_{k \in f_{Q_i^1}} \sum_{j \in H} 2^{\alpha p} \int |M_{X_k}^{1} w| \leq C \sum_{k \in f_{Q_i^1}} |M_{X_k}^{1} w| \leq C \sum_{k \leq f_{Q_i^1}} |M_{X_k}^{1} w| < \infty
\]
since \( m < p \) and \( \int |M|^{1/p} w < \infty \) (otherwise there is nothing to prove). On the other hand if \( k = L \), then \( \Omega_{k} \subset 2^{L-k} \cdot Q \) and (15) of Lemma 3 yields
\[
\sum_{k \in f_{Q_i^1}} \sum_{j \in H} 2^{\alpha p} \int |M_{X_k}^{1} w| \leq C \sum_{k \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1-L} w| \leq C \sum_{k \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w| < \infty
\]
since \( \sum_{k \leq f_{Q_i^1}} 2^{-np} |M_{X_k}^{1} w| \leq C_{p, \alpha} |M_{X_k}^{1} w| \) for \( p > 0 \). Thus \( S_M < \infty \) for all \( M \) and (17) now yields
\[
\int |M_{X_M}^{1} w| \leq C \sum_{k \in f_{Q_i^1}} S_M \leq C \int |M|^{1/p} w
\]
and this completes the proof of Lemma 4.

Proof of Lemma 2. Let \( \Omega_{k} = (T^k f > 2^p) = \bigcup_i Q_i^1 \) be as in the Whitney covering lemma with \( R = 20 \). A fundamental inequality of R. Coifman and C. Fefferman states (11), (8), p. 248
\[
|a \in \bigcup_{i \in Q_i^1} (T^k f > 2^p)| \leq C \cdot 2^{-N} |Q_i^1|^{-1}
\]
whenever \( 10^{Q_i^1} \in \{ Mf > 2^{k-N} \}, \quad N \geq 1. \)

Let \( \{Mf > 2^p\} = \bigcup H \) be as in the Whitney covering lemma with \( R = 20 \). We observe that for each cube \( Q_i^1 \) there are two cases (\( N \) will be chosen later).

Case (1). \( 10^{Q_i^1} \subset \{ Mf > 2^{k-N} \} \) in which case \( 10^{Q_i^1} \subset \{ T^k f > 2^p \} \) for some \( L \) where \( c_L \approx 15 \cdot 2^{10N} \approx 3000 \cdot 2^{10N} \) (choose \( R = 20 \) to contain the centre of \( Q_i^1 \)).

Case (2). \( 10^{Q_i^1} \subset \{ Mf > 2^{k-N} \} \) in which case (18) implies \( \sum_{j \in f_{Q_i^1}} |Q_i^1| \leq C \cdot 2^{-N} |Q_i^1|^{-1} \).

Now let
\[
S(k) = \sum_{i \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w| \quad \text{and}
\]
\[
S(k; i) = \sum_{j \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w| \leq \sum_{j \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w|.
\]
The last inequality follows from the fact that \( Q_i^1 \subset \{ Mf > 2^{k-N} \} \) whenever \( Q_i^1 \subset \{ Mf > 2^{k-N} \} \) (property (i) of the Whitney lemma). Thus
\[
S(k; i) \leq \sum_{j \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w| = \sum_{Q_i^1 \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w|.
\]

By (14) of Lemma 3 we have
\[
I \leq C(\delta) 2^{\alpha p} \sum_{i \in f_{Q_i^1}} \int |M_{X_k}^{1} w| \leq \delta \int |M_{X_k}^{1} w|
\]
where \( \delta > 0 \) is at our disposal and if \( z_{Q_i^1} \) denotes the centre of \( Q_i^1 \), then
\[
II \leq C_{2}^{\alpha p} \int_{|w - z_{Q_i^1}|^{1-p} w(z)} dV \leq C_{2}^{\alpha p} \int_{|w - z_{Q_i^1}|^{1-p} w(z)} dV
\]
in case (2)
\[
\leq C_{2}^{\alpha p} \int_{|w - z_{Q_i^1}|^{1-p} w(z)} dV
\]
Combining the estimates for \( I \) and \( II \) we obtain
\[
S(k; i) \leq C_{2}^{\alpha p} \sum_{j \in f_{Q_i^1}} 2^{\alpha p} \int |M_{X_k}^{1} w| \leq C_{2}^{\alpha p} |z_{Q_i^1}|^{2^{p-N}R} \int |M_{X_k}^{1} w|
\]
whenever \( Q_i^1 \subset \{ Mf > 2^{k-N} \} \) is a case (2) cube. Thus
\[
S(k) \leq \sum_{Q_i^1 \in f_{Q_i^1}} S(k; i) + \sum_{j \in f_{Q_i^1}} S(k; i) = III + IV.
\]
Now since each $Q_i^j$ intersects at most $C$ of the $Q_i^{j-1}$,
\[
III \leq \sum_{Q_i^j} \sum_{Q_k} 2^{2p} \int |M_{\chi_{Q_i^j}}|^p w \leq C \sum_{Q_k} 2^{2p} \int |M_{\chi_{Q_k}}|^p w
\]
by (15) of Lemma 3 and the inequality $M_{\chi_{Q_k}} \leq C M_{\chi_{Q}}$. For the remaining term we have by (19)
\[
IV \leq C 2^{2p} \int \left( \sum_{Q_k} 2^{2p} |\chi_{Q_k} |^p \right) w + (\delta + C 2^{-N}) \sum_{Q_k} 2^{2p} \int |M_{\chi_{Q_k}}|^p w
\]
\[
\leq C 2^{2p} \left( \Omega_{Q_k} \right)^{1/p} + (\delta + C 2^{-N}) \sum_{Q_k} |M_{\chi_{Q_k}}|^p w
\]
by property (ii) of the Whitney covering lemma (with $R = 20$) and upon choosing $\delta$ small enough and $N$ large enough. Combining III and IV we have
\[
(20) \quad S_k \leq (1/2) S_{k-1} + C 2^{2p} \Omega_{Q_k} |w| + C 2^{2p} \sum_{Q_k} |M_{\chi_{Q_k}}|^p w.
\]
Now let $S_M = \sum_{k=0}^M S_k$ and sum inequality (20) over $k \leq M$ to obtain
\[
(21) \quad S_M \leq (1/2) S_M + C \int |T^* f|^p w + C \int |M_{\rho_d}(f)|^p w
\]
\[
\leq (1/2) S + C \left( \int |T^* f|^p w + \int |M_{\rho_d}(f)|^p w \right)
\]
by Lemma 4.

Now the argument used at the end of the proof of Lemma 4 to show that $S_M < \infty$ can also be used here to obtain $S_M < \infty$ for all $M$ (use the fact that $T^* f \leq C M f$ outside $2Q$ if supp $f = Q$). Thus (21) yields
\[
\int |M_{\rho_d}(T^* f)|^p w \leq C \sup_M S_M \leq C \left( \int |T^* f|^p w + \int |M_{\rho_d}(f)|^p w \right)
\]
and this completes the proof of Lemma 2.

**Appendix.** We sketch a proof of Lemma 1. As already mentioned, the case $n = 1$ is in [7] and the proof given there extends to $n > 1$ with minor modifications. As that proof is fairly long, we limit ourselves here to a brief discussion of the required modifications, assuming that the reader is familiar with Sections 5 and 6 of [7].

Clearly $Q_0$ implies (5) so we now assume that (5) holds. Lemma 5.1 of [7] extends to $E^n$ without any essential change in the proof. Thus we can find $0 < \delta < 2^{-n}$ so small that whenever $(Q_i)$ is a collection of disjoint subcubes of a cube $Q$ with $\sum_i Q_i \leq 2 \delta |Q|$, then
\[
\int \left( \sum_i |M_{\chi_{Q_i}}|^p \right) w \leq (1/2) \int |M_{\chi_{Q}}|^p w.
\]
Now given $E \subset Q$ a cube in $E^n$, let $N$ be the least integer satisfying $\delta^N |Q| \leq |E|$. Define $E_1 = E$ and $E_j = \{ z \in E_{j-1} : \delta^j f \}$ for $1 \leq j \leq N$ where $M_{\rho_d}$ denotes the dyadic maximal operator $M_{\rho_d}(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int |f|$. Now $E_j = \bigcup_{i} Q_i^j$ where the $Q_i^j$ are the maximal dyadic cubes $I$ satisfying $|I|^{-1} \int_{2I} f > \delta^j$. Thus $\delta^j |E \cap Q_i^j| \leq |Q_i^j| \leq 2 \delta^j |Q_i^j|$ for $1 \leq j \leq N$ and all $k$.

Using (a), (b) and (22) we obtain
\[
\int A_j w \leq (1/2) \int A_{j-1} w, \quad 2 \leq j \leq N
\]
where $A_j(w) = \sum_i |M_{\chi_{Q_i}}(w)|^p$ and the proof can now be completed by iterating this inequality as in Section 6 of [7].

**References**


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